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CHAPTER ONE

Conformal Mappings and Harmonic Functions

1. Conformal Mappings

In this section we discuss the geometrical properties of analytic functions. First we calculate the gradient of a smooth path in the complex plane.

If $z_0 \neq z_1$, then $\theta = \arg(z_1 - z_0)$ $(-\pi < \theta \leqslant \pi)$ is the angle between the real axis and the directed line from z_0 to z_1. Suppose that $z(t) = x(t) + iy(t)$ $(\alpha \leqslant t \leqslant \beta)$ is a smooth path† and $z_0 = z(t_0)$, $z_1 = z(t)$ are two distinct points on its track, then $\theta = \arg(z(t) - z(t_0))$ is the angle between the real axis and the directed chord from $z(t_0)$ to $z(t)$.

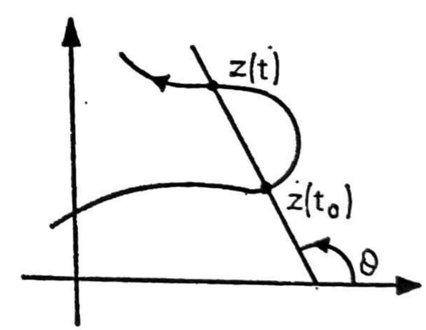

Figure 1

Now if c is a positive real number, then $\arg cz = \arg z$. If we assume that $t > t_0$ then $\dfrac{1}{t - t_0} > 0$ and so

$$\theta = \arg(z(t) - z(t_0)) = \arg\left\{\frac{z(t) - z(t_0)}{t - t_0}\right\}.$$

† i.e. $z'(t) = x'(t) + iy'(t)$ exists and is continuous for $\alpha \leqslant t \leqslant \beta$.

Let $t \to t_0$, then the chord tends to the tangent at t_0 directed in the sense t increasing. Also $\dfrac{z(t)-z(t_0)}{t-t_0} \to z'(t_0)$. From this we may infer that the angle between the real axis and the directed tangent is $\arg z'(t_0)$, provided that $z'(t_0) \neq 0$.

The case $z'(t_0) = 0$ is omitted because arg 0 is not well-defined. The proof in other cases is not trivial because $\arg z$ denotes the *principal* value $-\pi < \arg z \leqslant \pi$, and arg is not continuous on the negative real axis. Let $w = \dfrac{z(t)-z(t_0)}{t-t_0}$, $w_0 = z'(t_0)$. Since arg is continuous in the cut-plane†, when $-\pi < \arg w_0 < \pi$ we have $w \to w_0$ implies $\arg w \to \arg w_0$. Thus $\theta \to \arg z'(t_0)$. However if $\arg w_0 = \pi$, i.e. if w_0 is on the negative real axis, then although $\arg w_0 = \pi$, a point near w_0 but below the real axis has $\arg w$ nearly $-\pi$. If w tends to w_0 from below the real axis then $\arg w_0 \to -\pi$. Worse still, if w tends to w_0 in a spiral path, going round and round and getting ever closer to w_0 then $\arg w$ jumps from nearly $-\pi$ to π and back again ad infinitum so that $\arg w$ does not tend to a limit. Thus it is blatantly untrue to say that $w \to w_0$ implies $\arg w \to \arg w_0$ in the case of the principal value. If $\arg w_0 = \pi$, we choose the value of $\arg w$ in the range $0 < \arg w \leqslant 2\pi$. This value is continuous near w_0 and as $w \to w_0$, we have $\arg w \to \pi$, as required.

Now suppose f is an analytic function defined on a domain D. Let γ be a smooth path in D given by $z(t) = x(t)+iy(t)$ $(\alpha \leqslant t \leqslant \beta)$, then f transforms γ into a smooth path Γ given by $w(t) = f(z(t))$ $(\alpha \leqslant t \leqslant \beta)$. Suppose that z_0 is a point in D where $f'(z_0) \neq 0$ and z_0 lies on the track of γ, i.e. $z_0 = z(t_0)$ for some t_0.

We compare the directions of the tangent to γ at z_0 and the

† *Functions of a Complex Variable I*, p. 18.

tangent to Γ at $w_0 = f(z_0)$. Let $\phi = \arg z'(t_0)$, $\psi \equiv \arg w'(t_0)$. Since

$$w'(t_0) = f'(z(t_0))z'(t_0)$$

we have $\arg w'(t_0) = \arg f'(z(t_0)) + \arg z'(t_0)$ up to a multiple of 2π and so $\psi = \arg f'(z_0)+\phi$ up to a multiple of 2π.

