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Derivation of the solar geometric relationships using vector analysis

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Abstract

The standard mathematical approach used in deriving equations to describe the apparent motion and position of the Sun is spherical trigonometry. Additionally, the derivation of the equations for the intensity of the direct beam radiation, incident on the surface of a solar collector or architectural surface, also generally relies on the same approach. An alternative approach utilizing vector analysis is used to derive all of these equations. The technique greatly simplifies the derivation of equations for quantities such as the declination, altitude and azimuth of the Sun, and the intensity of the direct beam radiation on a tilted panel with an arbitrary orientation. Additionally, it allows a simple derivation of the equations needed to accurately describe the Equation of Time and the right ascension.

Keywords: Solar vector; Solar geometry; Vector analysis

1. Introduction

Spherical trigonometry was developed very early in the history of mathematics to understand the motion of astronomical objects. The derivation of equations that describe the position and apparent motion of the Sun usually relies upon the use of spherical trigonometry to determine the equations for the Sun's declination, altitude, azimuth and right ascension; and to determine the Equation of Time [1–5]. Similarly, the derivation of

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the equations describing the intensity of the direct beam of the solar radiation on a tilted surface of arbitrary tilt angle and orientation for solar energy applications relies upon the same spherical trigonometry approach [2,6]. Undoubtedly, the resulting equations which are derived in this way describe the solar geometry and direct beam intensity very accurately. However, for many students and practitioners in the field of solar energy, spherical trigonometry is a subject that is generally not well known. In this case, the equations are taken as given and as a result, there is very little understanding or appreciation of the origin of these equations, which may hinder a fuller understanding of the subject.

There have been a few authors who have utilised vector or direction cosine approaches to derive some aspects of the solar geometry and solar intensity on panels. However, typically a mixture of spherical trigonometry and the vectorial approach has been used to derive some of the necessary equations (for example, see [1,7]). In other cases, only some of the equations have been derived using either a direction cosine approach [8] or a vectorial approach [9].

Therefore, it is the purpose of this paper to present a comprehensive treatment of the vector derivation of all the equations required to describe the Earth–Sun geometry accurately (altitude, azimuth and intensity of the direct beam on a tilted plane). Further, utilising vector analysis, and building on previous work [10], this present paper presents the derivation of the geometric relationships needed to describe the declination and right ascension of the Sun that are consistent with the standard relationships used in accurate algorithms for the Sun's apparent position [11,12]. The vector approach is also used to derive the equation that accurately describes the Equation of Time. Additionally, this paper seeks to clarify the advantages and disadvantages of the various equations that are presented in the literature for determining the Sun's azimuth. Lastly, the nomenclature used in this work attempts to be consistent with standard texts in the field [6,11].

Essentially, the vector approach utilises the concepts of unit vectors, vector dot products and the transformation of reference frames (e.g. Cartesian to spherical co-ordinates) (for example, see [13]). Unlike spherical trigonometry, the aforementioned are all mathematical tools that should be familiar to anyone with a background in Engineering or the Physical Sciences.

2. Solar geometry

The Earth's orbit about the Sun is almost circular at an average distance of 149.6 million km. The Earth's axis of rotation is tilted by an angle $\varepsilon \approx 23.44^\circ$ with respect to the normal to the plane of the Earth's orbit around the Sun (see Fig. 1) [14]. The plane of the Earth's orbit is referred to as the plane of the ecliptic or simply the ecliptic. Alternately this can be expressed by saying that the plane passing through the Earth's equator is inclined obliquely to the plane of the ecliptic, at an angle ε (which is sometimes referred to as the obliquity of the ecliptic or simply as the obliquity). Due to the conservation of angular momentum, the Earth's axis of rotation points essentially in a fixed direction in space as the Earth orbits the Sun. This means that for the same location on Earth, at a fixed time (say midday as determined by solar time), the altitude of the Sun (the angular height above the horizon) will vary throughout the year. The above description is somewhat simplistic but it contains the essential elements of the Earth's orbit about the Sun.

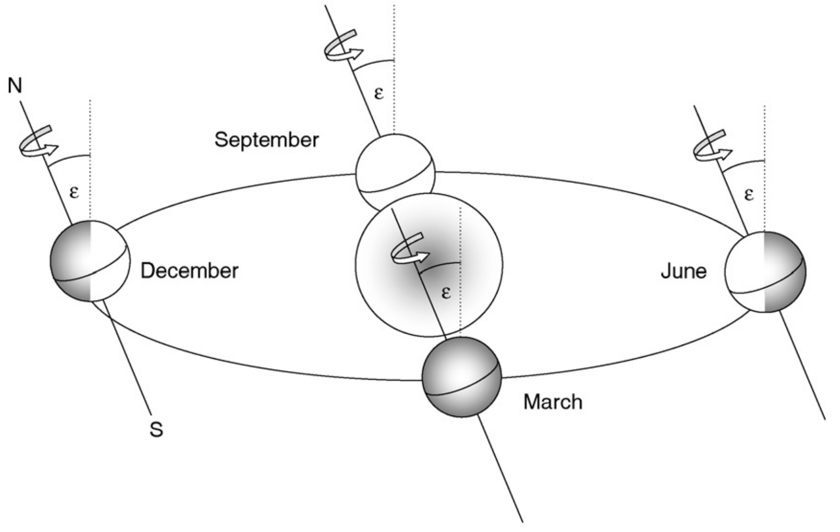

Fig. 1. Basic geometry of the Earth orbiting the Sun is schematically shown. The North pole–South pole axis is also shown, as are the Equator of the Earth and the direction of the Earth's daily rotation.

As the Earth's orbit is actually slightly elliptical, with an eccentricity of only 0.0167, the Earth–Sun distance varies from its average value by about $\pm 1.7\%$ throughout the year [6,14]. The Earth is closest to the Sun around January 1, and furthest around July 4 [6,14]. Also the Earth's orbit is influenced by the gravitational pull of the Moon. Strictly it is the centre-of-mass of the Earth–Moon system that travels in an elliptical orbit about the Sun [15]. Both the Earth and the Moon orbit about the centre-of-mass. However, for the Earth, the deviation of its orbit from the elliptical orbit of the centre-of-mass about the Sun is only a matter of ~4000 km. This is inconsequential in comparison to the average Earth–Sun distance of ~150 million km, so this effect is not usually significant for most solar applications.

The inclination of the Earth's axis of rotation, ε, and the orientation of this axis vary slightly with time. The direction of the axis is in fact precessing at a rate of 50 arc seconds per century, completing a 360° circuit approximately every 26,000 years [14]. In addition, there is an oscillation of this axis (nutation or "nodding" of the axis). This "nodding" is produced by the gravitational interaction between the Earth and the Moon as they orbit the Sun. The amplitude of this nutation is 9 arc seconds with a period of 18.6 years, and is superimposed on the precession [14].

However, these effects are small. Additionally, they do not alter the fundamental geometric relationships derived below. If high-precision formulas are required (as in some solar concentrating applications), the above effects can be accounted for by adding additional terms to the expressions describing the angular position of the Sun as a function of time [11,12].

