TRAVAIL DE DIPLÔME

Statistiques fractionnaires pour un modèle quantique à une dimension

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Résumé

Dans ce travail, on étudie un modèle quantique unidimensionnel de N fermions soumis à un potentiel à longue portée dans le but de réinterpréter sa thermodynamique comme étant celle d'un système de particules libres, mais obéissants à une statistique compliquée, dite statistique fractionnaire. Ces statistiques, intermédiaires entre celles de Bose-Einstein et Fermi-Dirac, ont été introduites suite à la découverte de spins fractionnaires dans certains problèmes bidimensionnels, mais leur définition originelle se référait spécifiquement au nombre de dimensions spatiales, ce qui empêchait leur utilisation dans des problèmes autres que bidimensionnels. En 1991, Haldane introduisit une nouvelle définition [1], basée sur le comptage des états,qui ne faisait aucune réference à la dimension de l'espace, ce qui ouvrit la voie à l'utilisation de ces statistiques dans des problèmes unidimensionnels.

Le système ici étudié a été introduit par Sutherland. Il est composé de N fermions sans spin, de masse m et soumis à un potentiel à deux corps répulsif en $\sinh^{-2}(r)$. Ce problème est un problème de diffusion complètement intégrable, au sens qu'il existe N constantes du mouvement deux à deux compatibles, ce qui permet de le résoudre asymptotiquement par la méthode dite de l'Ansatz de Bethe. Cette méthode, introduite en 1931 par H.Bethe [6], conduit à un système de N équations couplées, dites équations de Bethe, dont les inconnues sont les N nombres d'ondes indiçant l'état du système. Leurs solutions donnent le spectre d'énergie, ce qui permet de dériver la thermodynamique du système. Ce sont ces équations que l'on va utiliser pour réinterpréter l'interaction comme une interaction statistique intervenant entre des particules libres.

Le premier chapitre est consacré à la présentation des éléments théoriques nécessaires à la suite du travail: Ansatz de Bethe, intégrabilité quantique et statistiques fractionnaires. Dans le chapitre 2, on présente le modèle dont il est question, on donne une démonstration de l'intégrabilité du système et on le résoud à partir de l'Ansatz de Bethe asymptotique. Ces résultats sont déjà connus, mais peu développés dans la littérature, ce qui fait l'intérêt de leur présentation détaillée. Le chapitre 3 est consacré à la réinterprétation du système et à la discussion des résultats, dont on donne un résumé ci-après.

Les particules considérées étant des fermions, la répulsion due au principe d'exclusion de Pauli s'ajoute à celle due au potentiel répulsif, ce qui donne une statistique pour laquelle l'occupation de chaque état est inférieure à une particule! De plus, l'interaction statistique entre deux états à une particule indicés par des nombres d'onde k différents est non nulle. L'occupation d'un état dépend donc de façon non triviale de l'occupation des états voisins. On voit donc que la statistique fractionnaire de ce modèle est complexe et n'autorise pas d'interprétation intuitive.

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1. Introduction, general background

In this work, we study a model, first proposed and solved by Sutherland, of N fermions on a line which interact through a $\sinh^{-2}(r)$ pairwise potential. This model is integrable and its thermodynamics can be obtained through asymptotic Bethe Ansatz method. The aim of this work is to reinterpret the system as one made of free particles obeying a so-called fractional statistics.

In chapter 1.1, we make a quick review of Bethe Ansatz method and quantum integrability and then present Haldane's definition of fractional statistics with all the thermodynamics induced by it. In chapter 2, we present the model, prove its integrability and solve it using asymptotic Bethe Ansatz. In chapter 3, we derive the thermodynamics of the system using fractional statistics and we show that the interaction can be seen as a statistical one.

1.1 Quantum integrability and the Bethe Ansatz

One speaks about an exact model when it's possible to obtain an expression for some physical quantities, such as free energy or correlations, or at least when their evaluation can be reduced to a standard analysis problem. Exactly solvable models in one dimension don't seem to be very interesting because they are too simple and too far from reality. Actually, they are very useful because they illustrate some principles and general theorem rigorously established, and they allow to control approximations or perturbative methods that can be applied to more realistic and complex models. They can also provide realistic models for the physics of one dimensional conductors. A large class of these exactly solvable models can be solved either exactly or asymptotically using the Bethe Ansatz. This method was first introduced in 1931 by H. Bethe and will be presented in section 1.1.1. The models solvable by Bethe Ansatz are scattering ones and should be integrable systems. We'll see some points about quantum integrability in section 1.1.2.

1.1.1 The Bethe Ansatz

The so-called Bethe Ansatz method was first introduced in 1931 by H. Bethe [6] to solve the Heisenberg model, which is the simplest model for magnetism in one dimension. It's a chain of N spins one-half on a lattice with nearest neighbours interactions. The method was able to give the ground state and excitations of the antiferromagnet. The solution of this model made it easier to understand the microscopic origin of magnetism. The model was then extended to non-isotropic or long range interactions. The method was also used for continuous problems, such as the many-body problem with delta function interaction, which is not physical but has the great advantage of being entirely solvable. It was also used to obtain asymptotic solutions for models with potential such as

$$V(r) = \frac{g}{r^2}, \qquad V(r) = \frac{g}{\sin^2(r)}, \qquad V(r) = \frac{g}{\sinh^2(r)}$$

The latter is the one we will treat in this work.

The main point of the method is to make an Ansatz for the wavefunction, that is to say to assume that, in the region of space defined by $x_1 < x_2 < \cdots < x_N$, all the eigenfunctions of the hamiltonian take the general form

$$\psi(x_1, \ldots, x_N) = \sum_{P \in \mathcal{S}_N} A_P \exp\left(i \sum_{i=1}^{N} k_{P_i} x_i \right) \tag{1.1}$$

where $\sum_P$ is the sum of the $N!$ permutations of the wavenumbers $k_1 > k_2 > \cdots > k_N$. The A_P's are to be determined so that $\psi(x_1, \ldots, x_N)$ is an eigenfunction of the hamiltonian and the allowed values of the k's will be fixed by the boundary conditions. Other arrangements of particles are given by symmetry considerations.
By imposing that (1.1) is an eigenfunction of the hamiltonian, one can relate the coefficients A_P and $A_{P'}$, for P and P' differing by a transposition of j and $(j+1)$, through the two-body scattering matrix $S(k)$:

$$A_{P_1 \ldots P_{j+1} P_j \ldots P_N} = \underbrace{-e^{-i\theta(k_{P_j} - k_{P_{j+1}})}}_{= S(k_{P_j} - k_{P_{j+1}})} A_{P_1 \ldots P_j P_{j+1} \ldots P_N} \tag{1.2}$$

where $\theta(k)$ is the two-body phase-shift. From this, and letting $A_{\mathbb{1}} = 1$, one can determine all the coefficients A_P. For a given permutation P, there are many way to decompose it into a product of transpositions; from (1.2), each will give a determination of the coefficient A_P and we have to check that all these determinations are equivalent. This is ensured by Yang-Baxter equations

$$\begin{aligned} Y_{ij}^{ab} Y_{ji}^{ab} &= \mathbb{1} \\ Y_{jk}^{ab} Y_{ik}^{bc} Y_{ij}^{ab} &= Y_{ij}^{bc} Y_{ik}^{ab} Y_{jk}^{bc} \end{aligned} \tag{1.3}$$

where Y_{ij}^{ab} is the two-body S-matrix which describes the collision of particles a and b with momentum k_i and k_j. One can convince himself that they are sufficient conditions for the mutual consistency of all equations (1.2).

Placing the N particles system in a box of length L and imposing the periodic boundary conditions

$$\psi(0, x_2, \ldots, x_N) = \psi(L, x_2, \ldots, x_N) (\sigma)^{N-1} \psi(x_2, \ldots, x_N, L)$$

and

$$\frac{d}{dx}\psi(x, x_2, \ldots, x_N)\Big|_{x=0} = (\sigma)^{N-1}\frac{d}{dx}\psi(x_2, \ldots, x_N, x)\Big|_{x=L},$$

where $\sigma = +1$ for bosons and $\sigma = -1$ for fermions, one finds the so-called Bethe Ansatz equations (BAE)

$$e^{ik_j L} = (-\sigma)^{N-1}\prod_{i\neq j} e^{-i\theta(k_i - k_j)}, \quad \text{for } j = 1, \ldots, N \tag{1.4}$$

which determine the k's. A more convenient form of these equations is obtained by taking the logarithm of (1.4):

$$k_j L = 2\pi I_j - \sum_{i\neq j}\theta(k_i - k_j), \quad \text{for } j = 1, \ldots, N \tag{1.5}$$

where the I_j's are integers or half-integers depending on the parity of N and the type of particles. The phase-shift $\theta(k)$ can be obtained through the two-body problem and equations (1.5) can be solved for the k's. The energy spectrum is then given by

$$E = \frac{\hbar^2}{2m}\sum_{i=1}^{N} k_i^2.$$

The Bethe Ansatz form of the wavefunction describes a scattering state; the term with $k_1 > k_2 > \cdots > k_N$ is the incoming arrangement of momentum and the other terms describe rearrangement of particles by two-body collisions. The same k's are involved in each term, so that the collisions are purely elastic; the system is said to be non-diffractive. Since all k's are conserved quantities, there are N constants of motion and the system is integrable. Actually, the property of no diffraction is sufficient to ask that at least the asymptotic wavefunction is of Bethe Ansatz type.

1.1.2 Quantum integrability

A quantum system is said to be integrable if there exist a set of N independent constants of motion which commute one with another. Quantum integrability can be proved for many systems in one dimension using Lax technique: one must find two hermitian $N \times N$ matrices of operators, labelled by A and L, which satisfy the Lax equation (with $\hbar = 1$)

$$\frac{d}{dt}L = i[H, L] = i[A, L] \tag{1.6}$$

These matrices can be chosen with the general form

$$L_{jk} = \delta_{jk}p_j + i(1 - \delta_{jk})\alpha_{jk} \tag{1.7}$$

$$A_{jk} = \delta_{jk}\sum_{l\neq j}\gamma_{jl} + (1 - \delta_{jk})\beta_{jk} \tag{1.8}$$

where p_j is the usual impulsion operator for particle j and $\alpha_{jk} = \alpha(x_j - x_k), \beta_{jk} = \beta(x_j - x_k)$ and $\gamma_{jk} = \gamma(x_j - x_k)$ are real functions, which are determined by imposing that L and M satisfy equation (1.6). Imposing a supplementary condition to A, one can show that the operators

$$I_n = \frac{1}{n}\sum_{j,k}(L^n)_{jk}, \quad n = 1, \ldots, N \tag{1.9}$$

commute with the hamiltonian and are then constants of motion. Using a straightforward calculation, one can then check that the I_n commute one with another and are independent. This shows the integrability of the system. We'll see more details about this in section 2.2.1.

1.1.3 Asymptotic Bethe Ansatz

For a system that supports scattering, quantum integrability implies the conservation of the individual momentum, and then the property of no diffraction. Since the interaction is negligible at long distance, the motion is approximately free in the asymptotic region $|x_i - x_j| \to \infty$ for every i, j. The integrability and the asymptotic freedom allow us to give the wavefunction the Bethe Ansatz form (1.1) and expect that it will give an energy spectrum close to the exact one. This approximation is called the asymptotic Bethe Ansatz. Assuming that the relations between the A_P's have the same form that for Bethe Ansatz (1.2), with the phase-shift $\theta(k)$ obtained from the two-body problem, we can then place the system in a large box of length L and impose periodic boundary conditions to it. Since the form (1.1) of the wavefunction is only valid in the asymptotic region, we will have to take the thermodynamic limit so that we can reach the asymptotic region within the box of length L. The boundary conditions will lead us to the asymptotic Bethe Ansatz equations which are exactly the same that for the exact Bethe Ansatz (see section 2.3.2 for details). We can see this from the following discussion: if a particle of momentum k goes around the circle, it takes a total phase-shift made of a kinetic part kL, and of phase-shifts $\theta(k - k_i)$ due to scattering over each other particles. The total phase-shift must be 2π times an integer, since the final state is the same that the initial one. This gives equation (1.5). The ABA method gives the entire spectrum with degeneracies and the thermodynamics. In the case of $1/r^2$ potential, it has been shown to give exact results, even in the non-dilute limit.

1.2 Fractional statistics

1.2.1 Bosons, fermions, anyons

Considering the wavefunction of a two-body system with identical particles, it must satisfy

$$P_{12}^2\psi(x_1,\sigma_1;x_2,\sigma_2)=\psi(x_1,\sigma_1;x_2,\sigma_2)$$

and hence

$$P_{12}\psi(x_1,\sigma_1;x_2,\sigma_2)=\pm\psi(x_1,\sigma_1;x_2,\sigma_2)$$

where x_i,σ_i are respectively the position and spin of particle i, and P_{ij} is the operator that interchanges particles and spins i and j. The wavefunction of N identical particles must then have a definite symmetry under permutation of particles. Until recently, it was thought that there were only two types of particles: those with a completely symmetric wavefunction, called bosons, and those with an antisymmetric wavefunction, called fermions. Due to the antisymmetry of the wavefunction, two fermions can't have the same quantum numbers; this is Pauli's exclusion principle. Fermions are observed to have half-integer spin in units of $\hbar$. They obey Fermi-Dirac statistics. Bosons don't obey any exclusion principle: there can be any number of bosons in the same state. They have integer spins and obey Bose-Einstein statistics. The fact that fermions have half-integer spins and bosons integer ones suggests a connection between spin and statistics.

The limitation of allowed values of kinetic momentum to integer and half-integer values in units of $\hbar$ is due to the non-zero commutators between the generators of the rotation J_x, J_y and J_z. In three dimensions, the group of rotations $SO(3)$ is non-abelian and there can only exist fermions and bosons. In two dimensions, however, the rotations group $SO(2)$ is abelian, and then there is no more restriction on the possible values of the kinetic momentum. Thus, there can be particles with fractional spins, which have to obey intermediate statistics if the connection between spin and statistics is correct. These statistics are called fractional and the particles submitted to them are called anyons, because they may obey to any statistics. Fractional statistics were first defined through the appearance of a phase-shift different from ± 1 in the wavefunction of a system if two particles are interchanged. This phase-shift was related to the winding angle of the two particles, which is well defined only in two dimensions. This limited the use of fractional statistics to two dimensional systems.

