Physics 331 Class Notes for Week 11
Lecture 1
TOPICS

- Multipole expansion of the magnetic vector potential.
- Magnetization.
- Sample Problems.

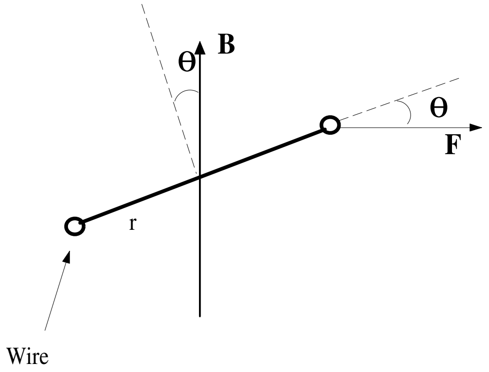
Multipole expansion.

- We can expand the vector potential in a similar manner as the scalar potential.
  - Start with a loop of wire carrying a current I.
  - $\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{1}{\mathscr{r}} d\mathbf{l}'$
    - Here we're only interested in the current in a wire.
  - For the scalar potential, we saw that
    - $\frac{1}{\mathscr{r}} = \frac{1}{r} \sum_{n=0}^{\infty} (\frac{r'}{r})^n P_n(\cos\theta')$
  - Expand about $\frac{1}{r}\frac{1}{\sqrt{1+r'/r}}$.
    - The vector potential can be written as:
    - $\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \left[\frac{1}{r} \oint d\mathbf{l}' + \frac{1}{r^2} \oint r' \cos\theta' d\mathbf{l}' + ... \right]$
  - Note, that each term goes as $(\frac{1}{r})^n$.
  - The first term is the monopole term.
    - The integral over the loop is always zero.
  - The lowest order term is the dipole term.
    - $\mathbf{A}_{dip}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\theta' d\mathbf{l}'$.
  - Note that:
    - $r' \cos\theta = \hat{\mathbf{r}} \cdot \mathbf{r}'$
    - $A_{dip} = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{\mathbf{r}} \cdot \mathbf{r}') d\mathbf{l}'$
  - We can show that $\oint (\hat{\mathbf{r}} \cdot \mathbf{r}') d\mathbf{l}' = -\hat{\mathbf{r}} \times \int d\mathbf{a}'$.
    - Let $\mathbf{c}$ be a constant vector and T a scalar function.
    - First let's show that $\int \nabla T \times d\mathbf{a} = -\int T d\mathbf{l}$
    - Stoke's Thm: $\int \nabla \times (\mathbf{c}T) \cdot d\mathbf{a} = \int \mathbf{c}T \cdot d\mathbf{l}$.
      - $\nabla \times \mathbf{c}T = T\nabla \times \mathbf{c} - \mathbf{c} \times \nabla T$
      - Since $\mathbf{c}$ is a constant, $T\nabla \times \mathbf{c} = 0$
      - $\int -(\mathbf{c} \times \nabla T) \cdot d\mathbf{a} = \int T\mathbf{c} \cdot d\mathbf{l}$.
      - $\int (\mathbf{c} \times \nabla T) \cdot d\mathbf{a} = -\mathbf{c} \cdot \int T d\mathbf{l}$
      - Use triple product identity.
      - $\int \mathbf{c} \cdot \nabla T \times d\mathbf{a} = -\mathbf{c} \cdot \int T d\mathbf{l}$

            - $\int \nabla T \times d\mathbf{a} = -\int T d\mathbf{l}$
        - Let $T = \mathbf{c} \cdot \mathbf{r}$.
            - $\int \nabla(\mathbf{c} \cdot \mathbf{r}) \times d\mathbf{a} = -\int (\mathbf{c} \cdot \mathbf{r}) d\mathbf{l}$
            - $\int [\mathbf{c} \times (\nabla \times \mathbf{r}) + (\mathbf{c} \cdot \nabla)\mathbf{r}] \times d\mathbf{a} = -\int \mathbf{c} \cdot \mathbf{r} d\mathbf{l}$
            - Note: $(\mathbf{c} \cdot \nabla)\mathbf{r} = (c_x \frac{\partial}{\partial x} + c_y \frac{\partial}{\partial y} + c_z \frac{\partial}{\partial z})(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) = \mathbf{c}$
            - And, $\nabla \times \mathbf{r} = 0$.
            - $\int \mathbf{c} \times d\mathbf{a} = -\int \mathbf{c} \cdot \mathbf{r} d\mathbf{l}$
        - Let $\mathbf{c} = \hat{\mathbf{r}}$.
            - Then, $\oint (\hat{\mathbf{r}} \cdot \mathbf{r}') d\mathbf{l}' = -\hat{\mathbf{r}} \times \int d\mathbf{a}'$
    - $A_{dip}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi r^2} (\hat{\mathbf{r}} \times \int d\mathbf{a}')$
    - $A_{dip}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$.
        - The magnetic dipole moment is defined as: $\mathbf{m} = I\mathbf{a}$
        - $\mathbf{a}$ is the area of the loop perpendicular to the current.

