Quantum Echoes at Exceptional Points in Microwave Billiards

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We experimentally investigated the time decay behavior of resonances near and at exceptional points, where two or more complex eigenvalues and also the associated eigenfunctions coalesce. The measurements were performed with a dissipative microwave billiard, whose shape depends on two parameters. First, exceptional points of the measured resonance spectrum were localized in the parameter space. Then, the time dependence of the decay of the corresponding pairs of resonances was studied. The $t^2$-dependence predicted at the exceptional point on the basis of a matrix model for the two states which coalesce could be verified. Outside the exceptional point the predicted quantum echoes were detected. To our knowledge this is the first time that quantum echoes related with exceptional points were observed experimentally.

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In quantum mechanics many dynamical processes are dominated by (avoided) level crossings. A crossing of two eigenvalues requires the variation of at least two parameters. It has been known for many years, that near a crossing the two energy surfaces form two sheets of a double cone [1, 2]. The apex of the double cone is associated with a singularity and called diabolic point (DP) [3, 4]. A DP occurs in Hermitian Hamiltonians. Phenomena related with a DP, as e.g. geometric phases, have been studied theoretically in various generalizations of Berry’s original paper (see e.g. [5, 6] and references therein) and experimentally e.g. in [7] with a microwave billiard. For non-Hermitian Hamiltonians, as those used for the theoretical description of dissipative systems, a topologically different singularity may appear: an exceptional point (EP) [8] – there the eigenvalues do not only degenerate, they coincide, because the two associated eigenstates coalesce [9]. Thus EPs are not only singularities of the spectrum but also of the eigenstates. Exceptional points have been observed in laser induced ionizations of atoms [10], crystals of light [11], electronic circuits [12], the propagation of light in dissipative media [13, 14] and in microwave billiards [15–17]. They also appear in many theoretical models: e.g. in that used for the decay of superdeformed nuclei [18], phase transitions and avoided level crossings [19, 20], geometric polarity reversal [21], tunneling between quantum dots [22], and in the context with the crossing of two Coulomb blockade resonances [23].

Exceptional points give rise to interesting phenomena such as level crossings and geometric phases [4, 17, 24]. In this article we present novel experimental results on the time decay of resonances in the vicinity of and at an EP. Close to an EP, the time spectrum exhibits – beside the decay of the isolated resonances – oscillations with a fixed frequency; these are called quantum echoes [25]. Quantum echoes occur due to the transfer of energy between the two nearly degenerate resonances. At the EP their vanishing and a quadratic time dependence of the resonance amplitude was predicted. This time dependence is a characteristic property of EPs with exactly two coinciding eigenvalues. In general more than two eigenvalues may coincide at an EP, however the coincidence of two eigenvalues is the most probable. Therefore we focus on these EPs.

For wavelengths longer than twice the height of the microwave billiard the scalar Helmholtz equation of the electric field strength in a cylindric microwave billiard is equivalent to the Schrödinger equation for the wave functions in a quantum billiard of corresponding shape (see e.g. [26, 27]). Thus, aside from their intrinsic interest, flat microwave billiards yield a possibility to gain experimental insight into properties of the eigenvalues and eigenfunctions of two-dimensional quantum billiards.

We used a coupled pair of dissipative, cylindric and flat microwave billiards manufactured of copper to mimic the two-level system required for an EP to occur [15–17]; a sketch is shown in Fig. 1. The pair of microwave billiards is obtained by dividing a circular microwave billiard into two approximately equal parts differing in their areas by about 5%. Due to this slight difference in size the twofold degeneracy of the eigenfrequencies of the circular billiard is lifted, that is the eigenfrequencies are split into two nearly degenerate ones. Their crossing behavior has been analyzed in [15–17] with a slightly modified experimental setup. In order to control the coalescence of the two states, we must be able to control frequency differences and width differences. We achieved this by tuning two parameters, the opening length $s$ between the two parts of the microwave billiard and the position $δ$ of a semi-circular Teflon® disc (see Fig. 1). The parameter $s$ con-