3. Reference frames

In order to derive the solar geometry equations, it is important to define suitable reference frames. Three principal reference frames will be used in this paper, the ecliptic, the equatorial and the horizon reference frames. These reference frames are all geocentric reference frames in that they are centred on, or referenced to, the geometric centre of the Earth. All are commonly found in astronomy textbooks (for example, see [14]) and in the solar literature [1,5]. In these reference frames, the apparent motion of the Sun is considered. The Sun and other celestial bodies are envisaged to lie on the *celestial sphere* (Fig. 2), a sphere with an arbitrarily large radius. Often the celestial sphere is imagined to rotate about the fixed Earth to describe the daily, apparent motion of the Sun and other celestial bodies [1,5]. An equivalent viewpoint, which will be adopted in this paper, is to consider the Earth rotating on its North-South (N–S) axis once per day, and let the rectangular reference frames of Fig. 2 remain fixed in space relative to the distant stars. In this treatment, the apparent daily motion of the Sun is handled by the Earth's rotation, while the actual yearly orbit of the Earth around the Sun is treated by the Sun appearing to orbit the Earth once per year. The Sun's apparent orbit lies is in the *plane of the ecliptic* (Fig. 2). The ecliptic is tilted at an angle of ε (≈23.44°) to the plane of the *celestial equator*, which is the projection of the Earth's equatorial plane onto the celestial sphere. Note that the ecliptic crosses the equatorial plane at two points, which correspond to the March and September equinoxes.

For the ecliptic reference frame, the Sun's position is defined relative to the (x', y', z') axes (see Fig. 2). The Sun's angular position is specified by the *ecliptic longitude* λ which

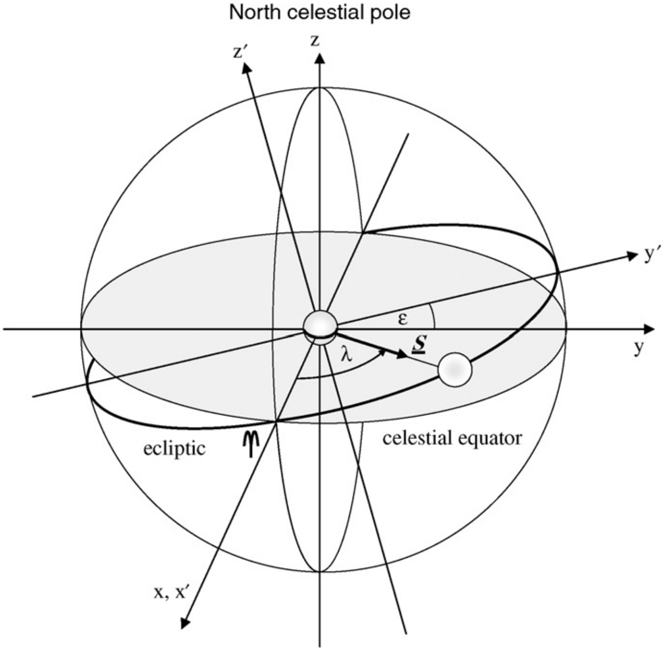

Fig. 2. Celestial sphere, showing the geometry of the Sun and Earth (Earth-based reference frame). In this frame of reference, the origin of the (x, y, z) axes, coincides with the centre of the Earth. The (x, y) plane (shaded) coincides with the equatorial plane of the Earth. The Sun "orbits" the Earth once per year, in the "plane of the ecliptic" (a plane tilted at an angle ε to the Earth's equatorial plane). The Earth rotates about the z-axis once per day. The projection of the Earth's North pole onto the celestial sphere is known as the North celestial pole.

varies from $\lambda = 0^\circ$ at the March equinox,[1] completing 360° after 1 year (that is one revolution relative to the "fixed" distant stars, i.e. 1 *tropical* year). When $\lambda = 90^\circ$, the Sun will be directly overhead at a latitude of 23.44°N (the June solstice) while at the September equinox, $\lambda = 180^\circ$ and the Sun is directly overhead at the Equator. At the December solstice, $\lambda = 270^\circ$ and the Sun is directly overhead at a latitude of 23.44°S. Numerous expressions for the ecliptic longitude can be found in the literature[2] [6,11,12]. The simplest expressions treat λ as uniformly varying throughout the year (e.g. [6]). However, this would only be true if the Earth orbited the Sun with a perfectly circular orbit and hence its orbital speed was constant. More accurate expressions for λ take into account the fact that the Earth's orbital speed varies due to its elliptical orbit, faster when it is closer to the Sun and slower when further away [11,12].

In Fig. 2, the plane of the ecliptic lies in the (x', y') plane. To undertake the vector analysis of the solar geometry, a unit vector $\underline{\boldsymbol{S}}$ is defined which at all times points towards the centre of the Sun with its origin at the geometric centre of the Earth. This vector is given by:

$$\underline{\boldsymbol{S}} = \cos\lambda \boldsymbol{i}' + \sin\lambda \boldsymbol{j}', \tag{1}$$

where $\boldsymbol{i}'$ and $\boldsymbol{j}'$ are unit vectors in the x' and y' directions, respectively. Note that any other vector parallel to this vector will have the same essential properties as $\underline{\boldsymbol{S}}$, due to the large Earth–Sun distance. Also, note that the computations of solar geometry are with respect to the vector from the Earth's geometric centre to the centre of the solar disk, which has a mean angular diameter of 0.53° [14].

Also shown in Fig. 2 is the equatorial reference frame, which is defined by the axes (x, y, z) where the z-axis is aligned with the N–S axis of the Earth. The equatorial reference frame is related to the ecliptic reference frame by a rotation through an angle ε about the x'-axis. This reflects the fact that the Earth's N–S axis is tilted by an angle ε from the normal to the plane of the Sun's apparent orbit (i.e. the z'-axis).

In carrying out the rotation, the relationship between the unit vectors of the two reference frames is given by:

$$\boldsymbol{i} = \boldsymbol{i}', \tag{2}$$

$$\boldsymbol{j} = \cos\,\varepsilon \boldsymbol{j}' - \sin\,\varepsilon \boldsymbol{k}', \tag{3}$$

$$\boldsymbol{k} = \sin\,\varepsilon \boldsymbol{j}' + \cos\varepsilon \boldsymbol{k}'. \tag{4}$$

It is useful to express the vector $\underline{\boldsymbol{S}}$ in terms of the equatorial reference frame. One way this can be done is to take the dot product of the vector $\underline{\boldsymbol{S}}$ with the unit vectors ($\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$) to determine the magnitude of the components of $\underline{\boldsymbol{S}}$ (S_x, S_y, S_z) in the (x, y, z) directions. For

[1]The direction of the March equinox is essentially fixed relative to the background of distant stars. When the Babylonians first realised this, the direction of the March equinox was pointing towards the constellation of Aries. So it is commonly refereed to as the "First point of Aries" and given the symbol of the rams head ♈. However, the position of the equinox is slowly precessing and will move about the equatorial plane in a compete revolution about once every 26,000 years.

[2]The Almanac contains a set of "low precision" formulas to describe the Sun's apparent position to an accuracy of 0°.01 and the Equation of Time to a precision of $0^{\mathrm{m}}.1$ between 1950 and 2050.

