Physics 511

Problem Set 2
Due September 22, 2006

Problem 1. Show that the geodesics of a spherical surface are great circles, i.e., circles whose centers lie at the center of the sphere.

Problem 2. Suppose it is known experimentally that a particle fell a given distance y_0 in a time $t_0 = \sqrt{2y_0/g}$. The times of fall for distances other than y_0 are not known. Suppose further that the Lagrangian for the problem is known, but that instead of solving the equation of motion for y as a function of t, it is guessed that the functional form is

$$y = at + bt^2.$$

If the constants a and b are adjusted always so that the time of fall y_0 is correctly given by t_0, show directly that the integral

$$\int_0^{t_0} L\, dt$$

is an extremum for real values of the coefficients only when $a = 0$ and $b = g/2$.

Problem 3. A point mass is constrained to move on a massless hoop of radius a fixed in a vertical plane that rotates about its vertical symmetry axis with constant angular speed ω. Obtain the Lagrange equations of motion assuming the only external forces arise from gravity. What are the constants of motion? Show that if ω is greater than a critical value ω_0, there can be a solution in which the particle remains stationary on the hoop at a point other than at the bottom, but that if $\omega < \omega_0$, the only stationary point for the particle is at the bottom of the hoop. What is the value of ω_0?

Problem 4. A particle of mass m slides without friction on a wedge of angle α and mass M that can move without friction on a smooth horizontal surface. Treating the constraint of the particle on the wedge by the method of Lagrange multipliers find the equations of motion for the particle and the wedge. Also obtain an expression for the forces of constraint. Calculate the work done in time t by the forces of constraint acting on the particle and on the wedge. What are the constants of motion for the system? Contrast the results you have found with the situation when the wedge is fixed. [Suggestion: For the particle you may either use a Cartesian coordinate system with y vertical, or one with y normal to the wedge or, even more instructively, do it in both systems.]

Problem 5. The one-dimensional harmonic oscillator has the Lagrangian $L = m\dot{x}^2/2 - kx^2/2$. Suppose you did not know the solution to the motion, but realized that the motion must be periodic and therefore could be described by a Fourier series of the form

$$x(t) = \sum_{j=0} a_j \cos j\omega t,$$

(taking $t = 0$ at a turning point) where ω is the (unknown) angular frequency of the motion. This representation of $x(t)$ defines a many-parameter path for the system point in configuration space. Consider the action integral I for two points, t_1 and t_2 separated by the period $T = 2\pi/\omega$. Show that with this form for the system path, I is an extremum for nonvanishing x only if $a_j = 0$, for $j \neq 1$, and only if $\omega^2 = k/m$.

Problem 6. A disk of radius R rolls without slipping inside the stationary parabola $y = ax^2$. Find the equations of constraint. What condition allows the disk to roll so that it touches the parabola at one and only one point independent of its position.

Problem 7.

(a) Show that if a particle describes a circular orbit under the influence of an attractive central force directed toward a point on the circle, then the force varies as the inverse-fifth power of the distance.

(b) Show that for the orbit described the total energy of the particle is zero.

(c) Find the period of the motion.

(d) Find $\dot{x}$, $\dot{y}$, and v as a function of angle around the circle and show that all three quantities are infinite as the particle goes through the center of force.

Problem 8.

(a) For circular and parabolic orbits in an attractive $1/r$ potential having the same angular momentum, show that the perihelion distance of the parabola is one-half the radius of the circle.

(b) Prove that in the same central force as in part (a) the speed of a particle at any point in a parabolic orbit is $\sqrt{2}$ times the speed in a circular orbit passing through the same point.

Problem 9. A particle of mass m is constrained to move under gravity without friction on the inside of a paraboloid of revolution whose axis is vertical. Find the one-dimensional problem equivalent to its motion. What is the condition on the particle's initial velocity to produce circular motion? Find the period of small oscillations about this circular motion.