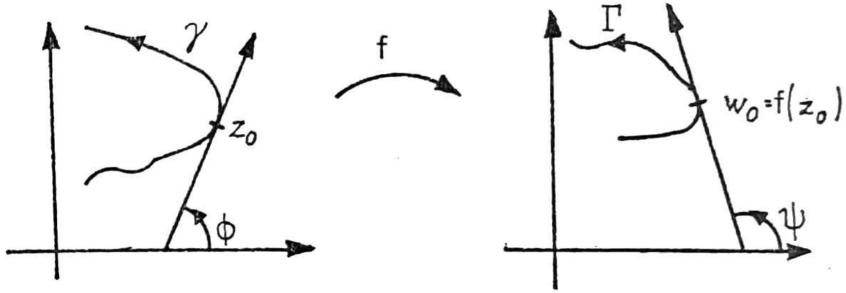

Figure 2

Hence the tangent to γ at z_0 is turned through an angle $\arg f'(z_0)$ upon transformation under f. This does not depend on the path γ and so if γ_1, γ_2 are two paths through z_0, then the transformed paths meet at the same angle† as γ_1, γ_2. (In each case the tangent is turned through the same angle $\arg f'(z_0)$, up to a multiple of 2π, upon transformation.)

A transformation preserving angles between curves is said to be *conformal*. An analytic function is conformal where $f'(z) \neq 0$. (It is certainly not conformal where $f'(z) = 0$. If z_0 is a zero of order m of f', then the angle between curves through z_0 is multiplied by $m+1$ upon transformation. The proof is omitted.)

We can find more information about analytic functions by considering the equation

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0).$$

† The angle between two paths through z_0 is the angle between their tangents (considered up to a multiple of 2π).

This implies
$$\lim_{z\to z_0}\left|\frac{f(z)-f(z_0)}{z-z_0}\right| = |f'(z_0)|$$
and so for z near z_0, we have
$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| \simeq |f'(z_0)|$$
$$\text{i.e. } |f(z)-f(z_0)| \simeq |f'(z_0)|\,|z-z_0|.$$

This says that f magnifies lengths by approximately $|f'(z_0)|$ near z_0.

Taking z_0, z_1, z_2 'close together', where $f'(z_0)\neq 0$, then conformality and the magnification property state that the small triangle with vertices z_0, z_1, z_2 is transformed into a similar triangle, with sidelengths multiplied approximately by $|f'(z_0)|$ and turned through an angle $\arg f'(z_0)$. The smaller the triangle, the better the approximation.

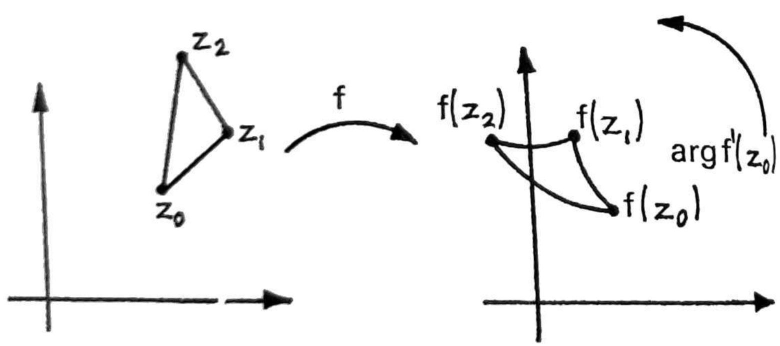

Figure 3

As an example of a conformal mapping†, we consider $f(z) = \dfrac{az+b}{cz+d}$ $(ad\neq bc)$ which is defined for all z if $c = 0$ and

† 'Mapping' is just another word for 'function'.

for all z except $z = -d/c$ otherwise. This is called a *bilinear mapping*. Note that $f'(z) = \dfrac{ad-bc}{(cz+d)^2}$ and so the condition $ad\neq bc$ ensures that $f'(z)\neq 0$ wherever f is defined and so f is conformal.

As particular cases we note:

EXAMPLE 1. A *translation* $w = z+\alpha$. Points in the w-plane correspond to those in the z-plane with a change in origin. Figures remain the same shape and size when transformed.

EXAMPLE 2. A *rotation* $w = e^{i\phi}z$ where ϕ is real. Since $\arg w = \arg z+\phi$ (up to a multiple of 2π) and $|w| = |z|$, we see that figures are rotated through an angle ϕ about the origin but lengths remain unchanged.