Of course, anyons can't be real particles since our world is three dimensional. However, there exists some condensed matter systems, for example an interface between two semiconductors, which can be regarded as two dimensional. The localised excitations of such systems, if they exist, are quasiparticles which may be anyons. In certain cases, they can be observed.

1.2.2 Haldane's fractional statistics

In 1991, Haldane introduced a new definition for fractional statistics based on a generalisation of the exclusion principle [1]. We will present it in this chapter and derive the thermodynamics induced by it [2, 5]. These fractional statistics are introduced through a definition of state counting and are formulated without reference to spatial dimension, so that their use is not limited to two dimensional systems, as the previous ones were.

1.2.2.1 Definition of the state counting. We first consider a system of N identical particles to be distributed in G states. Let $D(N)$ be the number of different possibilities to add one particle to the system. If the particles are bosons, then,

$$D_b(N) = G \ ,$$

if they are fermions,

$$D_f(N) = G - N \ .$$

This can be generalised to:

$$D(N) = G - \alpha N \ , \tag{1.10}$$

where α must be rational, so that $D(N)$ can be integer.
We now want to count the number $W(N)$ of different configurations one can have by placing N particles in G states. For bosons:

$$W_b = \frac{(N+G-1)!}{N!(G-1)!}$$

For fermions:

$$W_f = \left(\begin{array}{c} G \\ N \end{array} \right) = \frac{G!}{N!(G-N)!}$$

Again, we can generalise this for any statistics writing:

$$W_\alpha(N) = \frac{[D(N-1)+N-1]!}{N![D(N-1)-1]!} \tag{1.11}$$

where $D(N)$ is given by (1.10).

One can then consider the case of many different species of particles; different species could really be different particles, such as bosons and fermions, but also identical ones with different momentum. We have M species of particles and N_i particles of the i-th species to distribute in G_i states with a generalised statistics.
Let $D_i(\{N_j\})$ be the number of manners to add one particle of the i-th species when N_j particles of the j-th species ($j = 1, 2, \ldots, M$) are already present. Following Haldane, we define the statistical interactions (or statistical parameters) α_{ij} by

$$\boxed{D_i(\{N_j\}) = G_i - \sum_{j=1}^{M} \alpha_{ij} N_j} \tag{1.12}$$

We find again the bosons case for $\alpha_{ij} = 0$ and the fermions case for $\alpha_{ij} = \delta_{ij}$. Given a set of $\{N_j\}$, we get the number of configurations $W_i(\{N_j\})$ for the i-th species by generalising (1.11) to

$$W_i(\{N_j\}) = \frac{[D_i(\{N_j - \delta_{ij}\}) + N_i - 1]!}{N_i![D_i(\{N_j - \delta_{ij}\}) - 1]!}$$

We could have written $D_i(\{N_j - 1\})$ instead of $D_i(\{N_j - \delta_{ij}\})$, but it's not important since it gives the same thermodynamic limit. The total number of configurations is given by:

$$W(\{N_j\}) = \prod_{i=1}^{M} \frac{[D_i(\{N_j - \delta_{ij}\}) + N_i - 1]!}{N_i![D_i(\{N_j - \delta_{ij}\}) - 1]!}$$

Introducing (1.12), we have:

$$W(\{N_j\}) = \prod_{i=1}^{M} \frac{[G_i + N_i - 1 - \sum_{j=1}^{M} \alpha_{ij}(N_j - \delta_{ij})]!}{N_i![G_i - 1 - \sum_{j=1}^{M} \alpha_{ij}(N_j - \delta_{ij})]!} \tag{1.13}$$

1.2.2.2 Thermodynamics. We're now able to calculate the thermodynamics of the system. Assuming that every single-particle state of the i-th species has an energy ϵ_i, the total energy of the system is:

$$E(\{N_j\}) = \sum_{j=1}^{M} N_j \epsilon_j \tag{1.14}$$

We consider a grand canonical ensemble with temperature T and chemical potential μ_i for the i-th species. The grand canonical partition function is then

$$\mathcal{Z}(\{\mu_i\}, T, L) = \sum_{\{N_i\}} W(\{N_i\}) \exp\left\{ \frac{1}{k_B T} \sum_{j=1}^{M} (\mu_j - \epsilon_j) N_j \right\} \tag{1.15}$$

We assume that for large N_i and G_i, the summand of (1.15) has a very sharp peak around its maximal value, so that we can approximate $\mathcal{Z}(\{\mu_i\}, T, L)$ by the maximal term of the sum. Let $\{\bar{N}_i\}$ be the values which maximise the summand of (1.15). We can find these values by minimising the thermodynamic potential

$$\Omega\Big(\{\mu_i\}, T, L\Big) = -k_B T \ln \mathcal{Z}\Big(\{\mu_i\}, T, L\Big) \tag{1.16}$$

with fixed $\mu_i, i = 1, 2, \ldots, M$ and T.
We have $\Omega = -k_B T \ln \mathcal{Z} = -k_B T \ln W(\{\bar{N}_i\}) - \sum_j (\mu_j - \epsilon_j)\bar{N}_j$.
Let's calculate

$$\ln W(\{\bar{N}_i\}) = \sum_{i=1}^{M} \left\{ \ln \left[G_i + \bar{N}_i - 1 - \sum_{j=1}^{M} \alpha_{ij}(\bar{N}_j - \delta_{ij}) \right]! - \right.$$
$$\left. - \ln \bar{N}_i! - \ln \left[G_i - 1 - \sum_{j=1}^{M} \alpha_{ij}(\bar{N}_j - \delta_{ij}) \right]! \right\}$$

For large $\bar{N}_i$ and G_i, δ_{ij} and 1 are negligible in front of $\bar{N}_i$ and G_i, and we can use Stirling's formula $\ln N! \simeq N(\ln N - 1)$. So, we have

$$\ln W(\{\bar{N}_i\}) = \sum_{i=1}^{M} \left\{ \left(G_i + \bar{N}_i - \sum_{j=1}^{M} \alpha_{ij}\bar{N}_j \right) \left[\ln \left(G_i + \bar{N}_i - \sum_{j=1}^{M} \alpha_{ij}\bar{N}_j \right) - 1 \right] - \right.$$
$$\left. - \bar{N}_i(\ln \bar{N}_i - 1) - \left(G_i - \sum_{j=1}^{M} \alpha_{ij}\bar{N}_j \right) \left[\ln \left(G_i - \sum_{j=1}^{M} \alpha_{ij}\bar{N}_j \right) - 1 \right] \right\}$$

Introducing the occupation numbers

$$n_i \equiv \frac{\bar{N}_i}{G_i} \quad \text{and} \quad g_{ij} \equiv \frac{G_i}{G_j}, \tag{1.17}$$

we have

$$\ln W(\{n_j\}) = \sum_{i=1}^{M} G_i \left\{ \left(1 + n_i - \sum_{j=1}^{M} \alpha_{ij} n_j g_{ji} \right) \left[\ln G_i + \ln \left(1 + n_i - \sum_{j=1}^{M} \alpha_{ij} n_j g_{ji} \right) \right] \right.$$
$$\left. - n_i(\ln G_i + \ln n_i) - \left(1 - \sum_{j=1}^{M} \alpha_{ij} n_j g_{ji} \right) \left[\ln G_i + \ln \left(1 - \sum_{j=1}^{M} \alpha_{ij} n_j g_{ji} \right) \right] \right\} =$$
$$= \sum_{i=1}^{M} G_i \left\{ \left(1 + n_i - \sum_{j=1}^{M} \alpha_{ij} n_j g_{ji} \right) \ln \left(1 + n_i - \sum_{j=1}^{M} \alpha_{ij} n_j g_{ji} \right) - \right.$$
$$\left. - n_i \ln n_i - \left(1 - \sum_{j=1}^{M} \alpha_{ij} n_j g_{ji} \right) \ln \left(1 - \sum_{j=1}^{M} \alpha_{ij} n_j g_{ji} \right) \right\}$$

Since $\bar{N}_i$ and G_i are large, we can consider n_i as a continuous variable and find the minimum of Ω imposing its derivatives with respect to n_i to be zero.

$$0 = -k_B T \frac{\partial}{\partial n_l} \ln \mathcal{Z}(\{n_i\}) = \frac{\partial}{\partial n_l} \left\{ -k_B T \ln W(\{n_i\}) - \sum_{i=1}^{M} G_i n_i (\mu_i - \epsilon_i) \right\} =$$
$$= -k_B T \sum_{i=1}^{M} G_i \alpha_{il} g_{li} \left[\ln \left(1 - \sum_{j=1}^{M} \alpha_{ij} n_j g_{ji} \right) - \ln \left(1 + n_i - \sum_{j=1}^{M} \alpha_{ij} n_j g_{ji} \right) \right] -$$
$$- k_B T G_l \left\{ \ln \left(1 + n_l - \sum_{j=1}^{M} \alpha_{lj} n_j g_{jl} \right) - \ln n_l + \frac{\mu_l - \epsilon_l}{k_B T} \right\} \quad \Leftrightarrow$$
$$\ln n_l - \frac{\mu_l - \epsilon_l}{k_B T} = \ln \left(1 + n_l - \sum_{j=1}^{M} \alpha_{lj} n_j g_{jl} \right) +$$
$$+ \sum_{i=1}^{M} \frac{G_i}{G_l} \alpha_{il} g_{li} \ln \frac{1 - \sum_{j=1}^{M} \alpha_{ij} n_j g_{ji}}{1 + n_i - \sum_{j=1}^{M} \alpha_{ij} n_j g_{ji}}$$

Taking the exponential of this expression, we have

$$\exp\left(\frac{\epsilon_l-\mu_l}{k_BT}\right)=\left(1+\frac{1}{n_l}-\sum_{j=1}^{M}\alpha_{lj}\frac{n_j}{n_l}g_{jl}\right)\prod_{i=1}^{M}\left[\frac{\frac{1}{n_i}-\sum_{j=1}^{M}\alpha_{ij}\frac{n_j}{n_i}g_{ji}}{1+\frac{1}{n_i}-\sum_{j=1}^{M}\alpha_{ij}\frac{n_j}{n_i}g_{ji}}\right]^{\alpha_{il}}$$

Introducing

$$w_i\equiv\frac{1}{n_i}-\sum_{j=1}^{M}\alpha_{ij}\frac{n_j}{n_i}g_{ji}\ , \tag{1.18}$$

we have

$$\boxed{\exp\left(\frac{\epsilon_i-\mu_i}{k_BT}\right)=(1+w_i)\prod_{j=1}^{M}\left(\frac{w_j}{1+w_j}\right)^{\alpha_{ji}}\ ,}\quad i=1,\ldots,M \tag{1.19}$$

We can rewrite (1.18) as

$$n_iw_i+\sum_{j=1}^{M}\alpha_{ij}n_jg_{ji}=1 \qquad \Leftrightarrow$$

$$\boxed{\sum_{j=1}^{M}(w_j\delta_{ij}+\alpha_{ij}g_{ji})n_j=1,}\quad i=1,\ldots,M \tag{1.20}$$

Equations (1.19) allow us to find the w_i's, while equations (1.20) give the n_i's. We can now calculate the thermodynamic potential Ω:

$$\Omega=-k_BT\ln\mathcal{Z}(\{n_i\})=$$

$$=-k_BT\sum_{i=1}^{M}G_in_i\left\{\left(1+\frac{1}{n_i}-\sum_{j=1}^{M}\alpha_{ij}\frac{n_j}{n_i}g_{ji}\right)\left[\ln n_i+\ln\left(1+\frac{1}{n_i}-\sum_{j=1}^{M}\alpha_{ij}\frac{n_j}{n_i}g_{ji}\right)\right]\right.$$
$$\left.-\ln n_i-\left(\frac{1}{n_i}-\sum_{j=1}^{M}\alpha_{ij}\frac{n_j}{n_i}g_{ji}\right)\left[\ln n_i+\ln\left(\frac{1}{n_i}-\sum_{j=1}^{M}\alpha_{ij}\frac{n_j}{n_i}g_{ji}\right)\right]+\frac{\epsilon_i-\mu_i}{k_BT}\right\}$$

Using (1.18) and (1.19), we have:

$$\Omega=\ -\ k_BT\sum_{i=1}^{M}G_in_i\left\{(1+w_i)\ln(1+w_i)-w_i\ln w_i-\right.$$
$$\left.-\ln(1+w_i)-\sum_{j=1}^{M}\alpha_{ji}\ln\frac{w_j}{1+w_j}\right\}=$$
$$=\ -\ k_BT\sum_{i=1}^{M}G_in_i\left\{w_i\ln\frac{1+w_i}{w_i}+\sum_{j=1}^{M}\alpha_{ji}\ln\frac{1+w_j}{w_j}\right\}=$$
$$=\ -\ k_BT\sum_{i=1}^{M}n_i\sum_{j=1}^{M}\{G_jw_i\delta_{ij}+G_i\alpha_{ji}\}\ln\frac{1+w_j}{w_j}=$$
$$=\ -\ k_BT\sum_{j=1}^{M}G_j\ln\frac{1+w_j}{w_j}\left\{\sum_{i=1}^{M}(w_i\delta_{ij}+\alpha_{ji}g_{ij})n_i\right\}$$

Using (1.20), we finally get

$$\boxed{\Omega\Big(\{\mu_i\}, T, L\Big) = -\, k_B T \sum_{j=1}^{M} G_j \ln \frac{1+w_j}{w_j}} \tag{1.21}$$

From this, we are able to calculate all other thermodynamic quantities.

We should now verify our hypothesis about the summand of expression (1.15). For this, we should calculate

$$(\Delta N_i)^2 =< N_i^2 > - < N_i >^2 = (k_B T)^2 \frac{\partial^2}{\partial \mu_i^2} \ln \mathcal{Z} = -\, k_B T \frac{\partial^2}{\partial \mu_i^2} \Omega$$

and show that this is small in front of N_i^2. I wasn't able to do this in the general case, but I will do it later in a particular case.