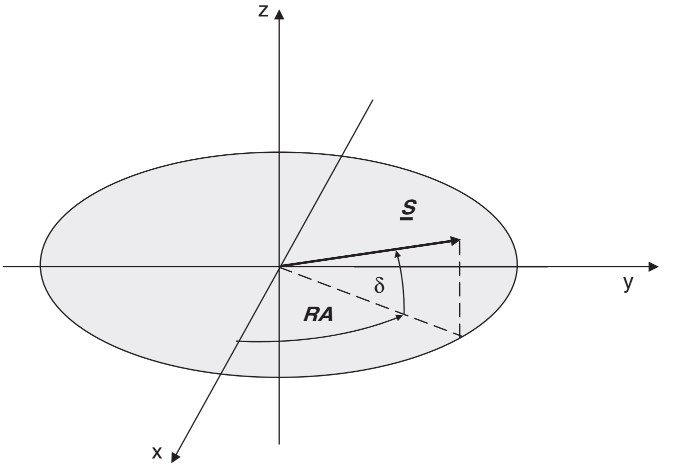

Fig. 3. Equatorial reference frame. The unit vector $\underline{\boldsymbol{S}}$ points towards the Sun and the Sun's angular position is described by the angle δ, the declination of the Sun, and RA, the right ascension.

example, $S_x = \underline{\boldsymbol{S}}.i$. Therefore, in the (x, y, z) equatorial reference frame, $\underline{\boldsymbol{S}}$ from Eq. (1) becomes

$$\underline{\boldsymbol{S}} = \cos\,\lambda\boldsymbol{i} + \sin\,\lambda\,\cos\,\varepsilon\boldsymbol{j} + \sin\,\lambda\,\sin\,\varepsilon\boldsymbol{k}. \tag{5}$$

Conventionally in the equatorial reference frame, the Sun's angular position is described utilising two angles: the declination δ and the right ascension RA (see Fig. 3). The declination is the angle that the vector $\underline{\boldsymbol{S}}$ makes with the equatorial plane, whilst the right ascension describes the angular displacement of the Sun in the equatorial plane, relative to the March equinox position. As such the vector $\underline{\boldsymbol{S}}$ can be written in terms of δ and RA giving

$$\underline{\boldsymbol{S}} = \cos\,\delta\,\cos\,\mathrm{RA}\,\boldsymbol{i} + \cos\,\delta\,\sin\,\mathrm{RA}\,\boldsymbol{j} + \sin\,\delta\,\boldsymbol{k}. \tag{6}$$

It is also useful in the following derivations to consider a rotating reference, frame (x'', y'', z''), which is defined in Fig. 4. In this frame of reference, the (x'', y'') plane still lies in the equatorial plane (that is the (x, y) plane of Fig. 2). However, the (x'', y'', z'') is a rotating frame of reference such that the vector pointing at the Sun, $\underline{\boldsymbol{S}}$, always lies in the (x'', z'') plane. The z''-axis is coincident with the N–S axis of the Earth. The (x'', y'', z'') reference frame is simply derived from the equatorial reference frame (x, y, z) by a rotation through an angle equal to the right ascension RA, about the z-axis (see Figs. 3 and 4). This reference frame will be referred to as the rotating equatorial reference frame. It rotates through 360° in 1 tropical year.

The horizon reference frame describes the Sun's position from the perspective of an observer on the surface of the Earth. In the horizon reference frame, the Sun's position is described relative to the axes (E, N, Z) as shown in Fig. 4. The angle ϕ is the latitude of the location being considered, $\underline{\boldsymbol{Z}}$ is the unit vector normal to the horizontal plane, $\underline{\boldsymbol{N}}$ is the unit vector pointing towards the North Pole and $\underline{\boldsymbol{E}}$ is the unit vector pointing in an Easterly direction. The angle ϕ is defined conventionally to be positive in the Northern hemisphere, zero at the Equator and negative in the Southern hemisphere. In this reference frame, the

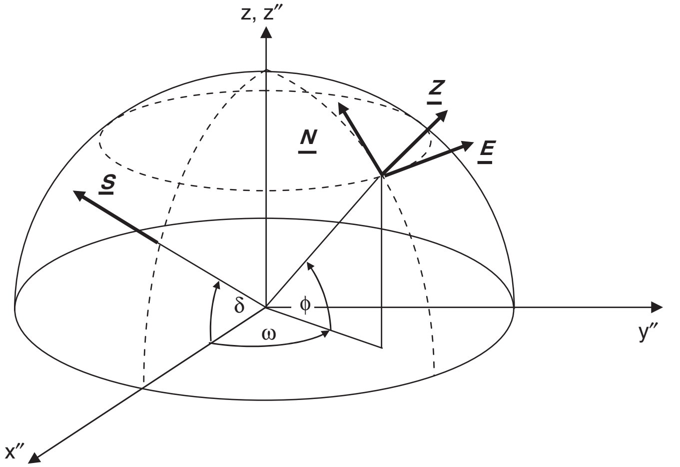

Fig. 4. Rotating equatorial (x'', y'', z'') and horizon (E, N, Z) reference frames. The hemisphere shown here describes the surface of the Earth (Northern hemisphere). Throughout 1 day, the hour angle ω, varies from 0° to 360° as the Earth rotates on its axis.

angular position of the Sun is described utilising the angles α_S and γ_S, the altitude and azimuth, respectively, of the Sun. Alternately instead of the altitude, the zenith angle, θ_Z, can be used. These angles are shown in Fig. 5. Often in the literature, α_S is evaluated from equations given in terms of $\sin\alpha_S$ or $\cos\theta_Z$ which are equivalent as $\alpha_S = 90 - \theta_Z$. Note that in this work, the Sun's azimuth γ_S is defined to be zero with respect to North and positive to the East of North. Alternate definitions are commonplace in the literature and care should be exercised when utilising azimuth angles.

The unit vectors $\underline{\boldsymbol{E}}$, $\underline{\boldsymbol{N}}$ and $\underline{\boldsymbol{Z}}$ vary with the hour angle ω, which is defined by

$$\omega = (\text{hour} - 12)\ 15^\circ, \tag{7}$$

where hour is the local apparent solar time (0–24 h) [3]. The hour angle varies as the Earth rotates relative to the Sun such that at solar noon, $\omega = 0^\circ$.

In subsequent derivations, it is necessary to consider the unit vectors $\underline{\boldsymbol{E}}$, $\underline{\boldsymbol{N}}$ and $\underline{\boldsymbol{Z}}$ in terms of their component vectors with respect to the x'', y'' and z'' directions (with unit vectors $\boldsymbol{i}''$, $\boldsymbol{j}''$ and $\boldsymbol{k}''$ respectively). This is simply a transformation from a rectangular co-ordinate system (x'', y'', z'') to a spherical polar co-ordinate system (E, N, Z), which is a standard transformation [13].

As such it is possible to write down expressions for the unit vectors in the horizon reference frame[3] ($\underline{\boldsymbol{E}}$, $\underline{\boldsymbol{N}}$, $\underline{\boldsymbol{Z}}$) in terms of the unit vectors of the rotating equatorial reference frame ($\boldsymbol{i}''$, $\boldsymbol{j}''$, $\boldsymbol{k}''$):

$$\underline{\boldsymbol{E}} = \sin\ \omega\ \boldsymbol{i}'' + \cos\ \omega\ \boldsymbol{j}'', \tag{8}$$

$$\underline{\boldsymbol{N}} = -\sin\ \phi\ \cos\ \omega\ \boldsymbol{i}'' - \sin\ \phi\ \sin\ \omega\ \boldsymbol{j}'' + \cos\ \phi\ \boldsymbol{k}'', \tag{9}$$

$$\underline{\boldsymbol{Z}} = \cos\ \phi\ \cos\ \omega \boldsymbol{i}'' + \cos\ \phi\ \sin\ \omega\ \boldsymbol{j}'' + \sin\ \phi\ \boldsymbol{k}''. \tag{10}$$

[3]Equivalently the southern direction could have been used in defining the horizon reference frame. The unit vector pointing South would simply be equal to $-\underline{\boldsymbol{N}}$.

