Phys 321, Theoretical Mechanics I, Spring 2006
Homework #3
Due Date: Friday March 10, 10am

The aim of this third homework set is to help you get comfortable with the concepts of Lagrangian and Hamiltonian Dynamics.

1. A pendulum consists of a rod of mass $m_{\rm r}$ and a ball of mass $m_{\rm b}$ attached to the rod's end. Find the frequency of small oscillations of the pendulum, when it is placed in a uniform vertical gravitational field $\vec{g}$.

2. Consider a simple plane pendulum consisting of a mass m attached to a string of length l. After the pendulum is set into motion, the length of the string is shortened at a constant rate

$$\frac{dl}{dt} = -a = constant \; . \tag{1}$$

   The suspension point remains fixed. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.

3. **Poisson Brackets:** Consider any two functions of the generalized coordinates and momenta $g(q_k, p_k)$ and $h(q_k, p_k)$. The Poisson brackets are defined by

$$[g, h] \equiv \sum_k \left(\frac{\partial g}{\partial q_k}\frac{\partial h}{\partial p_k} - \frac{\partial g}{\partial p_k}\frac{\partial h}{\partial q_k} \right) \; . \tag{2}$$

   Show the following identities:

$$\frac{dg}{dt} = [g, H] + \frac{\partial g}{\partial t} \tag{3}$$

$$\dot{q}_i = [q_i, H] \tag{4}$$

$$\dot{p}_i = [p_i, H] \tag{5}$$

$$[p_i, p_j] = [q_i, q_j] = 0 \tag{6}$$

$$[q_i, p_j] = \delta_{ij} \tag{7}$$

   where H is the Hamiltonian of the system. Argue that any quantity that does not depend explicitly on time but commutes with the Hamiltonian (i.e., when $[g, H] = 0$) is a constant of motion for the system.

4. [**For Honor Students Only:**] The ideas and formalism of Lagrange dynamics can be applied not to only to the motions of particles but also to the fundamental properties of forces, as well. We can obtain the (field) equation that describes a conservative force with a potential Φ by demanding that the potential extremizes the Lagrangian action

$$L = -\frac{1}{2}\left(\nabla\Phi\right)^2 - \frac{1}{2}m^2\Phi^2 \; , \tag{8}$$

   where m is the mass of the "particle" that mediates the force. Show that the Newtonian gravitational force can indeed be described by such a Lagrangian with the mass of the particle that mediates the force (the graviton) set equal to zero. *Hint:* You may solve this problem in one dimensions.