1.2.2.3 Particular cases. Let's consider the particular case where

$$\alpha_{ij} = \alpha_i \delta_{ij} \tag{1.22}$$

Equations (1.19) and (1.20) become:

$$\exp\left(\frac{\epsilon_i - \mu_i}{k_B T}\right) = (1+w_i)^{1-\alpha_i} w_i^{\alpha_i} \tag{1.23}$$

and

$$\begin{aligned} (w_i + \alpha_i) n_i &= 1 \qquad \Leftrightarrow \\ n_i &= \frac{1}{w_i + \alpha_i} \end{aligned} \tag{1.24}$$

Since the left-hand side of equation (1.23) is always positive, the right-hand side must be positive too. This imposes restrictions on the possible values of w_i. We find that it has to be positive. Through (1.24), we have then

$$n_i \leq \frac{1}{\alpha_i} \tag{1.25}$$

which is the expression of Haldane's exclusion principle.

In this case, we can verify that the fluctuations of N_i are small. We first calculate

$$N_i = \left.\frac{\partial \Omega}{\partial \mu_i}\right)_T = -\, k_B T \sum_{j=1}^{M} G_j \frac{1}{w_j(1+w_j)} \left.\frac{\partial w_j}{\partial \mu_i}\right)_T$$

Deriving the logarithm of (1.23) with respect to μ_i, we have

$$\left.\frac{\partial w_j}{\partial \mu_i}\right)_T = -\, \frac{\delta_{ij}}{k_B T} \frac{w_j(1+w_j)}{\alpha_j + w_j}$$

and then,

$$N_i = G_i \frac{1}{\alpha_i + w_i}$$

which is consistent with (1.24). From this, we get

$$(\Delta N_i)^2 = -k_B T \frac{\partial}{\partial \mu_i} \frac{\partial \Omega}{\partial \mu_i}\Big)_T = -k_B T \frac{\partial}{\partial \mu_i} \left(\frac{-G_i}{\alpha_i + w_i} \right)_T = \frac{G_i w_i (1 + w_i)}{(\alpha_i + w_i)^3}$$

Using (1.24), we have:

$$(\Delta N_i)^2 = N_i(1 - \alpha_i n_i)(1 + n_i - \alpha_i n_i)$$

Since $0 \leq n_i \leq \frac{1}{\alpha_i}$, the quantity $(1 - \alpha_i n_i)(1 + n_i - \alpha_i n_i)$ is finite and we have

$$\frac{\Delta N_i}{N_i} \propto \frac{1}{\sqrt{N_i}} \tag{1.26}$$

Thus, for large N_i, the relative fluctuations of N_i are negligible and our approximation is correct.

If all species of particles have the same chemical potential μ, we get Bose-Einstein distribution for $\alpha_i = 0$ and the Fermi-Dirac one for $\alpha_i = 1$. For large $e^{(\epsilon_i - \mu_i)/k_B T}$, we get the Boltzmann distribution independently of the statistical interactions.

In the zero temperature limit, we see from (1.23) that w_i is either 0 or ∞, depending on ϵ_i being respectively lower or larger than μ. From (1.24), we have then

$$n_i = \begin{cases} 1/\alpha & \text{if } \epsilon_i < \mu \\ 0 & \text{if } \epsilon_i > \mu \end{cases}$$

As in the case of fermions, there is a sharp Fermi surface, the difference being that in this case, the occupation of each state is $1/\alpha$ instead of one. One can obtain the Fermi energy ϵ_F from the condition

$$N = \sum_{i=1}^{M} N_i = \sum_{i=1}^{M} n_i G_i = \frac{1}{\alpha} \sum_{\substack{i=1 \\ \epsilon_i < \epsilon_F}}^{M} G_i$$

We see that the distribution is similar to the Fermi-Dirac one, except that there can be more than one particle in each state. For more complicated statistical interactions, the allowed number of particles in a state may depend on the occupation of other states, so that there can be no more intuitive interpretation of the statistics, as in the cases of bosons or fermions.

2. Known results about the model

In this chapter, we present the hamiltonian of the system in section 2.1. We then proove its integrability (section 2.2) and write the Bethe Ansatz equations of the system (section 2.3). In section 2.4, we derive the thermodynamics of the system using Yang's method.

2.1 Hamiltonian

In this work, we will study a model proposed by Sutherland [7]. It consists of N fermions of mass m, without spin, in one dimension.The hamiltonian of the system is

$$\mathcal{H} = \frac{1}{2m} \sum_{i=1}^{N} p_i^2 + \sum_{i<j} v_{ij} \tag{2.1}$$

where

$$v_{ij} = \frac{a^2 g}{\sinh^2\left(\frac{x_i - x_j}{a}\right)} \tag{2.2}$$

We choose units so that $\hbar = 1$ and $m = 1$. We find (see appendix A.1) that for $a^2 g < -1/4$, the problem has no physical meaning since there is no ground state and the particles fall one on another. So we restrict ourselves to $a^2 g > -1/4$ and we let

$$a^2 g = s(s+1) \qquad \text{with} \quad s \geq -1$$

The potential has the following asymptotic forms

$$a^2 g \sum_{i<j} \left[\sinh \frac{x_i - x_j}{a}\right]^{-2} \longrightarrow \begin{cases} g a^2 \sum_{i<j} \left(\frac{x_i - x_j}{a}\right)^{-2} & , |x_i - x_j| \to 0 \\ 4 a^2 g \sum_{i=1}^{N-1} \exp\left(-2 \frac{x_{i+1} - x_i}{a}\right) & , |x_i - x_j| \to \infty \end{cases}$$

In the second limit, the x's are ordered so that $x_1 < x_2 < \cdots < x_N$. The limit of high density corresponds to the inverse square potential solved by Sutherland [8], while the second limit corresponds to the Toda lattice [9]. In order to simplify the notations, we rescale all lengths by a, which is equivalent to let $a = 1$.
This model has been studied by Sutherland [7]. It was shown to be integrable and solved by asymptotic Bethe Ansatz. We'll present these results in sections 2.2, 2.3 and 2.4.

2.2 Integrability of the system

2.2.1 Construction of Lax matrices

In order to show the integrability of the system, we use Lax formalism. We must find two hermitian $N \times N$ matrices of operators, L and A, which satisfy

$$[\mathcal{H}, L_{jk}] = [A, L]_{jk} = \sum_{l=1}^{N}(A_{jl}L_{lk} - L_{jl}A_{lk}) \tag{2.3}$$

Using the notations

$$f_{ij} = f(x_i - x_j) \quad \text{and} \quad f'_{ij} = \frac{\partial}{\partial x_i} f(x_i - x_j),$$

we give L and A the general forms

$$L_{jk} = \delta_{jk}p_j + i(1-\delta_{jk})\alpha_{jk} \tag{2.4}$$

$$A_{jk} = \delta_{jk}\sum_{\substack{l=1\\ l\neq j}}^{N}\gamma_{jl} + (1-\delta_{jk})\beta_{jk} \tag{2.5}$$

where α, β, γ are real functions to be determined. We want that A and L are hermitian; this imposes conditions on the unknown functions:

$$\alpha_{ij} = -\alpha_{ji} \qquad \beta_{ij} = \beta_{ji} \qquad \gamma_{ij} = \gamma_{ji} \tag{2.6}$$

Introducing (2.4) and (2.5) in Lax equation (2.3), we obtain from respectively the diagonal and non-diagonal elements of it the two following equations (see appendix B.1 for details):

$$0 = \sum_{\substack{i=1\\ i\neq j}}^{N}(2\alpha_{ij}\beta_{ij} + \gamma'_{ji} - v'_{ji}) \tag{2.7}$$

$$\begin{aligned} 0 = i\sum_{\substack{i=1\\ i\neq j,k}}^{N}\Big(\alpha_{jk}(\gamma_{ji} - \gamma_{ki}) + \beta_{ji}\alpha_{ik} - \alpha_{ji}\beta_{ik}\Big) + \\ + (\alpha'_{jk} + \beta_{jk})' - (\alpha'_{jk} + \beta_{jk})(p_j - p_k) \end{aligned} \tag{2.8}$$

We then impose the sum-to-zero condition

$$\sum_{j=1}^{N} A_{jk} = 0 \tag{2.9}$$

Introducing (2.5), this becomes

$$0 = \sum_{j=1}^{N} A_{jk} = \sum_{j=1}^{N} \left\{ \delta_{jk} \sum_{\substack{l=1 \\ l \neq j}}^{N} \gamma_{jl} + (1 - \delta_{jk}) \beta_{jk} \right\} =$$

$$= \sum_{\substack{j=1 \\ j \neq k}}^{N} \Big(\beta_{jk} + \gamma_{jk} \Big) = 0 \tag{2.10}$$

We must then solve equations (2.7), (2.8) and (2.10) for α, β and γ so that v_{ij} is given by (2.2) with $a = 1$. Letting $\beta_{jk} = -\alpha'_{jk}$ cancels the last two terms of equation (2.8); if $\gamma_{jk} = -\beta_{jk} = \alpha'_{jk}$, condition (2.10) is satisfied. Equation (2.7) then becomes

$$0 = \sum_{\substack{i=1 \\ i \neq j}}^{N} \Big(2\alpha_{ji}\alpha'_{ji} + \alpha''_{ji} - v'_{ji} \Big)$$

which is satisfied if

$$v'_{ij} = 2\alpha_{ij}\alpha'_{ij} + \alpha''_{ij} \tag{2.11}$$

Condition (2.8) reduces to

$$0 = \sum_{\substack{i=1 \\ i \neq j,k}}^{N} \left\{ \alpha_{jk}\alpha'_{ij} - \alpha_{jk}\alpha'_{ik} - \alpha_{ij}\alpha'_{ik} - \alpha_{ij}\alpha'_{ik} \right\} \qquad \Rightarrow$$

$$\sum_{\substack{i=1 \\ i \neq j,k}}^{N} \underbrace{\alpha'_{ij}(\alpha_{jk} - \alpha_{ik})}_{= X_{ijk}} + \alpha'_{ik}(\alpha_{kj} - \alpha_{ij}) = \sum_{\substack{i=1 \\ i \neq j,k}}^{N} X_{ijk} + X_{ikj} = 0 \tag{2.12}$$

It remains to find a $\alpha(x)$ that satisfies (2.11) and (2.12). Deriving v_{ij}, we find that $\alpha(x)$ must satisfy

$$\frac{d^2\alpha(x)}{dx^2} + 2\alpha(x)\frac{d\alpha(x)}{dx} = -2s(s+1)\frac{\cosh x}{\sinh^3 x}$$

This suggest the use of the ansatz $\alpha(x) = A\frac{\cosh x}{\sinh x}$; we find that:

$$\begin{aligned} X_{ijk} &= -\frac{A^2}{\sinh^2(x_i - x_j)} \left[\frac{\cosh(x_j - x_k)}{\sinh(x_j - x_k)} - \frac{\cosh(x_i - x_k)}{\sinh(x_i - x_k)} \right] = \\ &= -\frac{A^2}{\sinh^2(x_i - x_j)} \left[\frac{\sinh(x_i - x_j)}{\sinh(x_j - x_k)\sinh(x_i - x_k)} \right] = \\ &= \frac{-A^2}{\sinh(x_j - x_k)\sinh(x_i - x_k)\sinh(x_i - x_j)} \end{aligned}$$

In this last expression, we see that $X_{ijk} = X_{ikj}$ and then condition (2.12) is satisfied. We check that $\alpha(x)$ is a solution of (2.11) for

$$A = -s \qquad \text{and} \qquad A = s + 1$$

We choose

$$\boxed{\alpha(x) = -s\frac{\cosh x}{\sinh x}} \tag{2.13}$$

With this, we have

$$\boxed{v_{ij} = \alpha_{ij}^2 + \alpha'_{ij} - s^2} \tag{2.14}$$

and Lax matrices are given by

$$\boxed{\begin{array}{l} L_{jk} = \delta_{jk}p_j + i(1-\delta_{jk})\alpha_{jk} \\ A_{jk} = \delta_{jk}\sum_{l\neq j}\alpha'_{jl} - (1-\delta_{jk})\alpha'_{jk} \end{array}} \tag{2.15}$$

This will allow us to construct constants of motion.

2.2.2 Proof of integrability

We define the operators

$$\boxed{I_n = \frac{1}{n}\sum_{j,k=1}^{N}(L^n)_{jk},} \qquad \text{for } n = 1,\ldots,N \tag{2.16}$$

We want to show that these operators commute with $\mathcal{H}$ and with each other. For this, we will use the formula

$$[\mathcal{H},(L^n)_{jk}] = [A, L^n]_{jk} \tag{2.17}$$

which is proved in appendix C.1. Let's calculate

$$\begin{aligned} [\mathcal{H}, I_n] &= \frac{1}{n}\sum_{j,k=1}^{N}[\mathcal{H},(L^n)_{jk}] \stackrel{(2.17)}{=} \frac{1}{n}\sum_{j,k=1}^{N}[A, L^n]_{jk} = \\ &= \frac{1}{n}\sum_{j,k=1}^{N}\sum_{l=1}^{N}\Big(A_{jl}(L^n)_{lk} - (L^n)_{jl}A_{lk}\Big) = \\ &= \frac{1}{n}\sum_{l,k=1}^{N}\Big(\sum_{j=1}^{N}A_{jl}\Big)(L^n)_{lk} - \frac{1}{n}\sum_{l,j=1}^{N}(L^n)_{jl}\Big(\sum_{k=1}^{N}A_{lk}\Big) \end{aligned}$$

Using (2.9), this implies

$$[\mathcal{H}, I_n] = 0 \tag{2.18}$$

This shows that the I_n's are constants of motion. Using Jacobi's relation, we find

$$0 = [\mathcal{H},[I_n,I_m]] + [I_n,\underbrace{[I_m,\mathcal{H}]}_{=0}] + [I_m,\underbrace{[\mathcal{H},I_n]}_{=0}]$$

and then

$$\frac{d}{dt}[I_n, I_m] = i[\mathcal{H}, [I_n, I_m]] = 0 \tag{2.19}$$

that is to say $[I_n, I_m]$ is a constant of motion.
In the asymptotic region $x_1 \ll x_2 \ll \cdots \ll x_N$, we have

$$L_{jk} \longrightarrow \delta_{jk} p_j - i(1-\delta_{jk})\text{sign}(j-k) \quad \text{for} \quad |x_j - x_k| \to \infty$$

Thus the L_{jk}'s depend only on the operators p_j and then commute with each other in this asymptotic region; this is also the case for the I_n's:

$$[I_n, I_m] \equiv 0 \quad \text{when} \quad |x_j - x_k| \to \infty$$

Since the system supports scattering, taking the limit $|x_j - x_k| \to \infty$ for all $j \neq k$ is equivalent to taking the limit $t \to \infty$; from (2.19), we deduce that, for all t, and thus for all x's,

$$[I_n, I_m] = 0 \tag{2.20}$$

Thus, there are N compatible constants of motion, and this shows the integrability of the system.

