

0.1 Physics 305 Assignment 10

Due: Monday, November 9 at 10:00 pm

In this week, we will focus on solving sets of linear differential equations with “normal modes”.

Consider a line of N pendulums, each with the same length L and mass m . Assuming that each pendulum swings only in one direction and that the angle of the swinging is small, then we can write the equation of motion of each one as

$$\frac{d^2x}{dt^2} = -\frac{g}{L}x \quad (1)$$

where x is the transverse displacement.

We now connect springs between each pair, possibly with varying spring constants. The pendulums move only in the direction of the springs. We label the pendulums as 0 to $N - 1$ and their displacements as x_0, x_1, \dots, x_{N-1} . The spring between x_j and x_{j+1} has spring constant K_j (so K_0 to K_{N-2} exist). The equations of motion are then

$$\frac{d^2x_0}{dt^2} = -\frac{g}{L}x_0 + \frac{K_0}{m}(x_1 - x_0) \quad (2)$$

$$\frac{d^2x_1}{dt^2} = -\frac{g}{L}x_1 - \frac{K_0}{m}(x_1 - x_0) + \frac{K_1}{m}(x_2 - x_1) \quad (3)$$

$$\frac{d^2x_2}{dt^2} = -\frac{g}{L}x_2 - \frac{K_1}{m}(x_2 - x_1) + \frac{K_2}{m}(x_3 - x_2) \quad (4)$$

$$\dots \quad (5)$$

$$\frac{d^2x_{N-1}}{dt^2} = -\frac{g}{L}x_{N-1} - \frac{K_{N-2}}{m}(x_{N-1} - x_{N-2}) \quad (6)$$

This set of equations has an important property, namely that all terms contain one and only one variable x_j . This means that the equations are linear. If we have a solution $\vec{x}(t)$, where \vec{x} is an N -dimensional vector $(x_0, x_1, \dots, x_{N-1})$, then any two solutions can be added to form another solution and any solution can be multiplied by a constant to give another solution. In other words, the solutions form a linear vector space.

This is an immense simplification. We already knew that the set of differential equations could be solved given N initial positions and N initial velocities. Now we find that this $2N$ -dimensional space of solutions can be treated as linear superpositions of the $2N$ -dimensional space of initial conditions. That means that if we just solve $2N$ independent choices in the initial conditions, we can build the solution for the arbitrary case.

In the case like the above problem, the coefficients in the equations are all independent of time. This is another major simplification, in which we seek solutions of the form

$$x_j(t) = \text{Re} [ae^{i\omega t}y_j] = [(\text{Re } a) \cos(\omega t) - (\text{Im } a) \sin(\omega t)] y_j. \quad (7)$$

where a is a complex number. Note that the y_j are just numbers, not functions of time. Inserting this guess into the above set of equations, we find that the time derivatives bring down a factor of $-\omega^2$ and the $Ae^{i\omega t}$ cancel out. This leaves

$$-\omega^2 y_0 = -\frac{g}{L}y_0 + \frac{K_0}{m}(y_1 - y_0) \quad (8)$$

$$-\omega^2 y_1 = -\frac{g}{L}y_1 - \frac{K_0}{m}(y_1 - y_0) + \frac{K_1}{m}(y_2 - y_1) \quad (9)$$

$$-\omega^2 y_2 = -\frac{g}{L}y_2 - \frac{K_1}{m}(y_2 - y_1) + \frac{K_2}{m}(y_3 - y_2) \quad (10)$$

$$\dots \quad (11)$$

$$-\omega^2 y_{N-1} = -\frac{g}{L}y_{N-1} - \frac{K_{N-2}}{m}(y_{N-1} - y_{N-2}) \quad (12)$$

This is no longer a differential equation, but simply a set of N coupled algebraic equations in N unknowns.

For most values of ω^2 , the only solution is the trivial solution $\vec{y} = 0$. However, there are special values of ω^2 where the solution is non-trivial. This is known as the eigenvalue problem. ω^2 is called the eigenvalue, and the corresponding solution \vec{y} is called the eigenvector. Of course, since this is a linear problem, any constant multiple of \vec{y} is also a solution. Conventionally, we pick \vec{y} to have unit normalization, $|\vec{y}| = 1$.

Without loss of generality, let's pick $m = 1$ and $g/L = \gamma$. Rewriting the above, we have

$$\omega^2 y_0 = (\gamma + K_0)y_0 - K_0 y_1 \quad (13)$$

$$\omega^2 y_1 = (\gamma + K_0 + K_1)y_1 - K_0 y_0 - K_1 y_2 \quad (14)$$

$$\omega^2 y_2 = (\gamma + K_1 + K_2)y_2 - K_1 y_1 - K_2 y_3 \quad (15)$$

$$\dots \quad (16)$$

$$\omega^2 y_{N-2} = (\gamma + K_{N-2})y_{N-1} - K_{N-2}y_{N-2} \quad (17)$$

In matrix notation, we have

$$\omega^2 \vec{y} = \mathbf{A} \vec{y} \quad (18)$$

where the N -dimensional matrix \mathbf{A} is

$$\mathbf{A} = \begin{pmatrix} \gamma + K_0 & -K_0 & 0 & 0 & \dots & 0 & 0 \\ -K_0 & \gamma + K_0 + K_1 & -K_1 & 0 & \dots & 0 & 0 \\ 0 & -K_1 & \gamma + K_1 + K_2 & -K_2 & \dots & 0 & 0 \\ 0 & 0 & -K_2 & \gamma + K_2 + K_3 & \dots & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \dots & \gamma_K + K_{N-3} + K_{N-2} & -K_{N-2} \\ 0 & 0 & 0 & 0 & \dots & -K_{N-2} & \gamma_K + K_{N-2} \end{pmatrix} \quad (19)$$

You may have seen equation 18 with ω^2 replaced by λ .

