Fast Fourier Transform (FFT)

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■ Goals

To

• understand the Fourier transform (FT)
• use the FFT computer implementation to measure the speed of sound in air.

■ Introduction

The Fourier transform (FT) is a nonlinear coordinate transformation that changes from space/time to wavenumber/frequency and vica verse.

Related to the FT is the Fourier series (FS) expansion, which is basically a FT with periodic boundary conditions. In this lab we will deal with the FS because we have a finite amount of discrete data.

The Fourier series (FS) expansion of a function f[x] with periodicity *L* is defined by

$$f[x] = \sum_{n=0}^{\infty} \left(a_n \operatorname{Cos}\left[\frac{n\,2\,\pi\,x}{L}\right] + b_n \operatorname{Sin}\left[\frac{n\,2\,\pi\,x}{L}\right] \right)$$

$$a_n = \frac{2}{L} \sum \int_0^L f[x] \operatorname{Cos}\left[\frac{n\,2\,\pi\,x}{L}\right] dx \qquad b_n = \frac{2}{L} \sum \int_0^L f[x] \operatorname{Sin}\left[\frac{n\,2\,\pi\,x}{L}\right] dx \quad .$$

The decomposition can also be written using complex exponentials: $e^{i\,k\,x} = \operatorname{Cos}[k\,x] + i \operatorname{Sin}[k\,x]$, which can be shown by Taylor expansion. When one changes to integral form, this transform is then called the Fourier transform/integral. It is defined by

$$f[x] = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{+\infty} A[k]\, e^{+i\,k\,x}\, dk$$

$$A[k] = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{+\infty} f[x]\, e^{-i\,k\,x}\, dx \quad .$$

The FT is different from the FS in that for the FS the position coordinate is modular L, while there is no such (built-in) periodicity in the case of the FT.
The preceding expression are written for a "space-scale" transformation. It is a simple matter to redefine the meanings of x and k to t (time) and ω (angular frequency) for the interpretation of a "time-frequency" transformation.

A computer is equipped with an analog to digital converter (ADC) card that allows it to read external voltages and record the values as numbers in computer memory. The card has an internal clock that allows it to sample input voltages at a specified rate up to 40,000 samples per second. By sampling the signal fast enough, the computer can plot the data and perform a FFT accurately (see Question 1.).

■ Procedure

■ A: Aliasing

You will investigate the effect of sampling a periodic signal (frequency f) with different sampling frequencies f_s . Intuitively, one can understand that in order to record a signal one needs to sample it at least once per period $1/f$. However, this is not often enough and the goal is to experimentally find the minimum f_s needed to still obtain the correct FT.

• Connect one function generator to the BNC connector on the breadbord and to the oscilloscope.
• Use the function generator to produce a sine wave of frequency f = 500 Hz and amplitude < 2.5 V.

• Turn on printer and computer (as password press Enter)
• Acquire data: double click on 'Signal' program
use a sample rate of 10,000 samples per second
do not use a filter for now
acquire the data
Data is now located in C:\fft\sig

• Plot data: double click on Wgnuplot
in the wgnuplot screen type:
cd 'c:\fft
plot 'sig
• Rescale the axes: use Alt-Tab to get back to the wgnuplot control screen
from the wgnuplot menu choose
Axes/Autoscale xy
display about 5 periods of the signal using
Axes/X Range
• Connect the points: from the wguplot menu choose
Styles/Data Styles/ Lines + Points
• Plot the FFT: in the wgnuplot control screen:
plot 'xform
you will need to (again) rescale the x-axis

• Notice that for every menu choice that you select, a command is placed on the screen and executed. It may be faster to type the command or use the history (see Additional hints).

Always print out all relevant graphs to illustrate your progress. Label the axes.
• To print a plot: type
screendump

• Determine the frequency of the signal and compare to the oscilloscope. Now, determine the minimum sample frequency needed to produce an accurate FFT of the *constant* input signal.

Find an explanation for the effects you observe. Possible keywords are: Fourier transform, sampling theorem, Nyquist frequency, aliasing,etc. A list of references is in the section 'Additional hints'.

■ B: Filtering

Next, you will filter frequency components of input signals. Mathematically, this is done by applying a ("filter") function to the FT and then reconstructing the signal by an inverse FT. On the computer, this is done by choosing a filter in the "Signal" program before you acquire data.

• Connect the additional function generator and set it to a sine wave of different frequency and amplitude.
• Choose from one of the pre-programmed filters and set the parameters carefully.
  - Step
  - Notch
  - Gaussian
  - Treshold
• Plot - the FT of the input data and the filter
  plot 'xform', 'filter'
  - the filtered data
  plot 'xfilter
  - the reconstructed signal
  plot 'sigfilt
• Try all filters and explain their function.

■ C: Square wave decomposition

So far all input signal were composed of pure sine waves. Therefore they produced each one sharp peak in frequency space. You will now decompose a square wave into its individual frequency components. By applying a filter and increasing its width, you will then reconstruct the square wave from the individual frequency components, one by one. Well, not exactly, because that would take an infinite amount of time.

• Disconnect one of the function generators.
• Use the other to produce a 500 Hz square wave, amplitude < 2.5 V, and plot its FT.
• Filter out all but the lowest frequency mode.
• By allowing more frequencies to pass through you should be able to see the progress from a sine wave to a square wave.
• Document this progress in your notebook.

Now for the fun part, calculate the amplitude of the Fourier coefficients for a square wave. See any of the texts suggested under "Additional hints". Compare your experimentally determined FT to theory.

■ D: Speed of sound in air

Lastly, you will determine the speed of sound in air by measuring the frequency of standing waves in an open ended pipe. There are several pipes in the room, a long one will work better.
• Connect the microphone (disconnect the function generator) and hold the microphone close to one end of the pipe.
• Gently blow across the other end of the pipe, acquire data, and print FT.
• Determine the speed of sound in air.

Hints to derive the working equation
• Use (refresh) your knowledge of standing waves to relate the frequency to the mode number and length of pipe. Plot the appropriate variables to determine the speed of sound from the slope of a linear least squares analysis.

• In addition, consider the end-effects of an open ended pipe. Give an argument why the effective length of the pipe is decreased at each open end by about a pipe radius.

■ Questions

1) What is a Fast Fourier transform (FFT)? What makes it fast? How much faster compared to what?
Information on FFT can be found in
William H. Press, et al, *Numerical Recipes in C : The Art of Scientific Computing*

■ Additional hints

• Introductions to FS and discrete and continuous FT can be found in:
Boas, *Mathematical Methods in the Physical Sciences*
Pedrotti and Pedrotti, *Introduction to Optics*
Robinett, *Quantum Mechanics*
Hecht, *Optics*
etc. etc. etc.

• If you want to save data, you can copy the following files from c:\fft\ to your disk

File	**Description**
sig	signal data
xform	Fourier transform of data
filter	filter data
xfilter	filtered Fourier transform data
sigfilt	filtered signal data

• Useful Wgnuplot stuff:
use ↑ key to get history of input commands
• Useful Windows stuff:
Alt-Tab to switch between "Signal" program and back