2.3 Solution by asymptotic Bethe ansatz

Since the system is integrable and supports scattering, it can be solved using asymptotic Bethe ansatz. To do this, we need to know the two-body phase shift, which can be obtained by solving the two-body problem. We will calculate it in section 2.3.1 and write asymptotic Bethe ansatz equations in section 2.3.2.

2.3.1 Solution of the two-body problem, the phase-shift $\theta(k)$

The hamiltonian of the two-body problem is, in appropriate units,

$$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2) + \frac{s(s+1)}{\sinh^2(x_1 - x_2)}, \quad s \geq -1 \tag{2.21}$$

We want to solve the eigenvalue problem

$$\mathcal{H}\Psi(x_1, x_2) = E\Psi(x_1, x_2) \tag{2.22}$$

We write this in the centre of mass frame

$$r = x_1 - x_2, \qquad R = \frac{x_1 + x_2}{2}$$

and use separation of variables $\Psi(R, r) = \phi(R)\varphi(r)$. We find the two equations

$$-\frac{1}{4}\frac{d^2}{dR^2}\phi(R) = \frac{\kappa^2}{4}\phi(R) \tag{2.23}$$

$$-\frac{d^2}{dr^2}\varphi(r) + \frac{s(s+1)}{\sinh^2 r} = k^2\varphi(r) \tag{2.24}$$

with $E = \frac{\kappa^2}{4} + k^2$. Since the particles are fermions, the wavefunction has to be antisymmetric under the exchange of the two particles, that is to say $\varphi(r)$ must be odd. We can then restrict ourselves to $r > 0$ and impose $\varphi(0) = 0$.
The centre-of-mass motion (2.23) is a free one; $\phi(R)$ is then given by

$$\phi(R) = e^{i\kappa R} \tag{2.25}$$

For the relative motion, we first study the asymptotic behaviours. For $r \to 0$, equation (2.24) behaves like

$$-\frac{d^2}{dr^2}\varphi(r) + \frac{s(s+1)}{r^2}\varphi(r) = 0$$

Letting $\varphi(r) \sim r^\alpha$, we find two possible values: $\alpha = -s$ and $\alpha = s+1$. The first one is not acceptable because it diverges, except in the case $-1 < s < 0$. Thus, for $r \to 0$, we have

$$\varphi(r) \sim r^{s+1}, \qquad r \to 0. \tag{2.26}$$

For $r \to +\infty$, equation (2.24) has the asymptotic behaviour

$$-\frac{d^2}{dr^2}\varphi(r) = k^2\varphi(r) \quad \Rightarrow \quad \varphi(r) = e^{\pm ikr}$$

Thus, we can write

$$\varphi(r) \sim e^{-ikr} + S(2k)e^{ikr}, \qquad r \to \infty, \tag{2.27}$$

$S(k)$ being the two-body S-matrix.
We now make the change of variables

$$\begin{cases} t = 1 + \sinh^2 r \\ \chi(t) = (t-1)^{-\frac{s+1}{2}}\varphi(r) \end{cases} \tag{2.28}$$

As $r \to 0$, i.e. $t \to 1$, $\chi(t)$ must behave like a constant. As $r \to \infty$, i.e. $t \to \infty$, t behaves like $1/4\, e^{2r}$ and thus

$$\begin{aligned} \chi(t) &\sim t^{-\frac{s+1}{2}}\left(e^{-ikr} + S(2k)e^{ikr}\right) \\ &\sim t^{-\frac{s+1}{2}}\left((4t)^{-\frac{ik}{2}} + S(2k)(4t)^{\frac{ik}{2}}\right) \\ &\sim 2^{-ik}t^{-\frac{s+1+ik}{2}} + 2^{ik}S(2k)t^{-\frac{s+1-ik}{2}} \end{aligned}$$

Equation (2.24) becomes

$$
\begin{aligned}
& -\frac{d^2}{dr^2}\varphi(r) + \frac{s(s+1)}{\sinh^2 r} = k^2\varphi(r) \qquad \Leftrightarrow \\
& \frac{d}{dr}\left[\frac{dt}{dr}\frac{d}{dt}\Big((t-1)^{\frac{s+1}{2}}\chi(t)\Big)\right] + \frac{s(s+1)}{t-1}(t-1)^{\frac{s+1}{2}}\chi(t) = k^2(t-1)^{\frac{s+1}{2}}\chi(t) \qquad \Leftrightarrow \\
& -(t-1)^{-\frac{s+1}{2}}\frac{d}{dr}\left[2\sinh r\cosh r\Big(\frac{s+1}{2}(t-1)^{\frac{s+1}{2}-1}\chi(t) + (t-1)^{\frac{s+1}{2}}\frac{d\chi(t)}{dt}\Big)\right] \\
& \qquad + \left[\frac{s(s+1)}{t-1} - k^2\right]\chi(t) = 0 \qquad \Leftrightarrow \\
& -(t-1)^{-\frac{s+1}{2}}\Big\{2[\cosh^2 r + \sinh^2 r]\Big((t-1)^{\frac{s+1}{2}}\frac{d\chi(t)}{dt} + \frac{s+1}{2}(t-1)^{\frac{s+1}{2}-1}\chi(t)\Big) + \\
& \qquad + 4\sinh^2 r\cosh^2 r\Big((t-1)^{\frac{s+1}{2}}\frac{d^2\chi(t)}{dt^2} + (s+1)(t-1)^{\frac{s+1}{2}-1}\frac{d\chi(t)}{dt} + \\
& \qquad + \frac{s^2-1}{4}(t-1)^{\frac{s+1}{2}-2}\chi(t)\Big)\Big\} + \left[\frac{s(s+1)}{t-1} - k^2\right]\chi(t) = 0 \qquad \Leftrightarrow \\
& -\Big\{2(2t-1)\Big(\frac{d\chi(t)}{dt} + \frac{s+1}{2(t-1)}\chi(t)\Big) + 4t(t-1)\Big(\frac{d^2\chi(t)}{dt^2} + \frac{s+1}{t-1}\frac{d\chi(t)}{dt} + \\
& \qquad + \frac{s^2-1}{4(t-1)^2}\chi(t)\Big)\Big\} + \left[\frac{s(s+1)}{t-1} - k^2\right]\chi(t) = 0 \qquad \Leftrightarrow \\
& t(1-t)\frac{d^2\chi(t)}{dt^2} + \Big(-\frac{2t-1}{2} - (s+1)t\Big)\frac{d\chi(t)}{dt} + \\
& \qquad + \frac{1}{4(t-1)}\Big(-(s+1)(2t-1) - t(s^2-1) + s(s+1) - k^2(t-1)\Big)\chi(t) = 0
\end{aligned}
$$

Finally, we get

$$
t(1-t)\frac{d^2\chi(t)}{dt^2} + \left[\frac{1}{2} - (2+s)t\right]\frac{d\chi(t)}{dt} - \frac{(s+1)^2+k^2}{4}\chi(t) = 0 \tag{2.29}
$$

This is an hypergeometric equation (see appendix D.1) with parameters $a = (s+1+ik)/2$, $b = (s+1-ik)/2$ and $c = 1/2$. We're interested in a solution for $r > 0$, i.e. $t > 1$; from (D.8) and (D.9), we get the general solution

$$
\begin{aligned}
\chi(t) = {} & At^{-\frac{s+1+ik}{2}}F\Big(\frac{1+s+ik}{2}; 1+\frac{s+ik}{2}; 1+ik; \frac{1}{t}\Big) + \\
& + Bt^{-\frac{s+1-ik}{2}}F\Big(\frac{1+s-ik}{2}; 1+\frac{s-ik}{2}; 1-ik; \frac{1}{t}\Big)
\end{aligned}
$$

In order to force the right behaviour near $t = 1$, we use the solution valid for $|t-1| < 1$ and impose the two solutions to be equal where they are both defined. This is done using formula (D.10). The solution for $|t-1| < 1$ with the right behaviour is (D.6):

$$
\chi(t) = C \cdot F\Big(\frac{1+s+ik}{2}; \frac{1+s-ik}{2}; \frac{3}{2}+s; 1-t\Big), \quad \text{for} \quad |1-t| < 1
$$

The connection of the two solutions gives, using (D.10)

$$-\frac{\sin i\pi k}{\pi\Gamma(3/2+s)}F\Big(\frac{1+s+ik}{2};\frac{1+s-ik}{2};\frac{3}{2}+s;1-t\Big)=$$

$$=\frac{t^{-\frac{s+1+ik}{2}}}{\Gamma(\frac{1+s-ik}{2})\Gamma(1+\frac{s-ik}{2})\Gamma(1+ik)}F\Big(\frac{1+s+ik}{2};1+\frac{s+ik}{2};1+ik;\frac{1}{t}\Big)-$$

$$-\frac{t^{-\frac{s+1-ik}{2}}}{\Gamma(\frac{1+s+ik}{2})\Gamma(1+\frac{s+ik}{2})\Gamma(1-ik)}F\Big(\frac{1+s-ik}{2};1+\frac{s-ik}{2};1-ik;\frac{1}{t}\Big)$$

Since $\Gamma(x)\Gamma(1/2-x)=\sqrt{\pi}2^{1-2x}\Gamma(2x)$, we have

$$\Gamma(\frac{1+s\pm ik}{2})\Gamma(\frac{1}{2}+\frac{1+s\pm ik}{2})=\sqrt{\pi}2^{-(s\pm ik)}\Gamma(1+s\pm ik)$$

and then

$$\chi(t)=C\Big\{\frac{2^{-ik}t^{-\frac{s+1+ik}{2}}}{\Gamma(1+s-ik)\Gamma(1+ik)}F\Big(\frac{1+s+ik}{2};1+\frac{s+ik}{2};1+ik;\frac{1}{t}\Big)-$$

$$-\frac{2^{ik}t^{-\frac{s+1-ik}{2}}}{\Gamma(1+s+ik)\Gamma(1-ik)}F\Big(\frac{1+s-ik}{2};1+\frac{s-ik}{2};1-ik;\frac{1}{t}\Big)\Big\}$$

In order to have the wanted behaviour for $t\to\infty$, we must let

$$C=\Gamma(1+s-ik)\Gamma(1+ik),$$

which implies

$$S(2k)=-\frac{\Gamma(1+s-ik)\Gamma(1+ik)}{\Gamma(1+s+ik)\Gamma(1-ik)}.$$

Finally, using (2.28), we have the wavefunction for the relative motion:

$$\varphi(r)=\Big(\frac{\sinh^2 r}{1+\sinh^2 r}\Big)^{\frac{s+1}{2}}\times \tag{2.30}$$

$$\times\Big\{(4+4\sinh^2 r)^{-\frac{ik}{2}}F\Big(\frac{1+s+ik}{2};1+\frac{s+ik}{2};1+ik;\frac{1}{1+\sinh^2 r}\Big)-$$

$$-S(2k)(4+4\sinh^2 r)^{\frac{ik}{2}}F\Big(\frac{1+s-ik}{2};1+\frac{s-ik}{2};1-ik;\frac{1}{1+\sinh^2 r}\Big)\Big\}$$

$$\text{with}\quad S(k)=-\frac{\Gamma(1+s-\frac{ik}{2})\Gamma(1+\frac{ik}{2})}{\Gamma(1+s+\frac{ik}{2})\Gamma(1-\frac{ik}{2})} \tag{2.31}$$

The complete wavefunction is, for $r>0$:

$$\Psi(R,r)=e^{i\kappa R}\varphi(r) \tag{2.32}$$

with eigenvalue $E=\frac{\kappa^2}{4}+k^2$.
Since $S(k)=-e^{-i\theta(k)}$, the two-body phase shift is given by

$$\boxed{\theta(k)=i\ln\frac{\Gamma(1+\frac{ik}{2})\Gamma(1+s-\frac{ik}{2})}{\Gamma(1-\frac{ik}{2})\Gamma(1+s+\frac{ik}{2})}} \tag{2.33}$$

From this, we can then apply the asymptotic Bethe Ansatz method.