Magnetization.

- In materials, the effects of the spin of the electrons and their orbits around the the nucleus, creates effective magnetic dipole moments.
    - Such magnetic dipoles can be affected by the presence of magnetic fields.
    - Unlike electric dipoles, there are three different behaviors.
        - paramagnetism: dipoles align parallel to the applied field.
        - diamagnetism: dipoles align anti-parallel to the applied field.
        - ferromagnetism: dipoles can become permanently aligned.
- Torque on a loop of wire.



    - Recall that the force on a wire is $\mathbf{F} = I\mathbf{l} \times \mathbf{B}$.
        - $l \perp \mathbf{B}$.
        - Let the length of the wire $\perp$ to page be a.
    - If a loop of wire is tilted in a magnetic field, the total force is zero.

  - The torque ($\tau = \mathbf{r} \times \mathbf{F}$) is:
    - $\tau = (Ia\frac{b}{2}\sin\theta + Ia\frac{b}{2}\sin\theta)B$.
    - $\tau = \mathbf{m} \times \mathbf{B}$.
  - The orientation of the dipole with the magnetic field must be associated with an energy.
    - You can prove to yourself that $U = -\mathbf{m} \cdot \mathbf{B}$.
    - i.e. when the dipole is aligned with the magnetic field, the energy is a minimum.
  - Although a dipole in a uniform magnetic field feels no force, a non-uniform field can produce a force.
    - i.e. the Stern-Gerlach experiment showed that a neutral beam of atoms can be split by a non-uniform magnetic field. (i.e. gradient).
    - The dipoles due to the electron spins responded to the gradient of the magnetic field and experienced a force.
    - The electron spins were either aligned parallel or anti-parallel to the field and therefore there were two distinct spots.
    - $\mathbf{F} = -\nabla U$.
    - $\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$

- Effect of a magnetic field on an atom.
  - We shall now show that the magnetic field affects the orbital speed of an atom.
  - This effect is responsible for diamagnetism.
    - i.e. it is anti-parallel to the applied field.
  - The change in the centripetal force is equal to the force due to the magnetic field.
    - $qvB = \frac{mv^2}{R} - \frac{mv_0^2}{R}$
    - $qvB = \frac{m}{R}(v + v_0)(v - v_0)$
  - In the case that the change in v is small,
    - $v + v_0 = 2v$
    - $v - v_0 = \Delta v$.
  - $\Delta v = \frac{qRB}{2m}$
    - So, v changes when the magnetic field is present.
  - Recall that $m = Ia$.
    - Assuming a circular orbit, $a = \pi R^2$.
    - $I = \frac{qv}{2\pi R}$.
    - $|\mathbf{m}| = Ia$
      - $|\mathbf{m}| = \frac{qv}{2\pi R}(\pi R^2)$
      - $|\mathbf{m}| = \frac{qvR}{2}$
      - For an electron $|\mathbf{m}| = \frac{-evR}{2}$
  - A change in the speed, can be related to a change in the magnetic dipole:

    - $\Delta\mathbf{m} = e\frac{e\Delta vR}{2} = -\frac{eR}{2}(\frac{eR\mathbf{B}}{2m})$
    - $\Delta\mathbf{m} = \frac{-e^2R^2}{4m}\mathbf{B}$.
  - An atom in a magnetic field will produce a dipole moment opposite to the applied magnetic field.
    - This is the source of diamagnetism.
    - It is generally weaker than paramagnetism.
      - Paramagnetism appears in atoms with odd numbers of electrons.
      - Diamagnetism appears in atoms with even numbers of electrons.
      - Why?

Sample Problem

- Calculate the torque on a square loop of wire of side b located a distance r from a circular loop of wire of radius a. The planes of the two loops are perpendicular to each other, and r is much greater than a and b. What will the final orientation be if the square loop is free to rotate? Each loop has a current I.