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FIG. 1: Sketch of the experimental setup: In the microwave billiard (right part) the parameter $s$ refers to the length of the coupling slit and $\delta$ to the position of the semi-circular Teflon® disc, measured from the billiard center $c$. The antenna positions are shown as crosses. The vectorial network analyzer (left part) measures the ratio of the power of the signals received at the left antenna 1 and emitted at the right antenna 2, $|S_{12}|^2$, and their relative phase. Shown is a part of a typical transmission spectrum measured in dB. The arrow points at two nearly degenerate resonances which for a certain choice of the parameters $s$ and $\delta$ coalesce at the exceptional point at $2.757 \pm 0.001$ GHz.

trols the coupling between the eigenmodes of the two microwave billiards, while the parameter $\delta$ mainly affects the resonance frequencies of that part of the microwave billiard which contains the Teflon® disc.

To measure the spectrum of the billiard system the vectorial network analyzer “HP-8510C” (VNA) produces a HF-signal, which is piped through a 30 dB amplifier and is forwarded to a dipole antenna with 0.5 mm in diameter and 2 mm in length. If the signal frequency is tuned to a resonance frequency of the microwave billiard, the dipole antenna excites the corresponding eigenmode such that power is radiated into the microwave billiard – otherwise the power is reflected at the antenna. After a short settling time (less than 1 ms) a standing wave has developed in the microwave billiard. While in a reflection measurement the same antenna which fed the power into the microwave billiard couples to this standing wave, in a transmission measurement one feeding and one receiving antenna are attached to the microwave billiard. The power coupled out of the microwave billiard is the system answer. It is lead back into the VNA, which compares the produced and the received signal in amplitude and phase. These two measured quantities are the basis of the data analysis.

To obtain the eigenfunctions we use the perturbation body method. This method is based on the Maier-Slater theorem [28], which states, that the amount of the frequency shift of a given resonance depends on the difference of the square of the electric field strength and the square of the magnetic induction at the position of the perturbing body. As perturbation we use a cylindrical magnetic rubber with about 2.2 mm in diameter and 3.5 mm in height which is placed inside the microwave billiard. Compared to iron the usage of a magnetic rubber has the advantage, that it suppresses the contribution of the magnetic induction to the frequency shift and still enables us to change the position of the perturbation body inside the closed microwave billiard with a guide magnet. By moving the perturbation body through the microwave billiard on a grid with spacing 5 mm and comparing the perturbed and unperturbed resonance frequency for each grid point we obtain an image of the intensity distribution of the electric field strength.

Quantum echoes are time recurrent signals. They have been seen e.g. in open microwave billiards [25] but they also appear in the gravity induced interference effect [29], the Aharonov-Bohm effect [30] as well as in NH$_3$-MASERs [29]. In the latter it is better known as a beating with a frequency called Rabi frequency. In the following we will analyze the decay behavior with time of two resonance states, i.e. the superposition of the energy transfer between two nearly degenerate resonances and that to the exterior, using a two-level Hamiltonian. Based on this two-level effective model we make predictions about the appearance of quantum echoes near and at an EP. The two-level model was used by Stafford and Barrett [18] in a different context. The theoretical results presented there may be mapped one to one onto our problem. This signifies, that the experimental results presented here are of general interest in many different fields of physics dealing with EPs.

The electric field strengths $\vec{E}_j(t) = E_j(t) \hat{e}_z$ with $j = 1, 2$ of the eigenmodes in the two parts of the microwave billiard can be described as a pair of coupled damped harmonic oscillators,

$$\left( \frac{\partial^2}{\partial t^2} + \gamma_1 \frac{\partial}{\partial t} + \omega_1^2 \right) E_1(t) = F_1(t) - sE_2(t)$$
$$\left( \frac{\partial^2}{\partial t^2} + \gamma_2 \frac{\partial}{\partial t} + \omega_2^2 \right) E_2(t) = F_2(t) - sE_1(t),$$

with angular frequencies $\omega_1$ and $\omega_2$, respectively, and decay rates $\gamma_1$ and $\gamma_2$. The driving terms $F_j(t)$ with $j = 1, 2$ describe the coupling of the antennae to the two eigenmodes.