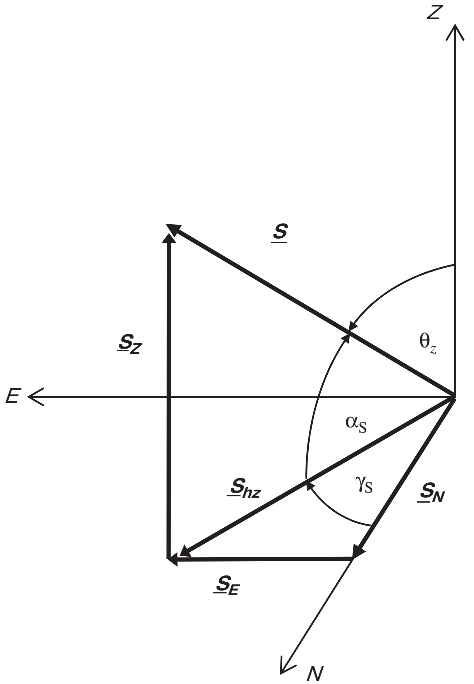

Fig. 5. The horizon reference frame for an observer positioned at the origin of the (E, N, Z) axes. The horizontal plane is defined by the plane (E, N). Also shown are the vector $\underline{\boldsymbol{S}}$, and its component vectors ($\underline{\boldsymbol{S}}_E, \underline{\boldsymbol{S}}_N, \underline{\boldsymbol{S}}_Z$), its horizontal component $\underline{\boldsymbol{S}}_{hz}$, the zenith angle θ_S, the altitude α_S and the azimuth γ_S. The azimuth is defined relative to North and is positive in an Easterly direction.

In terms of the rotating equatorial reference frame, the vector $\underline{\boldsymbol{S}}$ can be determined from Fig. 4 and is given by

$$\underline{\boldsymbol{S}} = \cos\,\delta\,\boldsymbol{i}'' + \sin\,\delta\,\boldsymbol{k}''. \tag{11}$$

It is useful to express the vector $\underline{\boldsymbol{S}}$ in terms of the unit vectors in the horizon reference frame (E, N, Z). As for the transformation from the ecliptic to the equatorial reference frame, the dot product of $\underline{\boldsymbol{S}}$ with the appropriate unit vectors, ($\underline{\boldsymbol{E}}$, $\underline{\boldsymbol{N}}$, $\underline{\boldsymbol{Z}}$), yields the magnitude of the component of the vector $\underline{\boldsymbol{S}}$, ($S_E$, S_N, S_Z) in the (E, N, Z) directions.

Hence, from Eqs. (8)–(11),

$$\begin{aligned}\underline{\boldsymbol{S}} = &-\cos\,\delta\,\sin\,\omega\,\underline{\boldsymbol{E}} + (\sin\,\delta\,\cos\,\phi - \cos\,\delta\,\sin\,\phi\,\cos\,\omega)\,\underline{\boldsymbol{N}}\\ &+ (\cos\,\delta\,\cos\,\phi\,\cos\,\omega + \sin\,\delta\,\sin\,\phi)\,\underline{\boldsymbol{Z}}.\end{aligned} \tag{12}$$

The vector $\underline{\boldsymbol{S}}$ can also be written in terms of α_S and γ_S. From examination of Fig. 5, $\underline{\boldsymbol{S}}$ can be written as

$$\underline{\boldsymbol{S}} = \cos\,\alpha_S\,\sin\,\gamma_S\,\underline{\boldsymbol{E}} + \cos\,\alpha_S\,\cos\,\gamma_S\,\underline{\boldsymbol{N}} + \sin\,\alpha_S\,\underline{\boldsymbol{Z}}. \tag{13}$$

Having defined suitable reference frames and expressed the vector $\underline{\boldsymbol{S}}$ in the ecliptic (Eq. (1)), equatorial (Eqs. (5) and (6)), rotating equatorial (Eq. (11)) and horizon (Eqs. (12) and (13)) reference frames, it is now possible to simply derive the usual geometric quantities required to describe the Sun's apparent motion and position.

4. Declination of the Sun

The Sun's declination was previously defined in Section 3 when discussing Eq. (6) and Fig. 3. The equation required to describe the declination at any time during the Earth's yearly orbit of the Sun can be easily derived using Eq. (5). The declination, δ, is simply the angle that the vector $\underline{\boldsymbol{S}}$ makes with the equatorial (x, y) plane (see Fig. 3) and hence:

$$\sin\ \delta = S_z/S, \tag{14}$$

where S_z is the magnitude of the z-component of $\underline{\boldsymbol{S}}$ and S is the magnitude of $\underline{\boldsymbol{S}}$ (which as it is a unit vector is unity). Therefore,

$$\sin\ \delta = \sin\ \lambda\ \sin\ \varepsilon, \tag{15}$$

where ε is the obliquity. Alternately, this equation could be derived from the above expressions for $\underline{\boldsymbol{S}}$: Eq. (5) (in terms of the ecliptic co-ordinate λ) and Eq. (6) (in terms of the equatorial co-ordinates δ, RA). Both of these equations are in the equatorial reference frame. Hence, by simply equating the z-components, Eq. (15) can be obtained.

Explicitly, δ is given by

$$\delta = \sin^{-1}(\sin\ \lambda\ \sin\ \varepsilon). \tag{16}$$

A slightly simpler expression for δ is very commonly used in the solar energy literature [2,3,6] given by

$$\delta = 23.44^\circ \sin\ \lambda. \tag{17}$$

As an approximation, Eq. (17) is quite good. In the worst case, it is in error by 0.26° when compared to Eq. (16). Note that this error is approximately the value of the mean angular radius of the solar disk (0.27° [14]). The use of Eq. (17) is widespread, yet it seems to offer no major simplification over Eq. (16), at the expense of accuracy. Eq. (16) is the usual form of the expression for the declination of the Sun for accurate calculations of solar position [11,12] and therefore its use in all types of solar energy calculations should be encouraged.

5. Altitude of the Sun

In a similar fashion, it is possible to derive an expression for the altitude of the Sun, α_S. To do this, consider Eq. (12) for $\underline{\boldsymbol{S}}$ in the horizon reference frame and Fig. 5. From inspection of Fig. 5, it can be seen that $\sin\ \alpha_S = S_Z/S$ and hence,

$$\sin\ \alpha_S = \cos\ \delta\ \cos\ \phi\ \cos\ \omega + \sin\ \delta\ \sin\ \phi, \tag{18}$$

where S_Z is the magnitude of the Z-component of $\underline{\boldsymbol{S}}$, and is obtained from Eq. (12). Alternately, Eq. (18) can be derived by considering the two expressions for $\underline{\boldsymbol{S}}$ in the horizon reference frame, Eqs. (12) and (13). Again the two expressions of $\underline{\boldsymbol{S}}$ are equivalent, so by equating the Z-components, Eq. (18) can be obtained.

This equation is identical to that derived via spherical trigonometry [3,4,16] or by conventional planar geometry (a complex and involved process) [3]. However, the steps outlined here are much more readily accessible to students and practitioners, involving only simple vector analysis and transformations of reference frames.

6. Azimuth of the Sun

To calculate the azimuth of the Sun, γ_S, consider Eq. (12) and Fig. 5. The azimuth is the angle between the horizontal component of $\underline{\boldsymbol{S}}$ (denoted by $\underline{\boldsymbol{S}}_{hz}$), and the unit vector $\underline{\boldsymbol{N}}$, which points North (sometimes the azimuth is defined relative to South [6]; however, the usual astronomical definition is with respect to North [14]).