EXAMPLE 3. A *magnification* $w = rz$ where r is real and positive. A figure remains similar and similarly situated when transformed, but lengths are multiplied by a factor r.

EXAMPLE 4. An *inversion* $w = 1/z$. If $z = re^{i\theta}$ then $w = \dfrac{1}{r}e^{-i\theta}$ and so $|w| = 1/|z|$, $\arg w = -\arg z$. Unlike the previous examples, this may change the shape of figures. For example a circle may be transformed either into a circle or into a straight line. However, by considering a line to be a 'circle of infinite radius'†, it may be shown that an inversion transforms 'circles' into 'circles'. Other curves may have their shape altered, but because of the conformal property, the angle between two paths remains unaltered (provided that their intersection is not the origin, where the transformation is not defined).

† See Exercise 4 at the end of this chapter.

The reader is encouraged to draw pictures for the above examples to help visualize them.

It is a remarkable fact that a general bilinear mapping may be expressed as a succession of the particular types described above. For $c \neq 0$, we write

$$\frac{az+b}{cz+d} = \frac{bc-ad}{c^2(z+(d/c))} + \frac{a}{c}.$$

Let $\frac{bc-ad}{c^2} = \lambda$, then $\lambda \neq 0$. We write $w_1 = z+(d/c)$, $w_2 = 1/w_1$, $w_3 = |\lambda| w_2$, $w_4 = (\lambda/|\lambda|)w_3$, $w = w_4 + (a/c)$. By successive substitution we find that w is obtained from z by a translation, then an inversion, a magnification, a rotation and another translation.

The case $c = 0$ is somewhat easier. We have $w = \frac{az+b}{d} = \alpha z + \beta$ where $\alpha = a/d$, $\beta = b/d$. Thus if $w_1 = |\alpha| z$, $w_2 = (\alpha/|\alpha|)w_1$, $w = w_2 + \beta$, we see that w is obtained from z by a magnification, a rotation and a translation.

Of the particular examples considered, only an inversion changes the shape of a figure and even this takes 'circles' into 'circles'. Thus a general bilinear mapping transforms 'circles' into 'circles'.

Bilinear mappings have many other interesting properties. The reader should consult the literature on the subject.†

2. Orthogonal Curves

As we have seen in the last section, the angle between two smooth paths is preserved under transformation by an analytic function where that function has non-zero derivative. The most important case occurs when the paths are orthogonal

† L. V. Ahlfors, *Complex Analysis*, McGraw Hill Book Co., pp. 76–88.

(i.e. intersect at right angles). If γ_1 is a line parallel to the x-axis and γ_2 is parallel to the y-axis, then they are orthogonal and so the transformed curves Γ_1, Γ_2 meet at right angles:

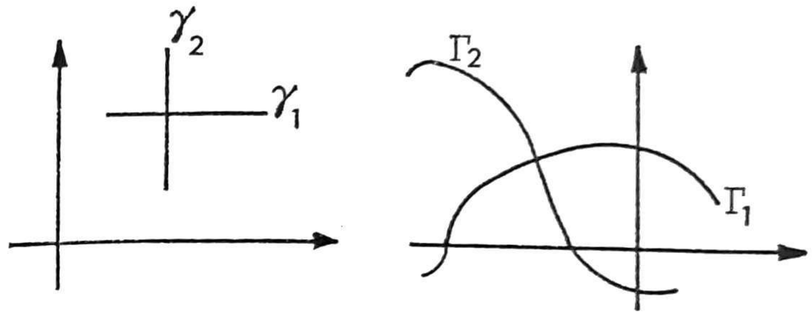

Figure 4

As an example of this phenomenon, consider the function $f(z) = e^z = e^{x+iy}$. Taking polar coordinates in the w-plane, $w = Re^{i\phi}$, then $w = f(z)$ gives $R = e^x$ and $\phi = y$ (up to a multiple of 2π). Thus the line $x =$ constant transforms into $R =$ constant, which is a circle centre the origin, and $y =$ constant transforms into $\phi =$ constant, which is a straight line through the origin. These evidently meet at right angles.