The coupled differential equations (1) can be solved with the Green’s function method. The inverse of the Green’s function is given by [29]

$$G^{-1}(\omega) = \omega^2 - \mathbb{H}(\omega)$$

with the non-Hermitian two-level Hamiltonian [16, 17]

$$\mathbb{H}(\omega) = \left( \begin{array}{cc} \omega_1^2 - i \omega \gamma_1 & s \\ s & \omega_2^2 - i \omega \gamma_2 \end{array} \right).$$

While the real parts of the diagonal elements ($\omega_1^2$ and $\omega_2^2$) are proportional to the square of the eigenfrequencies of the uncoupled billiard parts, the imaginary parts...
The Hamiltonian is symmetric since the system is time reversal invariant and it is non-Hermitian due to the antennas and the absorption in the billiard walls and the Teflon® disc. We denote the eigenvalues of Eq. (2b) by $\varepsilon_{\pm} := \tilde{\omega} - i \frac{\gamma}{2} \pm R$ with $R := \frac{1}{2} \sqrt{(\Delta \omega^2 - i \Delta \Gamma)^2 + 4 s^2}$, $\tilde{\omega} := \sqrt{(\omega_1^2 + \omega_2^2)/2}$, $\Gamma := (\Gamma_1 + \Gamma_2)/2$, $\Delta \omega^2 := \omega_2^2 - \omega_1^2$ and $\Delta \Gamma := \Gamma_2 - \Gamma_1$. Although in the experiment the physical quantities $\omega_1$, $\omega_2$, $\Gamma_1$ and $\Gamma_2$ are complicated functions of the Teflon® disc position $\delta$ and the opening length $s$ (see Fig. 1), these two parameters are sufficient to achieve coalescence of the two eigenvalues of the two-level Hamiltonian, i.e. the vanishing of the parameter $R$. At the EP $\Delta \omega^2$ vanishes and $\Delta \Gamma^2 = 4 s^2$. The parameter setting is denoted by $s = s^{EP}$ and $\delta = \delta^{EP}$. As we are interested in the decay behavior with time of the two-level system we need to apply a Fourier transformation from the frequency to the time domain on the inverse of Eq. (2a), that is on $G(\omega)$. A simple analytic expression is obtained with the following approximation: in the vicinity of $\omega \simeq \omega_1, \omega_2$ the terms $\omega_1$ and $\omega_2$ are small in comparison with $\omega^2$. Therefore $\omega$ in Eq. (2b) is replaced by the average value of the angular frequencies $\bar{\omega}$. This approximation takes into account that the major contribution comes from the poles of the Green's function. With

$$\bar{\omega} := \frac{R}{2 \omega}, \quad f := \frac{i}{2} \gamma, \quad \bar{\gamma} := \frac{\gamma_2 - \gamma_1}{2}$$

the Fourier transformation $\tilde{G}_{12}(t)$, which describes the transmission from one resonance to another, is given by [18, 22]

$$|\tilde{G}_{12}(t)|^2 \approx \frac{s^2}{\bar{\omega}^2} \frac{e^{-it\bar{\omega}}}{\sqrt{\xi + \xi^-}} \left[ \cos(\Omega t) + i f \frac{\sin(\Omega t)}{\Omega} \right]^2.$$  

(3b)

The signal decays with a decay constant of $\bar{\gamma}/2$ and oscillates with a high, not resolvable angular frequency $\bar{\omega}$. The physical value $\Omega/2\pi$ denotes the much lower ($R \ll \bar{\omega}$) echo frequency. It corresponds to the Rabi frequency which is well known in quantum optics [29, 31], nuclear physics [18] and quantum computing [32].