From inspection of Fig. 5, the magnitude of $\underline{\boldsymbol{S}}_{hz}$, S_{hz} is simply equal to $\cos\alpha_S$, as $\underline{\boldsymbol{S}}$ is a unit vector. Therefore, as γ_S is the angle between the vector $\underline{\boldsymbol{S}}_{hz}$ and the vector $\underline{\boldsymbol{S}}_N$, hence $\cos\gamma_S = S_N/S_{hz}$, and therefore:

$$\cos\ \gamma_S = \frac{\sin\ \delta\ \cos\ \phi - \cos\ \delta\ \sin\ \phi\ \cos\ \omega}{\cos\ \alpha_S}, \tag{19}$$

where S_N, the magnitude of $\underline{\boldsymbol{S}}$ in the $\underline{\boldsymbol{N}}$ direction, is obtained from Eq. (12).

Again this is the same equation derived using spherical trigonometry (see for example [3]), but is far more easily derived using the vector approach presented here. The above equation for azimuth is one of the many that appear in the solar literature. Alternate expressions for the Sun's azimuth can be easily derived utilising vector analysis.

Again, simply from inspection of Fig. 5, note that $\tan\gamma_S = S_E/S_N$, and $\sin\gamma_S = S_E/S_{hz}$. Hence,

$$\tan\ \gamma_S = \frac{\sin\ \omega}{\sin\ \phi\ \cos\omega - \tan\ \delta\ \cos\ \phi}, \tag{20}$$

$$\sin\ \gamma_S = \frac{-\sin\ \omega\ \cos\ \delta}{\cos\ \alpha_S}, \tag{21}$$

and the components S_E and S_N are obtained from Eq. (12).

Eq. (20) is identical to the expression quoted in Duffie and Beckman [6], and Eq. (21) is the same as derived by spherical trigonometry [16] after allowing for a change in sign.[4] Great care needs to be taken when calculating azimuth using Eqs. (19)–(21), as they involve the inverse trigonometric relationships. This has caused great confusion in the literature; however, from the analysis presented here, it is straightforward to see the strengths and weaknesses of the different methods for calculating which quadrant the azimuth lies in.

For example, Eq. (19) alone can determine that the Sun is North or South of the observer, as the component of $\underline{\boldsymbol{S}}$ pointing North, S_N changes sign between these two cases. However, Eq. (19) alone cannot determine whether the Sun is East or West of the observer, as S_{hz} has the same value for hour angles, $\omega = -\omega'$ and $\omega = \omega'$. That is, the Sun's altitude is the same in the morning and afternoon for times equally either side of solar noon.

To calculate γ_S with Eq. (19) requires the use of the inverse cos function, $\cos^{-1}$. Let γ_c denote the angle returned by the inverse cos function. That is,

$$\gamma_c = \cos^{-1}\left(\frac{\sin\ \delta\ \cos\ \phi - \cos\ \delta\ \sin\ \phi\ \cos\ \omega}{\cos\ \alpha_S}\right). \tag{22}$$

[4]Braun and Mitchell [16] define the Sun's azimuth with South having an azimuth of zero, and West being positive. This means that their azimuth differs from the azimuth in this paper by 180°. Hence, this changes the sign of the term $\sin\gamma_S$, but not the sign of the term $\tan\gamma_S$.

Typically, this function is implemented so that it will always return an angle γ_c such that $(0^\circ \leqslant \gamma_c{}^\circ \leqslant 180^\circ)$. However, the Sun's azimuth, γ_S, has a range $(0^\circ \leqslant \gamma_S \leqslant 360^\circ)$. That is, using Eq. (19) alone, it cannot determine if γ_S lies in the range $(0^\circ \leqslant \gamma_S \leqslant 180^\circ)$ or $(180^\circ \leqslant \gamma_S \leqslant 360^\circ)$, as $\cos(\gamma_S) = \cos(360^\circ - \gamma_S)$.

Fortunately, as noted by Szokolay [3], there is a simple solution. For γ_S such that $(0^\circ \leqslant \gamma_S \leqslant 180^\circ)$, then the Sun is always East of the observer and this corresponds with solar times between midnight and noon and hence $\omega < 0$ (using Eq. (7)). For γ_S such that $(180^\circ \leqslant \gamma_S \leqslant 360^\circ)$, then the Sun is always West of the observer and this corresponds with solar times between noon and midnight and hence $\omega > 0$. Hence when calculating γ_S using Eq. (19),

$$\begin{aligned} &\text{for } \omega < 0; \quad \gamma_S = \gamma_c, \\ &\text{whilst for } \omega > 0, \text{ then}: \quad \gamma_S = 360^\circ - \gamma_c, \end{aligned} \tag{23}$$

where γ_c is the angle returned by the $\cos^{-1}$ function from Eq. (22).

Similar issues regarding the determination of the correct quadrant for the Sun's azimuth are involved in using Eqs. (20) and (21). In particular, the use of Eq. (21) is very complicated [16]. This is simply because whilst Eq. (21) alone can determine whether the Sun is East or West of the observer (as S_E changes sign), it cannot determine whether the Sun is North or South of the observer with the same altitude (as S_{hz} will have the same value in both cases). Braun and Mitchell [16] propose a complex procedure for determining the correct quadrant for γ_S when using Eq. (21). However, use of Eq. (19) is a much simpler approach and its use is recommended.

The use of Eq. (20) is less complicated than the use of Eq. (21), as the tan (γ_S) function has a single value, for γ_S in the range: $(0^\circ \leqslant \gamma_S \leqslant 180^\circ)$. However, $\tan(\gamma_S) = \tan(\gamma_S + 180^\circ)$, so Eq. (20) alone, like Eq. (19), cannot determine if γ_S lies in the range $(0^\circ \leqslant \gamma_S \leqslant 180^\circ)$ or $(180^\circ \leqslant \gamma_S \leqslant 360^\circ)$. That is, the hour angle ω can be used to determine the correct quadrant, as was the case when using Eq. (19). However, care needs to be taken, as the inverse tan function is often implemented such that the angle returned (denoted by γ_t) lies in the range $(-90^\circ \leqslant \gamma_t \leqslant 90^\circ)$. In this case, negative values of γ_t need to be made positive by the addition of 180°. Thus, when calculating γ_S using Eq. (20)

$$\begin{aligned} &\text{for } \omega < 0: \\ &\text{if } \gamma_t < 0, \quad \gamma_S = \gamma_t + 180^\circ \\ &\text{if } \gamma_t > 0, \quad \gamma_S = \gamma_t. \\ &\text{For } \omega < 0: \\ &\text{if } \gamma_t < 0, \quad \gamma_S = \gamma_t + 360^\circ, \\ &\text{if } \gamma_t > 0, \quad \gamma_S = \gamma_t + 180^\circ, \end{aligned} \tag{24}$$

where γ_t is the angle returned by the $\tan^{-1}$ function when solving Eq. (20).

For azimuth calculations, care needs to be taken when $\phi = \delta$ and $\omega = 0$. In this case, all equations for determining γ_S are not defined, as the Sun is directly overhead of the observer, so no value of γ_S can be assigned. In conclusion, of the three approaches outlined above, the simplest approach is to use Eq. (19) and therefore its use is recommended.

7. Direct beam intensity on a panel of arbitrary orientation and tilt

In many solar engineering and architectural applications, it is important to be able to calculate the direct beam intensity of the solar radiation incident on a panel or surface which is oriented in any direction and tilted from the horizontal. The direct beam intensity incident on any surface, I_D, is given by

$$I_D = I_n \cos\theta, \quad (25)$$

where I_n is the direct beam intensity incident on a plane normal to the direct beam solar radiation, and θ is the angle of incidence of the direct beam radiation onto the surface (the angle the Sun's rays make with the surface normal) [17].