2.3.2 Asymptotic Bethe Ansatz, the ABA equations

As the system is integrable and supports scattering, we can use asymptotic Bethe ansatz, assuming it gives correct results even in the non-dilute limit. We assume that for $x_1 \ll x_2 \ll \cdots \ll x_N$, the asymptotic wavefunction takes the form (1.1) with $k_1 > k_2 > \cdots > k_N$, and the A_P's satisfy (1.2). We place the system on a large ring $[-L/2, L/2]$ of circumference L, and impose periodic boundary conditions:

$$\Psi(-\frac{L}{2}, x_2, \ldots, x_N) = \Psi(\frac{L}{2}, x_2, \ldots, x_N) = (-1)^{N-1}\Psi(x_2, \ldots, x_N, \frac{L}{2}) \quad \Rightarrow$$

$$\sum_{P\in\mathcal{S}_N} A_P \exp\Big(- ik_{P_1}\frac{L}{2} + i(k_{P_2}x_2 + \cdots + k_{P_N}x_N)\Big) =$$

$$= (-1)^{N-1} \sum_{P'\in\mathcal{S}_N} A_{P'} \exp\Big(ik_{P'_N}\frac{L}{2} + i(k_{P'_1}x_2 + \cdots + k_{P'_{N-1}}x_N)\Big) =$$

$$= (-1)^{N-1} \sum_{P'\in\mathcal{S}_N} A_{P'} \exp\Big(ik_{P_1}\frac{L}{2} + i(k_{P_2}x_2 + \cdots + k_{P_N}x_N)\Big)$$

where we have let

$$P' = \begin{pmatrix} 1 & 2 & \cdots & N-1 & N \\ P_2 & P_3 & \cdots & P_N & P_1 \end{pmatrix} \quad \forall\, P \in \mathcal{S}_N$$

The condition on the derivative of Ψ is exactly the same, so we don't write it. Due to the linear independence of the exponentials, we get out of this, using moreover equation (1.2),

$$\begin{aligned} A_P \exp\Big(- ik_{P_1}\frac{L}{2}\Big) &= (-1)^{N-1} A_{P'} \exp\Big(ik_{P_1}\frac{L}{2}\Big) = \\ &\overset{(1.2)}{=} \exp\Big(ik_{P_1}\frac{L}{2}\Big) \exp\Big(- i\sum_{j=2}^{N} \theta(k_{P_1} - k_{P_j})\Big) A_P \end{aligned}$$

for each A_P, and then

$$\exp\Big[i\Big(k_{P_1}L - \sum_{j=2}^{N}\theta(k_{P_1} - k_{P_j})\Big)\Big] = 1 \quad \Leftrightarrow$$

$$\boxed{k_i L - \sum_{\substack{j=1\\ j\neq i}}^{N} \theta(k_i - k_j) = 2\pi I_i} \tag{2.34}$$

where $i = 1, \ldots, N$ and the I_i's are integers. These equations are the asymptotic Bethe ansatz ones (ABA). Solving these equations allows to find the k's of each particle, from which one gets the energy levels through

$$E = \frac{1}{2}\sum_{i=1}^{N} k_i^2 \tag{2.35}$$

2.4 Thermodynamics of the model

In this chapter, we will derive the thermodynamics of the model using a method proposed by Yang and Yang for the delta-function interaction model [11]. From the Bethe ansatz equations (2.34), we define the function

$$Lh(p) = pL - \sum_{k_i} \theta(p - k_i) \tag{2.36}$$

where the sum goes over each k_i solution of the ABA equations. When $p = k_i$ is one of these solutions, $Lh(p) = 2\pi I_i$ where I_i is the integer appearing in (2.34). Such a p will be said to correspond to a particle. The values of p such that $Lh(p) = 2\pi J$, where J is a integer different from the I_i's, will be called holes. For a large system, we may define the densities of holes $\rho_h(k)$ and of particles $\rho(k)$ so that

$$L\rho_h(k)dk \text{ is the number of holes in } dk$$

$$L\rho(k)dk \text{ is the number of particles in } dk$$

From these definitions, we see that the number of particles and holes in dk is the number of times $Lh(p)$ goes through $2\pi I$, with I integer, in this interval, so that

$$\begin{aligned} \frac{d(Lh(k))}{2\pi} &= L(\rho(k) + \rho_h(k))dk \qquad \Rightarrow \\ \frac{dh(k)}{dk} &= 2\pi\Big(\rho(k) + \rho_h(k)\Big) \overset{def.}{\equiv} 2\pi f(k) \end{aligned} \tag{2.37}$$

We can rewrite (2.36) in the thermodynamic limit to obtain

$$h(k) = k - \int \theta(k - k')\rho(k')dk'$$

Introducing this in (2.37), we get

$$1 - \int \theta'(k - k')\rho(k')dk' = 2\pi f(k) = 2\pi\Big(\rho(k) + \rho_h(k)\Big) \tag{2.38}$$

where we have used the notation $\theta'(k) = \frac{d}{dk}\theta(k)$. Knowing $\rho(k)$, the mean density is given by

$$D \equiv \frac{N}{L} = \int \rho(k)dk \tag{2.39}$$

and the energy per particle is

$$\frac{E}{N} = \frac{1}{D}\int \frac{k^2}{2}\rho(k)dk \tag{2.40}$$

The entropy of the system is non-zero because the existence of holes allows many configurations with approximately the same energy for given $\rho(k)$ and $\rho_h(k)$. For given particles and hole densities, there is

$(\rho+\rho_h)Ldk$	holes and particles in dk, of which
ρLdk	are particles, and
$\rho_h Ldk$	are holes.

Thus, the number of configurations in dk is $\frac{[(\rho+\rho_h)Ldk]!}{(L\rho dk)!(L\rho_h dk)!}$ and the entropy is given by

$$\begin{aligned} S &= k_B \sum_{dk} \ln\left(\frac{[(\rho+\rho_h)Ldk]!}{(L\rho dk)!(L\rho_h dk)!}\right) = \cdots = \\ &= k_B \sum_{dk} \{(\rho+\rho_h)\ln(\rho+\rho_h) - \rho\ln\rho - \rho_h \ln\rho_h\} Ldk \end{aligned}$$

where we have used Stirling's formula. As dk tends to zero, this becomes

$$\frac{S}{N} = \frac{k_B}{D}\int \{(\rho+\rho_h)\ln(\rho+\rho_h) - \rho\ln\rho - \rho_h\ln\rho_h\}\, dk \tag{2.41}$$

We now want to find the equilibrium state at a given temperature T and density D. We then have to maximise the partition function $\exp(S-E/T)$ with respect to ρ and ρ_h under the condition (2.39). In order to do that, we define $X(\rho) = TS - E - A(N - L\int\rho(k)dk)$, where A is a Lagrange multiplier, and impose that its first variation is zero. For a given ρ, ρ_h is given by (2.38):

$$\rho_h(k) = \frac{1}{2\pi} - \rho(k) - \frac{1}{2\pi}\int \theta'(k-k')\rho(k')dk'$$

We have then, for each acceptable $\delta\rho(k)$:

$$\begin{aligned} 0 &= \frac{d}{d\lambda}X(\rho+\lambda\delta\rho)\Big|_{\lambda=0} = \\ &= k_B TL \int \Bigg\{\Big[-\frac{1}{2\pi}\int\theta'(k-k')\delta\rho(k')dk'\Big]\Big[1+\ln\underbrace{\frac{1}{2\pi}\Big[1-\int\theta'(k-k')\rho(k')dk'\Big]}_{=\rho(k)+\rho_h(k)} - \\ &\qquad -\delta\rho(k)[1+\ln\rho(k)] - \Big[-\delta\rho(k) - \frac{1}{2\pi}\int\theta'(k-k')\delta\rho(k')dk'\Big]\times \\ &\qquad \times\Bigg[1+\ln\underbrace{\Big[\frac{1}{2\pi}-\rho(k)-\frac{1}{2\pi}\int\theta'(k-k')\rho(k')dk'\Big]}_{=\rho_h(k)}\Bigg]\Bigg\}dk - \\ &\qquad -L\int\frac{k^2}{2}\delta\rho(k)dk + AL\int\delta\rho(k)dk = \\ &= -\frac{k_BTL}{2\pi}\iint \delta\rho(k')\theta'(k-k')\ln\left[\frac{\rho(k)+\rho_h(k)}{\rho_h(k)}\right]dkdk' - \\ &\qquad\qquad -L\int\delta\rho(k)\left[k_BT\ln\frac{\rho(k)}{\rho_h(k)} + \frac{k^2}{2} - A\right]dk = \\ &= -L\int\delta\rho(k)\left\{\frac{k_BT}{2\pi}\int\theta'(k'-k)\ln\left[1+\frac{\rho(k')}{\rho_h(k')}\right]dk' + k_BT\ln\frac{\rho(k)}{\rho_h(k)} + \frac{k^2}{2} - A\right\}dk \end{aligned}$$

Since this must be right for each acceptable $\delta\rho(k)$, we deduce from it

$$\frac{k_B T}{2\pi}\int \theta'(k'-k)\ln\left[1+\frac{\rho(k')}{\rho_h(k')}\right]dk' + k_B T\ln\frac{\rho(k)}{\rho_h(k)} + \frac{k^2}{2} - A = 0$$

Introducing the function $\epsilon(k)$ defined through $\frac{\rho_h(k)}{\rho(k)} = \exp\left(\frac{\epsilon(k)}{k_B T}\right)$, we can rewrite this as

$$\epsilon(k) = -A + \frac{k^2}{2} + \frac{k_B T}{2\pi}\int \theta'(k'-k)\ln\left[1+e^{-\frac{\epsilon(k')}{k_B T}}\right]dk' \tag{2.42}$$

Introducing $\epsilon(k)$ in (2.38), we have

$$2\pi f(k) = 2\pi\rho(k)\left(1+e^{\frac{\epsilon(k)}{k_B T}}\right) = 1 - \int \theta'(k-k')\rho(k')dk' \tag{2.43}$$

We can solve (2.42), and then (2.43), for $\epsilon(k)$ and $\rho(k)$ by iterations and obtain the energy, density and entropy from (2.39), (2.40) and (2.41).
The next step is to show that the Lagrange multiplier A is the chemical potential μ. For this, we multiply equation (2.42) by ρD^{-1}, and then integrate over k; this gives

$$\frac{1}{D}\int \epsilon(k)\rho(k)dk = -\frac{A}{D}\underbrace{\int \rho(k)dk}_{\overset{(2.39)}{=}D} + \frac{1}{D}\int \frac{k^2}{2}\rho(k)dk +$$
$$+\frac{k_B T}{2\pi D}\int dk' \underbrace{\int dk\rho(k)\theta'(k'-k)}_{\overset{(2.43)}{=}1-2\pi f(k')}\ln\left(1+e^{-\frac{\epsilon(k')}{k_B T}}\right) \quad \Rightarrow$$

$$A = \frac{1}{D}\int\left(\frac{k^2}{2}-\epsilon(k)\right)\rho(k)dk + \frac{k_B T}{D}\int\left(\frac{1}{2\pi}-f(k)\right)\ln\left(1+e^{-\frac{\epsilon(k)}{k_B T}}\right)dk \tag{2.44}$$

Introducing the function $\epsilon(k)$, we can rewrite the entropy (2.41) as

$$\frac{S}{N} = \frac{k_B}{D}\int\left(\rho(k)+\rho_h(k)\right)\ln\left(1+e^{-\frac{\epsilon(k)}{k_B T}}\right)dk + \frac{1}{DT}\int \epsilon(k)\rho(k)dk$$

From this and (2.40), we can write the free energy per particle

$$\frac{F}{N} = \frac{E}{N} - T\frac{S}{N} = \frac{1}{D}\int\left(\frac{k^2}{2}-\epsilon(k)\right)\rho(k)dk - \frac{k_B T}{D}\int f(k)\ln\left(1+e^{-\frac{\epsilon(k)}{k_B T}}\right)dk$$

Comparing this last equality with (2.44), we get

$$\frac{F}{N} = A - \frac{k_B T}{2\pi D}\int \ln\left(1+e^{-\frac{\epsilon(q)}{k_B T}}\right)dq \tag{2.45}$$

We know from thermodynamics that $F = -PL + \mu N$; if we can show that the pression of the system is given by

$$P = \frac{k_B T}{2\pi} \int \ln\left(1 + e^{-\frac{\epsilon(q)}{k_B T}}\right) dq, \tag{2.46}$$

then (2.45) will imply that A is exactly the chemical potential μ. We write the pression as $P = -\frac{\partial F}{\partial L})_T$ and introduce (2.45) in it to get

$$\begin{aligned} P &= -N\frac{\partial A}{\partial L})_T + \frac{k_B T}{2\pi} L \int \frac{\partial}{\partial L}\left[\ln\left(1 + e^{-\frac{\epsilon(q)}{k_B T}}\right)\right]_T dq + \frac{k_B T}{2\pi}\int \ln\left(1 + e^{-\frac{\epsilon(q)}{k_B T}}\right) dq = \\ &= -N\frac{\partial A}{\partial L})_T - \frac{N}{2\pi D}\int \frac{\partial A}{\partial L}\frac{\partial \epsilon}{\partial A}\left(1 + e^{-\frac{\epsilon(q)}{k_B T}}\right)^{-1} dq + \\ &\qquad + \frac{k_B T}{2\pi}\int \ln\left(1 + e^{-\frac{\epsilon(q)}{k_B T}}\right) dq \end{aligned} \tag{2.47}$$

We now have to obtain an expression of $\frac{\partial \epsilon}{\partial A}$ in terms of $\rho(k)$ and $\epsilon(k)$. In this purpose, we differentiate equation (2.42) with respect to A; this leads us to the following expression, where we have used the fact that, since $\theta(k)$ is an odd function, $\theta'(k)$ must be even:

$$1 = -\frac{\partial \epsilon}{\partial A} - \frac{1}{2\pi}\int \theta'(k-q)\frac{\partial \epsilon}{\partial A}\left(1 + e^{-\frac{\epsilon(q)}{k_B T}}\right)^{-1} dq$$

If we compare this with equation (2.43), assuming the uniqueness of its solution, we get to the conclusion that

$$\frac{\partial \epsilon}{\partial A} = -2\pi\rho(k)\left(1 + e^{-\frac{\epsilon(k)}{k_B T}}\right)$$

which can be introduced in (2.47) to give, using moreover the fact that $\frac{\partial A}{\partial L}$ is independent of k,

$$P = -N\frac{\partial A}{\partial L} + \frac{N}{D}\frac{\partial A}{\partial L}\underbrace{\int \rho(q)dq}_{\overset{(2.39)}{=} D} + \frac{k_B T}{2\pi}\int \ln\left(1 + e^{-\frac{\epsilon(q)}{k_B T}}\right) dq$$

which leads to (2.46) and shows that the Lagrange multiplier A is the chemical potential μ.
We can finally write the thermodynamic potential

$$\Omega(\mu, T, L) = -PL = -\frac{k_B T L}{2\pi}\int \ln\left(1 + e^{-\frac{\epsilon(q)}{k_B T}}\right) dq \tag{2.48}$$

which is the quantity we will obtain in next chapter using fractional statistics.