As was pointed out already in [18, 22] the imaginary part of the echo frequency $\Omega/2\pi$ adds to the decay, while the real part describes the oscillation between the two resonances. As a consequence, the quantum echoes vanish if $\Omega$ is purely imaginary. This happens for all subcritical couplings $s < s^{EP}$ at the critical Teflon® disc position $\delta = \delta^{EP}$. For overcritical couplings $s > s^{EP}$ the quantum echoes persist for all Teflon® disc positions, while exactly at an EP both the real and imaginary part of the echo frequency $\Omega/2\pi$ vanish. Then, with Eq. (3b) a quadratic time dependency of the echo amplitude

$$\lim_{(s, \delta) \to \mathbf{EP}} |\tilde{G}_{12}(t)|^2 \approx \frac{s^2}{\xi + \xi^-} t^2 e^{-\gamma t}.$$  

(4)

is obtained. This dependency can also be verified by an exact calculation based on Eq. (2b) evaluated at the EP. It is consistent with the result given in [33] for the coalescence of two eigenvalues.

The EP was localized in the parameter plane $(s, \delta)$ by measuring crossings and avoided crossings of the frequencies and widths of the resonances which coincide at the EP [34]. For this we measured resonance spectra by varying the position of the Teflon® disc and the opening length in order to extract the dynamics of the eigenfrequencies and widths (for a more detailed description see for example [17]). In parallel we performed measurements of the nodal domains of the eigenfunction. Since the time behavior is extremely sensitive in the vicinity of an EP, the Teflon® disc position and the opening of the slit were changed in steps of 0.5 mm. We were able to localize two EPs below 3 GHz, the first at $2.757 \pm 0.001$ GHz and the second at $2.806 \pm 0.001$ GHz. In this frequency range the pair of resonances which coalesce at the EP can be treated as isolated from the others. Hence the two-level model is applicable. Since both EPs exhibit the same decay behavior with time, we only show the results for the EP at $2.757 \pm 0.001$ GHz.

The decay of the resonances with time was deduced from the transmission spectra by a fast Fourier transformation. The square of this transformation and a fit of Eq. (3b) to these data are shown in Fig. 2 for the critical coupling $s = s^{EP}$ and a subcritical Teflon® disc position $\delta < \delta^{EP}$. The model Eq. (3b) describes the measured signal very well starting from a time larger than $30 \pm 5$ ns, where a peak indicates the time the signal needs to travel through the coaxial cables connecting the VNA with the antenna, up to times, where the noise level is reached (at about $-65$ dB). The echo frequency equals $\Omega/2\pi = 1.1 \pm 0.2$ MHz, which approximately corresponds to the spacing between the two eigenfrequencies.

The time decay of the resonances is shown in Fig. 3.
FIG. 3: At the EP the quantum echoes disappear and the predicted quadratic time decay of the echo amplitude could be verified. Shown is the result for the EP at $2.757 \pm 0.001$ GHz.

The quantum echoes disappear as they are an interference effect of two eigenmodes and, at the EP, just one eigenmode "exists". However, the coalescence of the two resonances has an influence on the time spectrum. An average over different TE positions. There remains a small but non-vanishing.

For subcritical couplings, that is $s < s_{EP}$, the quantum echoes persist for all Teflon® disc positions. There again Eq. (3b) provides a very good description of the measured time dependence. However, we were not able to verify that at the critical Teflon position $\delta = \delta_{EP}$ the quantum echoes vanish for subcritical couplings $s < s_{EP}$. It seems that the resonance frequencies $\omega_1$ and $\omega_2$ of the two uncoupled resonators are only approximately degenerate, such that the real part of $\Omega$ in Eq. (3b) is very small but non-vanishing.

In the present work we experimentally studied quantum echoes in the vicinity of and at two exceptional points (EPs). Although the system is very sensitive to small deviations of the critical parameters, we were able to control the system sufficiently well to verify the predicted $t^2$-dependence of the echo amplitude at an EP. We showed, that a $2 \times 2$-model describes the quantum echoes very well. To our knowledge this is the first experimental verification of quantum echoes in the vicinity of an EP.

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