A most useful technique is to write $f(z) = u(x, y) + iv(x, y)$ and consider the curves $u(x, y) = u_0 =$ constant and $v(x, y) = v_0 =$ constant. Suppose that these are smooth paths

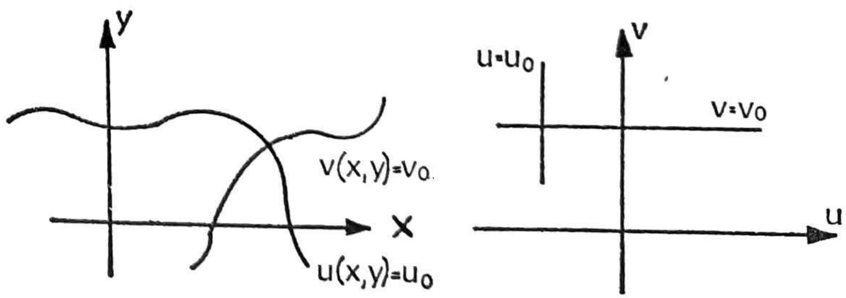

Figure 5

which meet in a point $z_0 = x_0+iy_0$ where $f'(z_0)\neq 0$. If $w = u+iv = f(z)$, then the curve $u(x, y) = u_0$ in the z-plane transforms into $u = u_0$ in the w-plane and $v(x, y) = v_0$ transforms into $v = v_0$. But $u = u_0$, $v = v_0$ are straight lines parallel to the axes in the w-plane and hence meet at right angles.

This means that $u(x, y) = u_0$, $v(x, y) = v_0$ are orthogonal curves.

For different values of u_0, v_0 we obtain two families of curves. Any curve of the first family meets one of the second family at right angles. These curves are called the *level curves* of f.

EXAMPLE. $f(z) = z^2$. Since $f'(z) = 2z$, the transformation is conformal where $f'(z)\neq 0$, i.e. at all points except the origin. We have

$$f(z) = (x+iy)^2 = x^2-y^2+2ixy$$

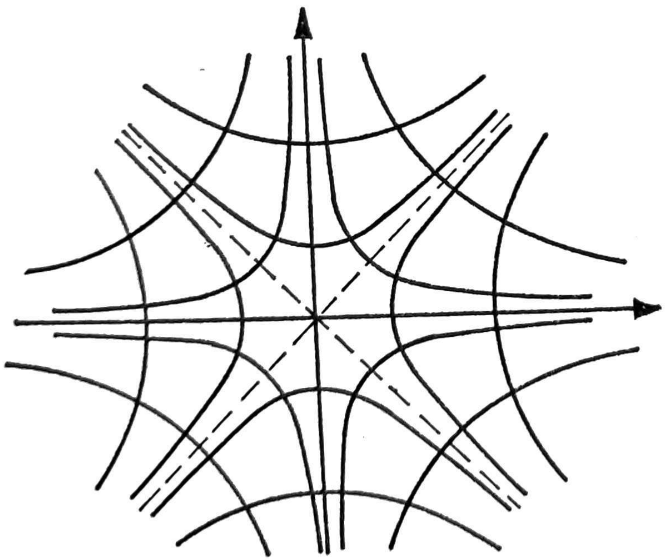

Figure 6

and so $u(x, y) = x^2-y^2$, $v(x, y) = 2xy$. The level curves are $x^2-y^2 = c$, $2xy = k$. For $c\neq 0$, $k\neq 0$, these do not pass through the origin and are hence orthogonal. Using coordinate geometry, for different values of c, the first set of curves are rectangular hyperbolae with asymptotes $x = y$, $x = -y$. Similarly the second set of curves are rectangular hyperbolae with the axes as asymptotes.

3. Harmonic Functions and Potential Theory

Suppose that $\phi(x, y)$ is a real valued function of two real variables x, y. If ϕ satisfies the differential equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = 0 \tag{1}$$

then ϕ is called a *harmonic function* or *potential function*. Equation (1) is called *Laplace's equation*.

Usually ϕ is only defined for those values of x, y where $x+iy$ lies in a domain D. If f is an analytic function defined in D and $f(z) = u(x, y)+iv(x, y)$, then it may be shown that both u and v are harmonic in D.

This follows from the Cauchy-Riemann equations† and Taylor's Theorem.‡ First note that

$$f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}. \tag{2}$$

From Taylor's Theorem, f'' exists throughout D and so f' is also analytic in D. Let $f' = U+iV$, then from the Cauchy-Riemann equations for U, V, the partial derivatives of U, V exist and satisfy:

† *Functions of a Complex Variable I*, p. 23.
‡ *Functions of a Complex Variable I*, p. 55.

B

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \qquad (3)$$

$$\frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}. \qquad (4)$$

Since $U = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $V = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, on substituting in (3) we have

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

and in particular,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0. \qquad (5)$$

Substituting in (4), we also find

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)$$

which gives

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = 0. \qquad (6)$$

This has applications in two-dimensional potential theory. If u is a potential function, then the curves $u(x, y) =$ constant are 'equipotential lines'. But we have seen in the last section that the curves $v(x, y) =$ constant are orthogonal to these. The curves $v(x, y) =$ constant are 'stream lines'.