To summarize, we are looking for vectors \vec{y} that when multiplied by A return a new vector that is a constant multiple of the original \vec{y} . An N -dimensional matrix can have up to N of these eigenvectors, each with a different eigenvalue, but it is not guaranteed to. Finding the eigenvalues and eigenvectors of a matrix is a non-trivial computational problem and there are many different methods.

We will focus on a special case, namely that the matrix A is symmetric and real-valued. You'll see that the above matrix does have this property. In this case, the eigenvalues and eigenvectors are real valued. We will find the eigenvalues using a code from Numerical Recipes, second edition. This is provided at `/home/doug/phys305/linalg305b.c`.

Returning to the physics problem, we have now found that there are N vectors that solve equation 18 and hence produce solutions of the form in 7. These solutions are called the “normal modes”. Each normal mode has a simple sinusoidal time dependence with its frequency ω . Since each mode has a complex number a that gives its amplitude, each mode actually has two pieces of initial data, so these N modes fully span the $2N$ -dimensional space of solutions. In other words, any solution of our pendulum problem can be decomposed into a linear superposition of our normal modes, each of which has a simple sinusoidal time dependence. If we label the modes by k , then we have

$$x_j = \sum_k Q_{jk} \operatorname{Re} [a_k e^{i\omega_k t}] \quad (20)$$

where the matrix \mathbf{Q} has the eigenvectors \vec{y} as the columns of the matrix.

In this week's assignment, you will construct the normal modes of the coupled pendulum problem and study the frequencies and behaviors of the modes as a function of the spring constants.

1) Begin by considering the case with $N = 3$, $\gamma = 0.25$, and all the spring constants $K_j = 1$. Write a program to construct the matrix \mathbf{A} , find its eigenvalues and eigenvectors, and print them out.

As stated in the notes, we're using a provided code. You will make matrices and vectors with the usual C arrays:

```
#define NDIM 3
double A[NDIM][NDIM], eigenvectors[NDIM][NDIM], eigenvalues[NDIM];
```

Then you find the eigenvalues with

```
eigensym(A, eigenvectors, eigenvalues, NDIM);
```

You will need the prototype

```
void eigensym(double [][] , double [][] , double [] , int n);
```

The function will turn `A` into garbage, so if you want to check that your input matrix was correct, print it before you call `eigensym`. The eigenvalues will be sorted from high to low, so `eigenvalues[0]` is the highest frequency mode and `eigenvalues[NDIM-1]` is the lowest frequency mode. The eigenvectors will be in the corresponding **columns** of the matrix, so that the eigenvector for eigenvalue i is in `eigenvectors[0..NDIM-1][i]`.

Take the square root of the eigenvalues to find the frequencies.

Print out the matrix A and the eigenvectors and frequencies. I recommend that you write simple utility functions to print out vectors and matrices.

Note that to pass your matrices to your own functions, you will need to include the dimension of the matrix in the function argument list. This is why you probably will want to use a global definition like `#define NDIM 3`. Then you have:

```
void myfunction(double mymatrix[NDIM][NDIM], int size);
```

to be called as `myfunction(A, NDIM)`; Including `NDIM` in the argument list explicitly is just a good practice, but in principle you could just code constant `NDIM` in your function instead of the variable `size`. Be sure to structure your code so that `NDIM` can be easily varied; we'll need that below.

2) Now animate the normal modes with Philsplot. Your vertical axis should be the mode number $1, 2, \dots, N$. Your horizontal axis should be the mode displacements y_j , offset by j , i.e., plot $j + b(t) * y_j$ where $b(t) = \cos(\omega t)$. Note that because this is a linear problem, the overall normalization of the y_j is arbitrary, but this scaling makes a nice graph. Your product should show a 3x3 set of dots that are moving horizontally, with each row representing the behavior of a single mode.

To be clear, you are not numerically solving the original differential equation, but rather plotting the derived time dependence of the normal modes.

In your report, discuss the behavior of the nodes. Describe the pattern of displacements for each mode. How does the pattern of displacements correlate with the frequency? Physically, this frequency corresponds to more energy for a given amplitude. Can you see why this is the case?

For the TA's benefit, you should set up the Philsplot animation so that the interesting behavior can be seen in about 15 seconds. That is, don't make the animation run very slowly!

3) Now change N to be 6 and $\gamma = 0.01$ and remake your animation. Discuss the pattern of displacements as a function of frequency.

If we took $N \rightarrow \infty$, we would imagine that this problem would turn into the propagation of compressional waves along a bar of fixed length. The modes would be the fundamental and harmonics thereof, in which each next higher mode has one additional node (point of zero displacement) in the bar. Interpret your $N = 6$ results in this light.

4) Next, consider a case in which the spring constants are not all the same. For $N = 6$ and $\gamma = 0.01$, leave $K_1 = K_3 = 1$ but make $K_0 = K_2 = K_4 = 1 + r$. Animate the modes for $r = 1$, $r = 10$, and $r = 100$. What happens to the modes and frequencies as r increases?

Compare the $r = 100$ case to the $N = 3$ case with $K = 1$ and $\gamma = 0.01$. What is happening to the low frequency modes as three of the springs become very forceful? Compare the lowest three frequencies in the two cases; explain physically why they differ (hint: remember that frequency in this problem scales as the inverse square root of mass).

This is a good example of how a sharp contrast between the strength of physical elements can give rise to a separation in the characteristic frequencies and energies of the modes. At low energies we probably wouldn't notice the high-frequency modes. These kinds of hierarchies are common in physics and underlie many of our approximations.