Shown in Fig. 6 is a panel, tilted at an angle β with respect to the horizontal—the (*E*, *N*) plane. The direction of the panel is specified by the unit vector $\underline{\boldsymbol{n}}$, which is normal to the panel. The horizontal component of $\underline{\boldsymbol{n}}$ makes an angle of γ with the *N*-axis (North) and is measured positive to the East, consistent with the solar azimuth γ_S. The angle γ is often referred to as the panel azimuth. In terms of its components in the horizon reference frame, $\underline{\boldsymbol{n}}$ is given by

$$\underline{\boldsymbol{n}} = \sin\beta\sin\gamma\,\underline{\boldsymbol{E}} + \sin\beta\cos\gamma\,\underline{\boldsymbol{N}} + \cos\beta\,\underline{\boldsymbol{Z}}. \quad (26)$$

To determine the angle of incidence θ is a simple matter with vector analysis. To obtain an expression for θ in terms of the derived quantities of altitude and azimuth, consider Eq. (13) for $\underline{\boldsymbol{S}}$. Taking the dot product of $\underline{\boldsymbol{n}}$ and $\underline{\boldsymbol{S}}$ gives

$$\cos\theta = \sin\beta\sin\gamma\cos\alpha_S\sin\gamma_S + \sin\beta\cos\gamma\cos\alpha_S\cos\gamma_S + \cos\beta\sin\alpha_S. \quad (27)$$

as $\underline{\boldsymbol{n}}\cdot\underline{\boldsymbol{S}} = \cos\theta\cdot$ Using the standard trigonometric identity for $\cos(\gamma-\gamma_S) = \cos\gamma\cos\gamma_S + \sin\gamma\sin\gamma_S$ gives

$$\cos\theta = \sin\beta\cos\alpha_S\cos(\gamma-\gamma_S) + \cos\beta\sin\alpha_S. \quad (28)$$

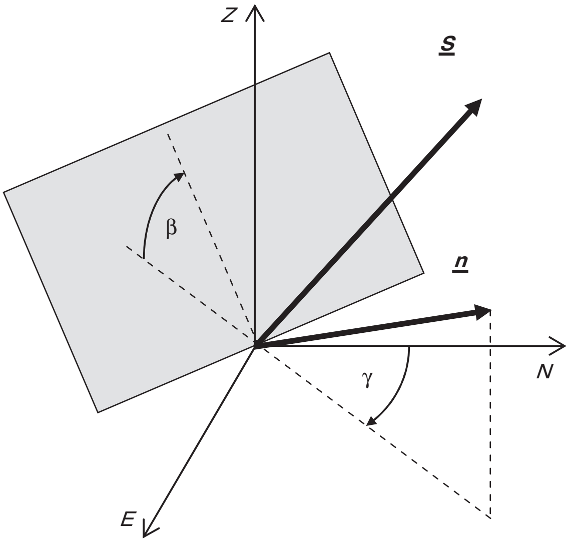

Fig. 6. An arbitrarily oriented surface, tilted by an angle β with respect to the horizontal—the (*E*, *N*) plane.

This expression is identical to that derived elsewhere by spherical trigonometry [16], and by direction cosines [8] and is independent of the definition of azimuth.

To derive the angle of incidence θ, in terms of more fundamental parameters (ϕ, δ, ω), is again straight forward utilising the vector approach. Again, taking the dot product of $\underline{\boldsymbol{n}}$ and $\underline{\boldsymbol{S}}$ (Eq. (12)) gives

$$\begin{aligned}\cos\,\theta = & -\sin\,\beta\,\sin\,\gamma\,\cos\,\delta\,\sin\,\omega \\ & +\sin\,\beta\,\cos\,\gamma\,\sin\,\delta\,\cos\,\phi - \sin\,\beta\,\sin\,\gamma\,\cos\,\theta\,\sin\,\omega \\ & +\cos\,\beta\,\cos\,\delta\,\cos\,\phi\,\cos\,\omega + \cos\,\beta\,\sin\,\delta\,\sin\phi, \end{aligned} \tag{29}$$

which is the same expression found elsewhere in the literature, apart from sign changes to the first three terms.[5] Eq. (29) is presented, unreferenced, in Duffie and Beckman [6], derived from a combination of vector and spherical trigonometric results by Jolly [7] and by direction cosines by Coffari [1].

8. Equation of time and right ascension

Time, as measured by an accurate clock, differs from the time determined from the Sun when using a sundial. Part of this difference has to do with the longitude of the observer's location, daylight saving and the remaining difference (termed the Equation of Time E) is due to factors associated with the Earth's orbit. The exact time difference in minutes is given by the following equation:

$$\text{Solar time} - \text{clock time} = 4(L_{\text{st}} - L_{\text{loc}}) + E + \text{DS}, \tag{30}$$

where L_{loc} (in degrees) is the longitude of the observer, in a local time zone with a standard longitude of L_{st} (in degrees) [6]. Each degree of longitude difference is equivalent to 4 min of time difference and the term DS depends on whether Daylight Saving is in operation (DS $= -60$ min) or not (DS $= 0$).

The Equation of Time arises because the length of a day (i.e. the time for the Earth to complete one revolution about its own axis with respect to the Sun) is not uniform throughout the year. Over the year, the average length of a day is 24 h, however, the length of a day varies due to two elements of the Earth's orbit. As such E can be described as a sum of two terms, E_1 and E_2. The term E_1 arises primarily from the eccentricity of the Earth's orbit and E_2 primarily from the tilt of the Earth's axis from the normal to the plane of its orbit.

To derive an expression for E, it is usual to define a fictitious mean Sun [18]. With respect to an Earth-bound observer, this mean Sun is envisaged to orbit the Earth uniformly in the equatorial plane (see Fig. 7). Its angular position is described by the "mean longitude of the Sun", L, which varies uniformly throughout the year, and is defined to be zero at the March equinox. Define a vector $\underline{\boldsymbol{S}}_m$ that points at this mean Sun, and rotates about the Earth once per year. As it lies in the (x, y) plane (the equatorial plane—see Fig. 8), it is described by the following equation:

$$\underline{\boldsymbol{S}}_m = \cos\,L\,\boldsymbol{i} + \sin\,L\,\boldsymbol{j}. \tag{31}$$

[5] Duffie and Beckman [6] and Coffari [1] define azimuth to be zero facing South and positive to the West. This means their expression for azimuth differs from that presented here by 180°, hence all terms involving $\cos\gamma_S$ or $\sin\gamma_S$ are of opposite sign.