3. Solution using fractional statistics

In this chapter, we will transform the asymptotic Bethe Ansatz equations (2.34) in order to compare them with the definition of the statistical interactions (1.12). This will allow us to reinterpret the system as one composed of free particles obeying fractional statistics.

3.1 Calculation of the statistical interactions

We divide the k axis in intervals of length Δk centred on k_i. Particles with momentum in the i-th interval will be said to be of the i-th species and given approximate energy $\epsilon_i = k_i^2/2$. Later, we will take the limit $\Delta k \to 0$ and there will be no more approximation on the energy. We will then define the different quantities introduced in section 1.2.
When there is no particles, equation (2.34) becomes

$$k_i L = 2\pi I_i$$

and then, the interval between two single-particle states is $2\pi/L$. Hence, the number of one-particle states of species i is

$$G_i = \frac{\Delta k}{2\pi/L} = \frac{L\Delta k}{2\pi} \tag{3.1}$$

Let $\rho(k)$ be the density of solutions of ABA equations. $\rho(k)$ is such that

$$L\rho(k_i)\Delta k \equiv N_i = \text{number of particles of the } i\text{-th species} \tag{3.2}$$

Let's define from (2.34),

$$\begin{aligned}
\frac{2\pi}{L} I(k) &= k - \frac{1}{L}\sum_{k' \neq k} \theta(k - k') = \\
&= k - \frac{1}{L}\sum_{k_i \neq k} \theta(k - k_i) N_i = \\
&\overset{(3.2)}{=} k - \sum_{k_i \neq k} \theta(k - k_i)\rho(k_i)\Delta k
\end{aligned} \tag{3.3}$$

Each time $I(k)$ is integer, the corresponding value of k is a possible solution of ABA equations, and then an accessible state. The number D_i of free one-particle states available for particles of species i when there is already N_i particles of this species is then, assuming that $I(k)$ is a increasing function of k,

$$D_i + N_i = I(k + \frac{\Delta k}{2}) - I(k - \frac{\Delta k}{2}) \quad \Rightarrow$$

$$\frac{2\pi}{L}(D_i + N_i) = \Delta k - \sum_{k_j \neq k_i} [\theta(k_i - k_j + \frac{\Delta k}{2}) - \theta(k_i - k_j - \frac{\Delta k}{2})]\rho(k_j)\Delta k$$

Let $h(k)$ be the density of free one-particle states:

$$Lh(k)\Delta k \equiv D_i \tag{3.4}$$

The last equation can then be rewritten using (3.2) and (3.4):

$$2\pi\Big(h(k_i) + \rho(k_i)\Big) = 1 - \sum_{k_j \neq k_i} \frac{\theta(k_i - k_j + \frac{\Delta k}{2}) - \theta(k_i - k_j - \frac{\Delta k}{2})}{\Delta k}\rho(k_j)\Delta k$$

Taking the limit $\Delta k \to 0$, we obtain, writing k and k' instead of k_i and k_j,

$$\begin{aligned} h(k) &= \frac{1}{2\pi} - \rho(k) - \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{d}{dk}\theta(k - k')\rho(k')dk' = \\ &= \frac{1}{2\pi} - \int_{-\infty}^{+\infty} \Big[\delta(k - k') + \frac{1}{2\pi}\frac{d}{dk}\theta(k - k')\Big]\rho(k')dk' \end{aligned} \tag{3.5}$$

In order to find the statistical interactions α_{ij}, we have to specialise equation (1.12) to our case. Introducing (3.1), (3.2) and (3.4) in it, we have:

$$\begin{aligned} Lh(k)\Delta k &= \frac{L\Delta k}{2\pi} - \sum_j \alpha_{ij} L\rho(k_j)\Delta k \quad \Leftrightarrow \\ h(k) &= \frac{1}{2\pi} - \sum_j \frac{\alpha_{ij}}{\Delta k}\rho(k_j)\Delta k \end{aligned}$$

Taking again the limit $\Delta k \to 0$, we find

$$h(k) = \frac{1}{2\pi} - \int_{-\infty}^{+\infty} \alpha(k, k')\rho(k')dk' \tag{3.6}$$

where we have let $\alpha(k_i, k_j) = \frac{\alpha_{ij}}{\Delta k}$.
Comparing (3.5) and (3.6), we obtain the statistical interactions

$$\boxed{\alpha(k, k') = \delta(k - k') + \frac{1}{2\pi}\frac{d}{dk}\theta(k - k')} \tag{3.7}$$

3.2 Thermodynamics

We can find the thermodynamics using equations (1.19), (1.20) and (1.21). Equation (1.19) gives

$$\frac{k_i^2/2-\mu}{k_BT} = \ln(1+w_i)+\sum_j \frac{\alpha_{ij}}{\Delta k}\ln\frac{w_j}{1+w_j}\Delta k \quad \Rightarrow$$
$$\frac{k^2/2-\mu}{k_BT} = \ln\Big(1+w(k)\Big)+\int \alpha(k',k)\ln\frac{w(k')}{1+w(k')}dk'$$

Where we have taken the limit $\Delta k \to 0$ to go from the first to the second line. Letting $w(k)=\exp\Big((\epsilon(k)-\mu)/k_BT\Big)$ and introducing (3.7), we have

$$\boxed{\frac{k^2/2}{k_BT} = \frac{\epsilon(k)}{k_BT} - \frac{1}{2\pi}\int \frac{d}{dk}\theta(k-k')\ln\Big(1+e^{-\frac{\epsilon(k')-\mu}{k_BT}}\Big)dk'} \tag{3.8}$$

Once equation (3.8) is solved, we get the density from equation (1.20):

$$\int \Big(w(k')\delta(k-k')+\alpha(k,k')\Big)\rho(k')dk' = \frac{1}{2\pi} \quad \Leftrightarrow$$
$$\rho(k)w(k)+\int \alpha(k,k')\rho(k')dk' = \frac{1}{2\pi}$$

Introducing (3.7), we have

$$\boxed{\rho(k) = \frac{1}{2\pi}\Big(1+e^{\frac{\epsilon(k)-\mu}{k_BT}}\Big)^{-1}\left[1-\int \frac{d}{dk}\theta(k-k')\rho(k')dk'\right]} \tag{3.9}$$

We then obtain the thermodynamic potential from equation (1.21):

$$\boxed{\Omega(\mu,T,L) = -\frac{k_BTL}{2\pi}\int \ln\left(1+e^{-\frac{\epsilon(k)-\mu}{k_BT}}\right)dk} \tag{3.10}$$

We can have the energy from equation (2.40) and the mean density from (2.39). All the thermodynamics comes from equations (3.8) and (3.10). We can see that, provided we replace $[\epsilon(k)-\mu]$ by $\epsilon(k)$, equations (3.8), (3.9) and (3.10) are exactly the same that equations (2.42), (2.43) and (2.48) obtained in section 2.4. The two methods give then the same results, which shows that fractional statistics gives correct results. Unfortunately, since the two methods are based on the asymptotic Bethe Ansatz, we have no confirmation of the validity of it.

3.3 Discussion of the results

First, we have to make a remark: all the constructions of sections 3.1 and 3.2 are based on the assumption that the function $I(k)$ defined in (3.3) is a non decreasing one. If this wasn't the case, the sum of holes and particles densities would be negative, which has no meaning, and the new interpretation of Bethe Ansatz equations would not be possible. Assuming there is no such problem, we write the statistical interactions, using the formulas of appendix E.1.

For $s = -1$ and $s = 0$, the interaction is turned off and we expect to recover the Fermi-Dirac statistics of an ideal gas. Since for these values, $\theta(k)$ is a constant, we have

$$\boxed{\alpha(k,k') = \delta(k-k')} \qquad \text{for } s=-1 \text{ and } s=0 \tag{3.11}$$

Equations (3.8), (3.9) and (3.10) lead to

$$\epsilon(k) = \frac{k^2}{2}, \qquad 2\pi\rho(k) = \frac{1}{1+\exp\left(\frac{k^2/2-\mu}{k_BT}\right)}$$
$$\Omega(\mu,T,L) = -\frac{k_BTL}{2\pi}\int \ln\left[1+\exp\left(-\frac{k^2/2-\mu}{k_BT}\right)\right]dk$$

which is what we were expecting.

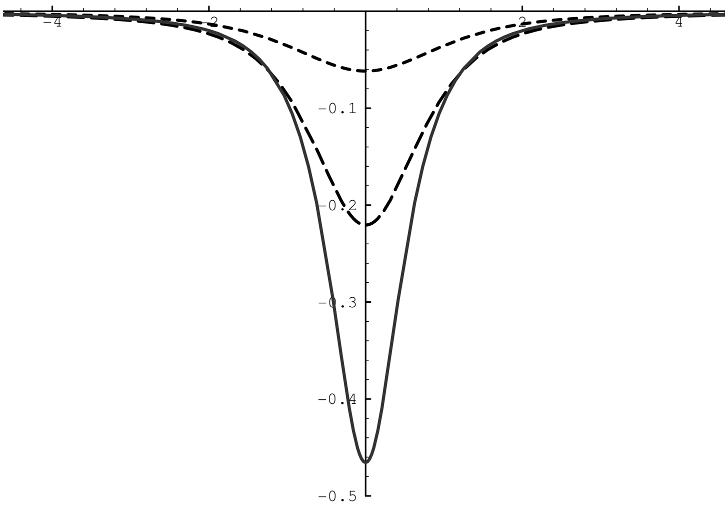

Fig. 3.1. $[\alpha(k,k')-\delta(k-k')]$ for, from top to bottom, $s=-0.2,-0.5,-0.7$ in function of $\Delta k = k-k'$

For $s<0$, we have, from (E.3),

$$\boxed{\alpha(k,k') = \delta(k-k') + \frac{1}{2\pi}\int_0^1 \cos\left(\frac{k}{2}\ln t\right)\frac{t^s-1}{t-1}dt} \tag{3.12}$$

Figure (3.1) presents $[\alpha(k,k')-\delta(k-k')]$ for different negative values of s. We see that the attractive potential diminishes the statistical interactions so that the particles obey a statistics intermediate between Fermi-Dirac and Bose-Einstein ones. However, for $-1 < s < 0$, the two values of the interaction parameter $s_1 = s$ and $s_2 = -(1+s)$ give the same hamiltonian, but lead to different statistical interactions. Moreover, they lead to different thermodynamics and then the two can't be simultaneously correct. Since there is no mean to know which one is correct and even whether one is correct, we have to give up this case.

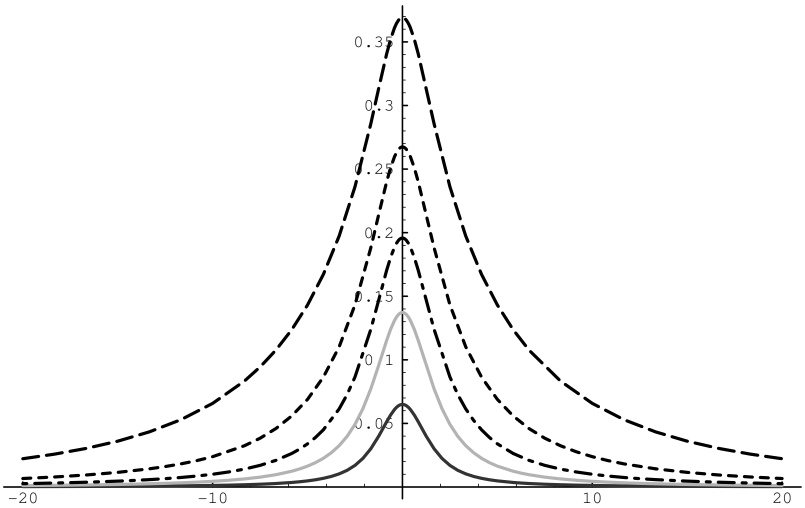

Fig. 3.2. $[\alpha(k,k') - \delta(k-k')]$ for, from bottom to top, $s = 0.3, 0.8, 1.4, 2.5, 5.2$ as a function of $\Delta k = k - k'$

For $s > 0$, the statistical interactions are, using (E.4),

$$\boxed{\begin{aligned}\alpha(k,k') = \delta(k-k') + \frac{1}{2\pi}\sum_{j=1}^{S}\frac{j+r}{(j+r)^2+(k/2)^2} &+ \\ + \frac{1}{2\pi}\int_0^1 \cos\left(\frac{k}{2}\ln t\right)\frac{t^r-1}{t-1}dt&\end{aligned}} \tag{3.13}$$

where we have let $s = S + r$ with S integer and $0 \leq r < 1$ and we understand that the sum cancels if $S = 0$. In this case, $[\alpha(k,k') - \delta(k-k')]$ is positive, as we can see in figure (3.2). This means that, with the repulsive potential, the statistical interaction is strongest than the one due to Pauli's exclusion principle. The presence of a particle in a state labelled by k would also prevent states with $k' = k + \delta k$ to be occupied. Moreover, a particle k occupies more than one single-particle state since, according to (1.12), the number of free states diminishes by more than one unity! The particles would then obey an exclusion principle more restrictive than Pauli's one.