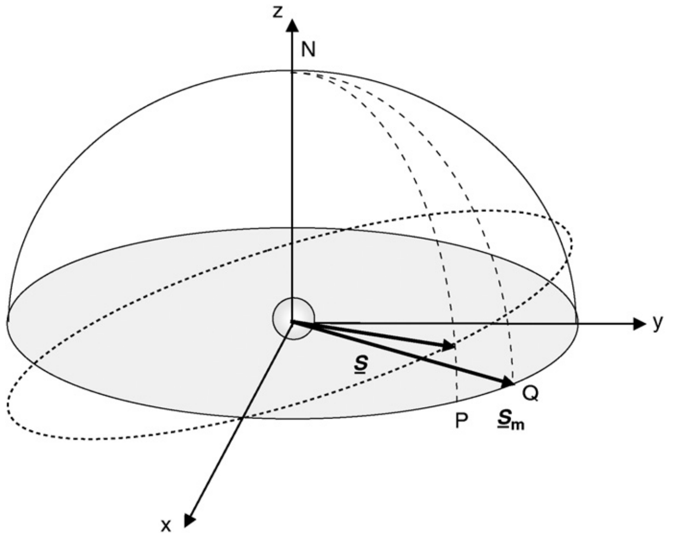

Fig. 7. The (x, y, z) frame of reference of Fig. 3 is redrawn showing the vectors $\underline{\boldsymbol{S}}$ and $\underline{\boldsymbol{S}}_{\mathrm{m}}$. The vector $\boldsymbol{S}$ lies in the (x', y') plane of Fig. 3 (the plane of the ecliptic), while the vector $\underline{\boldsymbol{S}}_{\mathrm{m}}$ lies in the (x, y) plane, the equatorial plane. Two arcs are shown which are coincident with a point N on the z-axis and intercept the (x, y) plane at points P and Q, respectively. These arcs are projections of lines of longitude on Earth, which is located at the origin. The vector $\underline{\boldsymbol{S}}$ lies in the plane passing through the point P and the z-axis, while the vector $\underline{\boldsymbol{S}}_{\mathrm{m}}$ lies in the plane defined by the point Q and the z-axis.

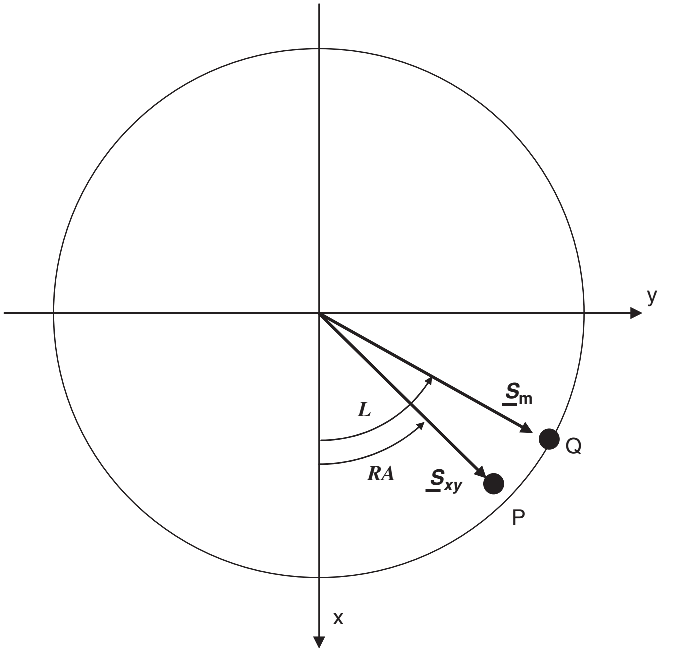

Fig. 8. The (x, y) plane is redrawn as viewed from the point N (i.e. directly above the North pole of the Earth). This figure shows the projection of the vector $\boldsymbol{S}$ onto the (x, y) ($\underline{\boldsymbol{S}}_{xy}$) plane and the vector $\underline{\boldsymbol{S}}_{\mathrm{m}}$.

One revolution of the Earth about its N–S axis takes exactly 24 h with respect to this mean Sun. This is how the real Sun would behave if (i) the Earth–Sun orbit was circular instead of elliptical and (ii) the Earth's axis was not tilted at $\varepsilon = 23.44^\circ$. It is this mean or average Sun that forms the basis for our uniform clock time, as each day has exactly 24 h.

The real Sun, however, does not behave quite so uniformly. As discussed previously, the real Sun appears to go around the Earth with a slightly elliptical orbit and this orbit lies in the plane of the ecliptic, with the Sun completing one orbit in 1 year. Its angular position from the March equinox, in the plane of the ecliptic, is described by the "ecliptic longitude" λ (as discussed previously with regard to Fig. 3). The real Sun's angular position does not vary uniformly throughout the year. Instead, due to the ellipticity of the orbit, the Earth is slightly closer to the Sun on January 1, and furthest from the Sun on July 4. As such, the Earth's orbiting speed is faster than its average speed for half the year (~October–March) and slower than its average speed for the remaining half of the year (~April–September), which results in λ varying non-uniformly throughout the year. For the real Sun, its associated unit vector $\underline{\boldsymbol{S}}$ is given by Eq. (5) in terms of the equatorial reference frame.

Once time differences due to the longitude of the observer and daylight saving are accounted for, then the only difference between clock time and solar time comes about because of the different positions the two Suns have relative to the equatorial plane. The Sun vectors move about 1° each day along their respective orbits. Consider the two Suns as shown in Figs. 7 and 8. Starting from the March equinox, L and λ increase through the year, and the two vectors $\underline{\boldsymbol{S}}$ and $\underline{\boldsymbol{S}}_{\mathrm{m}}$ move along their respective orbits as shown in Fig. 6 (their angular difference is greatly exaggerated in Figs. 7 and 8). For a given day, the vectors $\underline{\boldsymbol{S}}$ and $\underline{\boldsymbol{S}}_{\mathrm{m}}$ describe the position of the Sun and the mean Sun relative to the Earth. On any day, the Earth will rotate once about the z-axis (the N–S axis) in an anticlockwise direction when viewed from above the North pole. To simplify the discussion consider an observer whose longitude is equal to a standard longitude in any arbitrary time zone (i.e., $L_{\mathrm{loc}} = L_{\mathrm{st}}$), and daylight saving is not operating (i.e., DS = 0). For our observer on the Earth, they will determine solar noon to be the time when the Earth has rotated until their longitude, projected onto the celestial sphere, coincides with the arc NP where the vector $\underline{\boldsymbol{S}}$ is pointing at the real Sun (Fig. 7). However, for the case shown in Figs. 7 and 8, solar noon will not coincide with noon, as determined by clock time. This is because the mean Sun is located at point Q in the figures. According to clock time, the Earth will have to rotate further until the observer's longitude, projected onto the celestial sphere, coincides with the arc NQ. This determines noon, clock time. In this case, solar noon has occurred some minutes earlier. In terms of the equatorial plane, this angular (and hence time) difference is easily derived, by considering Fig. 8. The angular difference between the actual Sun's position and the mean Sun's position in the equatorial plane is equal to ($L-$RA), where RA is the right ascension of the real Sun, the angle between the x-axis and the vector $\underline{\boldsymbol{S}}_{xy}$. This vector, $\underline{\boldsymbol{S}}_{xy}$, is simply the sum of the x- and y-components of $\underline{\boldsymbol{S}}$, and hence $\underline{\boldsymbol{S}}_{xy}$ is the component of $\boldsymbol{S}$ in the equatorial plane (x, y). From Eq. (5), this yields

$$\underline{\boldsymbol{S}}_{xy} = \cos\lambda\,\boldsymbol{i} + \sin\lambda\,\cos\varepsilon\,\boldsymbol{j}. \tag{32}$$

For an angular difference equal to ($L-$RA), the time difference E (in minutes) is given by the following equation:

$$E = 4(L - \mathrm{RA}). \tag{33}$$

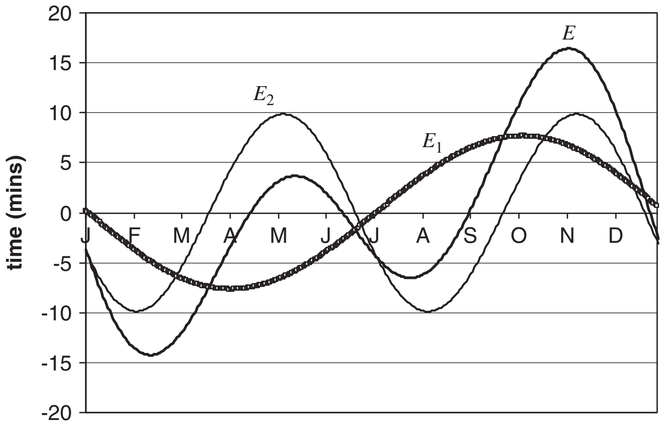

Fig. 9. Calculated values of the Equation of Time, E, using Eq. (33). Two "components" are also shown: E_1 is the Equation of Time for the case where $\varepsilon = 0$, while $E_2 = E - E_1$.