Expressions (3.11), (3.12) and (3.13) are too complicated to allow an explicit calculation of the thermodynamic properties of the system, so we have to use limiting cases (see appendix E.1). The only one that allows explicit calculations is $k \gg s$ for $s = 1$. In this case, equations (3.8), (3.9) and (3.10) give

$$k \gg s = 1: \qquad \epsilon(k) = \frac{k^2}{2} + k_B T \ln\left[\frac{1}{2} + \sqrt{\frac{1}{4} + \exp\left(-\frac{k^2/2 - \mu}{k_B T}\right)}\right] \tag{3.14}$$

$$2\pi\rho(k) = \frac{1}{2 + \exp\left(\frac{\epsilon(k)-\mu}{k_B T}\right)} \tag{3.15}$$

$$\frac{\Omega}{N}(\mu, T) = -\frac{k_B T}{2\pi D}\int \ln\left[1 + \exp\left(-\frac{\epsilon(k) - \mu}{k_B T}\right)\right] dk \tag{3.16}$$

where D is given by (2.39). In this case, we verify that the function $I(k)$ is increasing, so that our construction of the fractional statistics is consistent. Using thermodynamic relations, we can express the energy per particle as

$$\frac{E}{N} = \frac{1}{D}\left[\mu \frac{\partial P}{\partial \mu}\Big)_T + T\frac{\partial P}{\partial T}\Big)_\mu - P\right]$$

where $P = -\Omega/L$ is the pression. After some calculation, we find

$$\frac{E}{N} = \frac{1}{D}\int \frac{k^2}{2} \times \frac{1/2\pi}{2 + \exp\left(\frac{\epsilon(k)-\mu}{k_B T}\right) + \exp\left(-\frac{\epsilon(k)-\mu}{k_B T}\right)} dk$$

Since $k \gg 1$, the negative exponential in last expression can be neglected in front of the other terms, and we can rewrite the energy, using (3.15),

$$\frac{E}{N} = \frac{1}{D}\int \frac{k^2}{2}\rho(k) dk$$

which is consistent with (2.40). Ω is the thermodynamic potential of fermions with individual energies $\epsilon(k)$. If we introduce (3.14) in (3.16), we can rewrite it as

$$\frac{\Omega}{N}(\mu, T) = -\frac{k_B T}{2\pi D}\int \ln\left[1 + \left(\frac{1}{2} + \sqrt{\frac{1}{4} + \exp\left(-\frac{k^2/2 - \mu}{k_B T}\right)}\right)^{-1} \exp\left(-\frac{k^2/2 - \mu}{k_B T}\right)\right] dk$$

This can be seen as the thermodynamic potential of free particles with energy $k^2/2$ which obey a complicated statistics. We can see in this example that the thermodynamics we obtained seems to give consistent results, but the complexity of the expressions doesn't allow explicit calculation of thermodynamics without approximation.

4. Conclusion

In this work, we show the integrability of Sutherland's model with hyperbolic interaction, and then solve it using asymptotic Bethe Ansatz. We transform the asymptotic Bethe Ansatz equations in order to get statistical interactions out of them, using Haldane's definition of fractional statistics. This allows us to reinterpret the system as one composed of free particles obeying a fractional statistics. However, since the particles are fermions and the potential is repulsive, the statistical interactions of the system are stronger than those of fermions! This leads to a difficult, if not impossible, interpretation of the statistics.

It would be interesting to apply the same method either to a system of bosons with repulsive potential, in which case the interaction could induce a fermionic behaviour of the bosons as in the case of the delta-function interaction, or to a system of fermions with an attractive potential that would diminish the repulsion due to Pauli's exclusion principle.

Fractional statistics are a quite new subject that is not yet well understood. We see in this work that they can be very complex and difficult to interpret. I don't know if they will lead to the prediction of new observable effects or if they are just a mathematical curiosity, but anyway, their study is very interesting.

A.1 Restrictions on the coupling constant g

We consider the two-body problem in the centre of mass frame with a potential $gV(r)$. We impose that $V(r)$ is a potential that behaves like r^{-2} when $r \to 0$. The eigenvalues equation is, in appropriate units,

$$-\frac{d^2}{dr^2}\varphi(r) + gV(r)\varphi(r) = E\varphi(r) \tag{A.1}$$

For $g > 0$, the potential is repulsive and there is no difficulty. For $g < 0$, the potential is attractive, and since it has a singularity at $r = 0$, we have to check that it has a physical meaning. We let $\alpha = -g$ and write equation (A.1) for $r \ll 1$ and finite E:

$$\frac{d^2}{dr^2}\varphi + \frac{\alpha}{r^2}\varphi = 0 \tag{A.2}$$

We search a solution of the form $\varphi(r) = r^s$. Introducing this in equation (A.2), we find

$$s(s-1)+\alpha=0$$

which leads to

$$s_1=\frac{1}{2}+\sqrt{\frac{1}{4}-\alpha} \qquad \text{and} \qquad s_2=\frac{1}{2}-\sqrt{\frac{1}{4}-\alpha}$$

For $\alpha < 1/4$, the two solutions for s are real, with $s_1 > s_2$. The general solution is then

$$\varphi(r)=Ar^{s_1}+Br^{s_2}$$

To bypass the difficulty of an infinitely attractive potential, we let that, for $r < r_0 \ll 1$,

$$V(r)=\frac{1}{r_0^2}$$

Hence, we have for $r < r_0$,

$$\varphi(r)=C\sin kr + D\cos kr$$

with $k=\sqrt{\alpha}/r_0$. The particles are fermions and must obey Pauli exclusion principle; we must then have $\varphi(0)=0$ and thus,

$$\varphi(r)=C\sin kr$$

Asking for the continuity of the wavefunction and its first derivative, we find

$$\begin{cases} C\sin\sqrt{\alpha}=Ar_0^{s_1}+Br_0^{s_2} \\ C\sqrt{\alpha}\cos\sqrt{\alpha}=As_1r_0^{s_1}+Bs_2r_0^{s_2} \end{cases}$$

Solving this for B/A, we have

$$\frac{B}{A}\sim r_0^{s_1-s_2} \tag{A.3}$$

Taking the limit $r_0 \to 0$, we find that

$$\varphi(r)\sim Ar^{s_1}, \qquad \text{for } r\ll 1$$

For $\alpha > 1/4$, we have

$$s_1=\frac{1}{2}+i\sqrt{\alpha-\frac{1}{4}} \qquad \text{and} \qquad s_2=\frac{1}{2}-i\sqrt{\alpha-\frac{1}{4}}=s_1^*$$

Then (A.3) writes

$$\frac{B}{A}\sim r_0^{s_1-s_1^*}=r_0^{i\sqrt{4\alpha-1}}$$

which has no defined limit for $r_0 \to 0$. However, we can deduce of this that

$$
\begin{aligned}
\varphi(r) &\sim A\left(r^{\frac{1}{2}+i\sqrt{\alpha-1/4}} + r_0^{2i\sqrt{\alpha-1/4}} r^{\frac{1}{2}-i\sqrt{\alpha-1/4}}\right)\\
&= Ar^{1/2}\left(e^{i\sqrt{\alpha-1/4}\ln r} + e^{2i\sqrt{\alpha-1/4}\ln r_0 - i\sqrt{\alpha-1/4}\ln r}\right)\\
&= Ar^{1/2}e^{i\sqrt{\alpha-1/4}\ln r_0}\left(e^{i\sqrt{\alpha-1/4}\ln\frac{r}{r_0}} + e^{-i\sqrt{\alpha-1/4}\ln\frac{r}{r_0}}\right)\\
&= \text{const}\cdot\sqrt{r}\cos\left(\sqrt{\alpha-1/4}\ln\frac{r}{r_0}\right)
\end{aligned}
$$

For $r_0 \to 0$, this function has an increasing number of zeros. Since it is an eigenfunction for each finite value of E and the fundamental state has no zero, the fundamental state of the system is obtained for $E = -\infty$; this has no physical meaning and we must then restrict ourselves to $\alpha < 1/4$ and then $g > -1/4$.

B.1 Explicit form of Lax equation

We present here the detailed calculation of the Lax equation (2.3) with the Lax matrices (2.4) and (2.5).
We first remark that

$$[p_j, f_{ji}] = -if'_{ji}$$

Let's calculate separately the different terms of $[A, L]_{jk}$. The first one is

$$\begin{aligned}
\sum_l A_{jl}L_{lk} &= \sum_l \left[(1-\delta_{jl})\beta_{jl} + \delta_{jl}\sum_{i\neq j}\gamma_{ji}\right] \times \left[\delta_{lk}p_l + i(1-\delta_{lk})\alpha_{lk}\right] = \\
&= \sum_l \Big\{\delta_{lk}(1-\delta_{jl})\beta_{jl}p_l + \delta_{jl}\delta_{lk}\sum_{i\neq j}\gamma_{ji}p_l + i\delta_{jl}(1-\delta_{lk})\sum_{i\neq j}\gamma_{ji}\alpha_{lk} + \\
&\qquad + i(1-\delta_{jl})(1-\delta_{lk})\beta_{jl}\alpha_{lk}\Big\} = \\
&= (1-\delta_{jk})\beta_{jk}p_k + \delta_{jk}\sum_{i\neq j}\gamma_{ji}p_j + i(1-\delta_{jk})\sum_{i\neq j}\gamma_{ji}\alpha_{jk} + \\
&\qquad + i\sum_l (1-\delta_{jl})(1-\delta_{lk})\beta_{jl}\alpha_{lk}
\end{aligned}$$

The second one is

$$\begin{aligned}
\sum_l L_{jl}A_{lk} &= \sum_l \left[\delta_{jl}p_j + i(1-\delta_{jl})\alpha_{jl}\right] \times \left[(1-\delta_{lk})\beta_{lk} + \delta_{lk}\sum_{i\neq l}\gamma_{li}\right] = \\
&= \sum_l \Big\{\delta_{jl}(1-\delta_{lk})p_j\beta_{lk} + \delta_{jl}\delta_{lk}p_j\sum_{i\neq l}\gamma_{li} + i\delta_{lk}(1-\delta_{jl})\alpha_{jl}\sum_{i\neq l}\gamma_{li} + \\
&\qquad + i(1-\delta_{jl})(1-\delta_{lk})\alpha_{jl}\beta_{lk}\Big\} = \\
&= (1-\delta_{jk})p_j\beta_{jk} + \delta_{jk}p_j\sum_{i\neq j}\gamma_{ji} + i(1-\delta_{jk})\alpha_{jk}\sum_{i\neq k}\gamma_{ki} + \\
&\qquad + i\sum_l (1-\delta_{jl})(1-\delta_{lk})\alpha_{jl}\beta_{lk}
\end{aligned}$$

And then

$$\begin{aligned}
[A, L]_{jk} &= \sum_l (A_{jl}L_{lk} - L_{jl}A_{lk}) = \\
&= (1-\delta_{jk})[\beta_{jk}p_k - p_j\beta_{jk}] + \delta_{jk}\sum_{i\neq j}(\gamma_{ji}p_j - p_j\gamma_{ji}) + \\
&\qquad + i(1-\delta_{jk})\Big\{\sum_{i\neq j}\gamma_{ji}\alpha_{jk} - \sum_{i\neq k}\alpha_{jk}\gamma_{ki}\Big\} + i\sum_{l\neq j,k}(\beta_{jl}\alpha_{lk} - \alpha_{jl}\beta_{lk}) = \\
&= (1-\delta_{jk})\Big\{\beta_{jk}p_k - p_j\beta_{jk} + i\sum_{i\neq j,k}\alpha_{jk}(\gamma_{ji} - \gamma_{ki})\Big\} + \\
&\qquad + i\delta_{jk}\sum_{i\neq j}\gamma'_{ji} + i\sum_{i\neq j,k}(\beta_{ji}\alpha_{ik} - \alpha_{ji}\beta_{ik})
\end{aligned} \tag{B.1}$$

Next, we calculate the commutator of $\mathcal{H}$ and L_{jk}:

$$\begin{aligned}
[\mathcal{H}, L_{jk}] &= [\frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{n<m} v_{nm}, \delta_{jk}p_j + i(1-\delta_{jk})\alpha_{jk}] = \\
&= \frac{i}{2}(1-\delta_{jk})\sum_i [p_i^2, \alpha_{jk}] + \delta_{jk}\sum_{n<m}[v_{nm}, p_j] = \\
&= \frac{i}{2}(1-\delta_{jk})\Big\{[p_j^2, \alpha_{jk}] - [p_k^2, \alpha_{kj}]\Big\} + \delta_{jk}\Big\{\sum_{n<j}[v_{nj}, p_j] + \sum_{m>j}[v_{jm}, p_j]\Big\} = \\
&= \frac{i}{2}(1-\delta_{jk})\Big\{p_j[p_j, \alpha_{jk}] + [p_j, \alpha_{jk}]p_j - p_k[p_k, \alpha_{kj}] - [p_k, \alpha_{kj}]p_k\Big\} + \\
&\qquad + \delta_{jk}\Big\{-\sum_{i<j}[p_j, v_{ji}] - \sum_{i>j}[p_j, v_{ji}]\Big\} \\
&= \frac{1-\delta_{jk}}{2}\Big\{p_j\alpha'_{jk} + \alpha'_{jk}p_j - p_k\alpha'_{kj} - \alpha'_{kj}p_k\Big\} + i\delta_{jk}\sum_{i\neq j} v'_{ji} = \\
&= (1-\delta_{jk})\Big(\alpha'_{jk}(p_j - p_k) - i\alpha''_{jk}\Big) + i\delta_{jk}\sum_{i\neq j} v'_{ji} = [\mathcal{H}, L_{jk}] \qquad \text{(B.2)}
\end{aligned}$$

Introducing this in (2.3) and writing separately the diagonal and non-diagonal terms, we have

$$\begin{aligned}
0 &= [A, L]_{jj} - [\mathcal{H}, L_{jj}] = \\
&= i\sum_{i\neq j}\gamma'_{ji} + i\sum_{i\neq j}(\beta_{ji}\alpha_{ij} - \alpha_{ji}\beta_{ij}) - i\sum_{i\neq j} v'_{ji} = \\
&= \sum_{i\neq j}(2\alpha_{ij}\beta_{ij} + \gamma'_{ji} - v'_{ji}) = 0, \qquad \text{(B.3)}
\end{aligned}$$

for the diagonal elements, while the non-diagonal elements give

$$\begin{aligned}
0 &= [A, L]_{jk} - [\mathcal{H}, L_{jk}] = \\
&= \beta_{jk}p_k - p_j\beta_{jk} + i\sum_{i\neq j,k}\Big(\alpha_{jk}(\gamma_{ji} - \gamma_{ki}) + \beta_{ji}\alpha_{ik} - \alpha_{ji}\beta_{ik}\Big) - \\
&\qquad - \alpha'_{jk}(p_j - p_k) + i\alpha''_{jk} \qquad \Leftrightarrow \\
0 &= i\sum_{i\neq j,k}\Big(\alpha_{jk}(\gamma_{ji} - \gamma_{ki}) + \beta_{ji}\alpha_{ik} - \alpha_{ji}\beta_{ik}\Big) + \\
&\qquad + (\alpha'_{jk} + \beta_{jk})' - (\alpha'_{jk} + \beta_{jk})(p_j - p_k) \qquad \text{(B.4)}
\end{aligned}$$

C.1 Proof of equation (2.17)

We prove by recurrence equation (2.17)

$$[\mathcal{H}, (L^n)_{jk}] = [A, L^n]_{jk}.$$

For $n = 1$, this equation is Lax equation (2.3), which has already been verified. Assume now that

$$[\mathcal{H}, (L^{n-1})_{jk}] = [A, L^{n-1}]_{jk},$$

and let's calculate

$$[\mathcal{H}, (L^n)_{jk}] = [\mathcal{H}, \sum_{i=1}^{N} L_{ji}(L^{n-1})_{ik}] = \sum_{i=1}^{N} \Big\{ L_{ji}[\mathcal{H}, (L^{n-1})_{ik}] + [\mathcal{H}, L_{ji}](L^{n-1})_{ik} \Big\} =$$

$$= \sum_{i=1}^{N} \Big\{ L_{ji}[A, (L^{n-1})_{ik}] + [A, L]_{ji}(L^{n-1})_{ik} \Big\} =$$

$$= \sum_{i,l=1}^{N} \Big\{ L_{ji}\Big(A_{il}(L^{n-1})_{lk} - (L^{n-1})_{il}A_{lk} \Big) + \Big(A_{jl}L_{li} - L_{jl}A_{li} \Big)(L^{n-1})_{ik} \Big\} =$$

$$= \sum_{i,l=1}^{N} \Big\{ L_{ji}A_{il}(L^{n-1})_{lk} - L_{ji}(L^{n-1})_{il}A_{lk} + A_{jl}L_{li}(L^{n-1})_{ik} - L_{jl}A_{li}(L^{n-1})_{ik} \Big\} =$$

$$= \sum_{l=1}^{N} \Big\{ A_{jl}(L^n)_{lk} - (L^n)_{jl}A_{lk} \Big\} = [A, L^n]_{jk}$$

which proves equation (2.17).

D.1 Hypergeometric equation, hypergeometric function

The hypergeometric equation is the following

$$t(1-t)\frac{d^2\chi(t)}{dt^2} + [c-(1+a+b)]\frac{d\chi(t)}{dt} - ab\chi(t) = 0 \tag{D.1}$$

where a, b, c are constants. This equation has regular singularities at $t = 0, 1$ and ∞, that is to say that, if one writes the equation under the form $\chi''(t) + f(t)\chi'(t) + g(t)\chi(t) = 0$, then $f(t)$ and $g(t)$ have poles of order at worst one and two respectively. The behaviours of the solutions near these singularities are

$$\begin{array}{rlll} t=0: & \chi_1(t) \sim t^0 & \text{or} & \chi_2(t) \sim t^{1-c} \\ t=1: & \chi_1(t) \sim (t-1)^0 & \text{or} & \chi_2(t) \sim t^{c-a-b} \\ t\to\infty: & \chi_1(t) \sim t^{-a} & \text{or} & \chi_2(t) \sim t^{-b} \end{array} \tag{D.2}$$

One can solve equation (D.1) using a series expansion

$$\chi(t) = \sum_{j\geq 0} a_j t^j$$

Introducing this in the equation gives

$$\begin{aligned} 0 &= t(1-t)\sum_{j\geq 0} j(j-1)a_j t^{j-2} + [c-(1+a+b)t]\sum_{j\geq 0} j a_j t^{j-1} - ab\sum_{j\geq 0} a_j t^j \\ &= -\sum_{j\geq 0}[j(j-1) + j(1+a+b) + ab]a_j t^j + \sum_{j\geq 1}[j(j-1)+cj]a_j t^{j-1} \\ &\overset{(i=j-1)}{=} -\sum_{j\geq 0}(j+a)(j+b)a_j t^j + \sum_{i\geq 0}(i+1)(i+c)a_{i+1}t^i \\ &= \sum_{j\geq 0}[(j+1)(j+c)a_{j+1} - (j+a)(j+b)a_j]t^j \qquad \Rightarrow \\ a_{j+1} &= \frac{(j+a)(j+b)}{(j+1)(j+c)}a_j \end{aligned}$$

Using Pochhammer's symbol

$$(a)_0 = 1, \quad (a)_j = a(a+1)\cdots(a+j-1),$$

one can rewrite this as

$$a_j = \frac{(a)_j(b)_j}{(c)_j j!}a_0$$

The solution is then given by the series

$$\boxed{\chi(t) = a_0\sum_{j\geq 0}\frac{(a)_j(b)_j}{(c)_j}\frac{t^j}{j!} = F(a;b;c;t)} \tag{D.3}$$

Using d'Alembert criteria

$$\left|\frac{a_{j+1}}{a_j}\right| = \frac{(j+a)(j+b)}{(j+1)(j+c)} \to 1 \quad \text{as} \quad j \to \infty,$$

one finds that the series converges for $|t| < 1$. $F(a;b;c;t)$ is called hypergeometric function and is defined by (D.3). Using appropriated series expansions, one can find solutions valid for $|t| < 1$, $|t-1| < 1$ or $|t| > 1$ for each of the behaviours (D.2):

$$|t| < 1: \quad \chi(t) = F(a;b;c;t) \tag{D.4}$$

$$\chi(t) = t^{1-c} F(1+a-c; 1+b-c; 2-c; t) \tag{D.5}$$

$$|t-1| < 1: \quad \chi(t) = F(a;b;1+a+b-c;1-t) \tag{D.6}$$

$$\chi(t) = (1-t)^{c-a-b} F(c-a; c-b; 1+c-a-b; 1-z) \tag{D.7}$$

$$|t| > 1: \quad \chi(t) = t^{-a} F(a; 1+a-c; 1+a-b; t^{-1}) \tag{D.8}$$

$$\chi(t) = t^{-b} F(b; 1+b-c; 1+b-a; t^{-1}) \tag{D.9}$$

The solutions for $|t| > 1$ and $|t-1| < 1$ are connected through the formula

$$\frac{\sin \pi(b-a)}{\pi \Gamma(1+a+b-c)} F(a;b;1+a+b-c;1-t) = \tag{D.10}$$

$$= \frac{t^{-a}}{\Gamma(b)\Gamma(1+b-c)\Gamma(1+a-b)} F(a;1+a-c;1+a-b;t^{-1}) -$$

$$- \frac{t^{-b}}{\Gamma(a)\Gamma(1+a-c)\Gamma(1+b-a)} F(b;1+b-c;1+b-a;t^{-1})$$

All these results come from reference [10].

E.1 Calculation of $\theta'(k)$

In this appendix, we will obtain explicit expressions of $\frac{d}{dk}\theta(k)$ for different values of s. Starting from (2.33), we have

$$\begin{aligned}\frac{d}{dk}\theta(k) &= i\frac{d}{dk}\ln\frac{\Gamma(1+\frac{ik}{2})\Gamma(1+s-\frac{ik}{2})}{\Gamma(1-\frac{ik}{2})\Gamma(1+s+\frac{ik}{2})} = \\ &= i\left\{\frac{i}{2}\Psi(1+\frac{ik}{2}) - \frac{i}{2}\Psi(1+s-\frac{ik}{2}) + \frac{i}{2}\Psi(1-\frac{ik}{2}) - \frac{i}{2}\Psi(1+s+\frac{ik}{2})\right\}\end{aligned}$$

where $\Psi(k) = \frac{d}{dk}\ln\Gamma(k)$ is the psi function. Using the fact that $\Psi(\overline{z}) = \overline{\Psi(z)}$, we can rewrite this as

$$\frac{d}{dk}\theta(k) = \Re\left[\Psi(1+s+\frac{ik}{2}) - \Psi(1+\frac{ik}{2})\right]$$

where $\Re(z)$ is the real part of z. Using the integral representation of the psi function $\Psi(z) = \int_0^1 \frac{t^{z-1}-1}{t-1}dt + \Psi(1)$, valid for $\Re(z) > 0$ (see for example [12]), we have

$$\frac{d}{dk}\theta(k) = \Re\left[\int_0^1 t^{\frac{ik}{2}}\frac{t^s-1}{t-1}dt\right] \tag{E.1}$$

From this expression, we see that, for $s = -1$ and $s = 0$,

$$\frac{d}{dk}\theta(k) = 0, \qquad \text{for } s=-1 \text{ and } s=0 \tag{E.2}$$

For $-1 < s < 1$, we can rewrite (E.1) as

$$\frac{d}{dk}\theta(k) = \int_0^1 \cos\left(\frac{k}{2}\ln t\right)\frac{t^s-1}{t-1}dt \qquad \text{for } -1<s<1 \tag{E.3}$$

For $s > 1$, we separate s as $s = S + r$, where S is an integer and $0 \le r < 1$, and write

$$\frac{t^s-1}{t-1} = \sum_{j=1}^{S} t^{S-j} + \frac{t^r-1}{t-1}.$$

After integration, this gives

$$\frac{d}{dk}\theta(k) = \sum_{j=1}^{S}\frac{j+r}{(j+r)^2+(k/2)^2} + \int_0^1 \cos\left(\frac{k}{2}\ln t\right)\frac{t^r-1}{t-1}dt \tag{E.4}$$

This last expression is valid for $s > 0$ if we understand that the sum is zero when $S = 0$. From here we can obtain the following limiting cases.

s integer: when s is integer, $r = 0$ and then the integral in (E.4) cancels, so that

$$\frac{d}{dk}\theta(k) = \sum_{j=1}^{s}\frac{j}{j^2+(k/2)^2} \tag{E.5}$$

k, s → ∞: When k is large, the term $\cos\left(\frac{k}{2}\ln t\right)$ has great oscillation with a zero mean value, while the term $\frac{t^r-1}{t-1}$ is quasi-constant, so that the integral part of (E.4) cancels. The remaining term is

$$\frac{d}{dk}\theta(k) = \sum_{j=1}^{S}\frac{1}{k}\frac{\frac{j+r}{k}}{\left(\frac{j+r}{k}\right)^2+\frac{1}{4}}$$

For k very large, we can replace the sum by an integral over $x = \frac{j+r}{k}$:

$$\frac{d}{dk}\theta(k) \sim \int_{\frac{1+r}{k}}^{\frac{s}{k}} \frac{xdx}{x^2+1/4} \sim \frac{1}{2}\ln\left(x^2+\frac{1}{4}\right)\Big|_{x=0}^{s/k} = \frac{1}{2}\ln\left[1+\left(\frac{2s}{k}\right)^2\right] \quad \text{(E.6)}$$

k ≫ s: As before, the integral term of (E.4) cancels and we have, for s integer,

$$\frac{d}{dk}\theta(k) = \sum_{j=1}^{s}\frac{j}{j^2+(k/2)^2}$$

We let $k = \kappa k'$, where k' has the same magnitude as s, so that we can write

$$\frac{d}{dk}\theta(k) = \frac{1}{\kappa}\sum_{j=1}^{s}\frac{j/\kappa}{(j/\kappa)^2+(k'/2)^2}$$

For $k \gg s$ and then $\kappa \gg s$, we have $j/\kappa \ll 1$ and the summand of last expression tends to $\pi\delta(k'/2)$ so that, using $\delta(cx) = 1/|c|\delta(x)$,

$$\frac{d}{dk}\theta(k) = \frac{1}{\kappa}\sum_{j=1}^{s}\pi\delta(k'/2) = 2\pi s\delta(k)$$

From the continuity of $\mathcal{H}$ as a function of s, we assume that, even if s is not integer,

$$\frac{d}{dk}\theta(k) = 2\pi s\delta(k) \qquad \text{for } k \gg s \quad \text{(E.7)}$$

References

[1] F.D.M. Haldane, *“Fractional statistics” in arbitrary dimensions: a generalization of the Pauli principle*, Phys. Rev. Lett. 67(8), 937, (1991).

[2] Y.S. Wu, *Statistical distribution for particles obeying fractional statistics*, Phys. Rev. Lett. 73(7), 922, (1994)

[3] B. Sutherland, R.A. Römer, *Exciton, spinon, and spin waves in a soluble one-dimensional many-body system*, Phys. Rev. Lett. 71(17), 2789, (1993).

[4] M.L. Ge, Y. Wang, *Long-range interaction models and a deformed loop symmetry*, Phys. Rev E 51(4), 2919, (1995)

[5] D. Bernard, Y.-S. Wu, *A note on statistical interactions and the thermodynamic Bethe ansatz*, unpublished

[6] H. Bethe, *Zur Theorie der Metalle I: Eigenwerte und Eigenfunktionen der linearen Atomkette*, Z. Phys. 71, 205-226 ,(1931).

[7] B. Sutherland, *A brief history of the quantum soliton with new results on the quantization of the Toda lattice*, Rock. Mtn. J. Math. 8(1-2), 413, (1978)

[8] B. Sutherland, *Quantum many-body problem in one dimension: ground state*, J. Math. Phys. 12, 246-250, (1971)
B. Sutherland, *Quantum many-body problem in one dimension: thermodynamics*, J. Math. Phys. 12, 251-256, (1971)
B. Sutherland, *Exact results for a quantum many-body problem in one dimension*, Phys. Rev. A4, 2012-2021, (1971)
B. Sutherland, *Exact results for a quantum many-body problem in one dimension II*, Phys. Rev. A5, 1372-1376, (1972)

[9] M. Toda, *Studies on a nonlinear lattice*, Phys. Reports 18C, 1-124, (1975)

[10] F.W.J. Olver, *Asymptotics and special functions*, Academic press, New York and London, (1974)

[11] C.N. Yang, C.P. Yang, *Thermodynamics of a one-dimensional system of bosons with repulsive delta-function interaction*, J. Math. Phys. 10(7), 1115, (1969)

[12] I.S. Gradshteyn, I.M. Ryzhik, *Table of integrals, series, and products*, Academic press, fifth edition