The factor of 4 arises, as there are 4 min per degree, as discussed previously. The right ascension, RA, can be readily evaluated using Eq. (32) for $\underline{\boldsymbol{S}}_{xy}$ and Fig. 8, as tan RA $= S_y/S_x$. These expressions yield

$$\tan \text{RA} = \tan \lambda \cos \varepsilon. \tag{34}$$

Eqs. (33) and (34) are identical in form to the equations used in *The Astronomical Almanac*, and are typically derived using spherical trigonometry [4,5]. The derivation of these relationships using vector analysis is presented here for the first time and is significantly easier than the spherical trigonometry approach.

Numerous papers give simplified expressions for L, and λ allowing RA and hence E to be easily calculated [11,12]. Shown in Fig. 9 are the calculated values of E using expressions for L, and λ from *The Astronomical Almanac* [11]. The Equation of Time is also calculated for the case where ε is set to zero, and is denoted by E_1. This shows the impact of only the ellipticity of the Earth's orbit on E. Note that E_1 is essentially sinusoidal and has a period of 1 year, an amplitude of about 7.6 min and is zero in early January and July, corresponding to the times when the Earth's orbital velocity is at its fastest (early January) and at its slowest (early July). The shape of the E_1 curve is easily understood, as half the year the Sun's orbital velocity is faster than average and the remaining time it is slower. Correspondingly, half the year the Sun's angular position in the equatorial plane (RA) is greater than that of the mean Sun (L) (January–June), so E_1 is negative, whilst for the remainder of the year RA $< L$ (July–December), so E_1 is positive.

The second component of E, E_2, is also shown in Fig. 9. In this figure, it is calculated by simply subtracting E_1 from E. It too is sinusoidal, however, with a period of approximately half a year, amplitude of ~10 min, and is zero at the equinoxes and the solstices.

For these calculations, the ecliptic longitude λ varies non-uniformly throughout the year (due to the ellipticity of the orbit), so the calculated E_2 values are influenced by both the ellipticity of the orbit and the tilt of the ecliptic plane. However, the dominant effect on E_2 arises due to the tilt of the ecliptic plane. This can be demonstrated by calculating E with an ecliptic longitude λ, which varies uniformly and is equal to the mean longitude of the Sun, L (so there would be no effect on E due to the Earth's orbital speed varying, i.e. no E_1

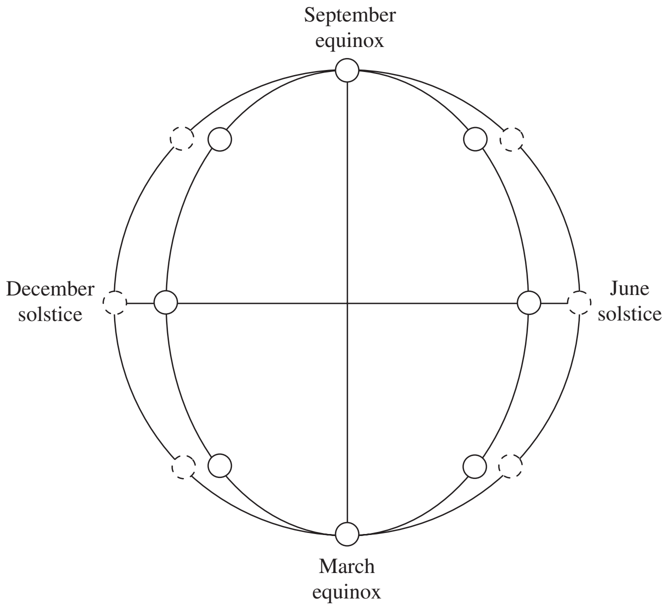

Fig. 10. Real Sun's orbit (inner curve) in the plane of the ecliptic and the mean Sun's orbit (outer curve) in the equatorial plane, for the special case where $\lambda = L$. The diagram shows the projection of the plane of the ecliptic onto the equatorial plane (i.e. the (x, y) plane). The positions of the Sun (solid circles) and the "mean" Sun (dashed circles) are shown at various times of the year. Each position corresponds to an angular step of 45° in their respective orbits.

component). In this case, there would only be a contribution to E arising from the tilt of the ecliptic. Calculating E in this way gives values very similar to those shown in Fig. 9 for E_2, indicating that the two components of E are almost independent of one another.

Further understanding of the second component of E can be gained by considering Fig. 10. In this figure, the orbits of the Sun and the mean Sun are viewed from directly above the equatorial plane, as for Fig. 8. Essentially, this results in a projection of the real Sun's apparent orbit in the tilted plane of the ecliptic onto the equatorial plane. In terms of the Sun's vector, the inner orbit is the path traced by the vector $\underline{\boldsymbol{S}}_{xy}$ (i.e. the component of $\underline{\boldsymbol{S}}$ that lies in the equatorial plane). This results in the very elliptical-looking orbit for the real Sun. Snapshots of the angular position in the equatorial plane of the real Sun and the "mean" Sun are shown throughout the year at 45° intervals of λ and L. For this diagram, λ and L are assumed to be equal and uniformly varying with time (so both the Sun and the mean Sun appear to have a constant orbital speed). As such for Fig. 10, there will be no contribution to E from E_1, only E_2.

In the equatorial plane, the angular displacement of the Sun (RA) and the mean Sun (L), throughout the year will be as shown in Fig. 10. As E is given by Eq. (33), and E_1 for this case is zero, it is possible to see how the component E_2 arises. From Fig. 10, it can be seen that the mean Sun and the real Sun are coincident at the March equinox, as $L = \lambda = RA = 0°$. Between the March equinox and the June solstice, $L > \text{RA}$, so E_2 is positive. At the June solstice, $L = \text{RA} = 90°$ and so $E_2 = 0$. Between the June solstice and the September equinox, $L < RA$, and so E_2 is negative. Again at the September equinox,

$L = \mathrm{RA} = 180^{\circ}$ and again $E_2 = 0$. For the remainder of the year, this pattern repeats itself. This gives a qualitative description of E_2 in agreement with the exact curve shown in Fig. 9.

9. Conclusions

Vector analysis has been used to derive all equations required to describe the apparent motion of the Sun: that is, the Sun's declination, altitude, azimuth and right ascension. For the first time, vector analysis has been used to derive accurate equations describing the Sun's declination and right ascension. This has led to a simple derivation of the geometric equations required to describe the Equation of Time. Also in using the vector approach to derive the equations for the Sun's azimuth, this has allowed far greater insight into the strengths and weaknesses of these equations in determining the azimuth. Additionally, the angle of incidence of the direct solar radiation on a surface has been derived. In the majority of textbooks dealing with these topics, these relationships have predominantly been derived using spherical trigonometry, which is far less accessible to students and practitioners in the field of solar energy. The derivations presented here can be easily understood by anyone with a first-year university background in mathematics or physics, allowing greater understanding of these basic relationships.

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