Suppose that we are given a potential function u in a domain D. Is it possible to determine the equations of the stream lines from this? Under suitable conditions this problem may be solved by looking for a real-valued function v such that $f = u+iv$ is analytic in D. The function v is called the

harmonic conjugate of u. If a harmonic conjugate exists, then from the Cauchy-Riemann equations for f we have

$$f' = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}. \qquad (7)$$

From this we may attempt to find f. Sometimes the solution is obvious by inspection, otherwise we may use contour integration. The latter method would require restrictions on the nature of the domain D. For example, if D were a star-domain† with star-centre z_0, then we may adopt the method of Volume I, Chapter Two, proposition 4.1 to find

$$f(z_1) = \int_{[z_0,\, z_1]} f'(z)dz$$

$$= \int_{[z_0,\, z_1]} \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)dz$$

where $[z_0, z_1]$ is the straight line from z_0 to z_1.

Note that a solution of (7) is unique up to an additive constant, for if f_1, f_2 are both solutions, then $f_1' = f_2'$. Hence $\frac{d}{dz}(f_1 - f_2) = 0$, and since D is a domain, $f_1 - f_2$ is constant throughout D (Volume I, Chapter One, theorem 5.1). This also implies that the harmonic conjugate v is unique up to an additive constant.

EXAMPLE 1. $u(x, y) = x^2 - y^2$, defined in the whole plane. Note first of all that u satisfies Laplace's equation. If f exists, then $f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 2x - i(-2y) = 2z$, hence

$$f(z) = z^2 + \text{constant} = u + iv$$

and so $v(x, y) = 2xy + \text{constant}$.

† *Functions of a Complex Variable I*, p. 46.

EXAMPLE 2. $u(x, y) = \log\sqrt{(x^2+y^2)}$ defined in the whole plane except the origin. Since $\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$, $\frac{\partial^2 u}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$, $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$, $\frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$, we see that u satisfies Laplace's equation throughout its domain of definition. If u were the real part of an analytic function f, then we would require $f'(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{1}{z}$. As we have shown (Volume I, page 43), no such f exists which is defined for all points except the origin. However, in the cut-plane (with the negative real axis removed) a solution is $f(z) = \text{Log } z$ and $v(x, y) = \arg(x+iy)$.

EXERCISES ON CHAPTER ONE

1. Consider the paths $z(t) = t$ $(-1 \leqslant t \leqslant 1)$, $z(t) = t(1+i)$ $(-1 \leqslant t \leqslant 1)$. Write down the equations of the transformed curves under the following functions: (i) e^z (ii) $\sin z$ (iii) z^2+z. In each case verify that the function has non-zero derivative at the origin and that the angle between the curves is preserved under the transformation.

2. Consider the line segments $z(t) = t$ $(0 \leqslant t \leqslant 1)$, $z(t) = te^{i\alpha}$ $(0 \leqslant t \leqslant 1)$ where $-\pi < \alpha \leqslant \pi$. Find the equations of the transformed curves under the function $f(z) = z^n$ where n is a positive integer. Show that on transformation the angle between the two curves is multiplied by n (up to a multiple of 2π).

3. Find the equations of the level curves of $f(z) = \frac{1}{z}$ and draw a sketch of them.

4. Show that the equation of any circle or straight line may be written as

$$\varepsilon(x^2+y^2)+px+qy+r = 0 \qquad (*)$$

where p, q, r are real.

If $\varepsilon \neq 0$, show that this is a circle of radius $\left(\frac{p^2+q^2-4r}{4\varepsilon}\right)^{\frac{1}{2}}$ and if $\varepsilon = 0$ then it is a line. (This demonstrates why we regard a line as a 'circle of infinite radius'.)

Show that an inversion $w = 1/z$ transforms (*) into

$$r(u^2+v^2)+pu-qv+\varepsilon = 0$$

where $w = u+iv$.

Hence show that under an inversion

(i) a straight line or circle through the origin transforms into a straight line,

(ii) any other straight line or circle transforms into a circle.

5. Find the most general cubic form

$$u(x, y) = ax^3+bx^2y+cxy^2+dy^3 \; (a, b, c, d \text{ real})$$

which satisfies Laplace's equation, and find an analytic function f which has u as its real part.

6. Verify that

$$u(x, y) = 2\sin x\cosh y - 2\cos x\sinh y + x^2 - y^2 - 4xy$$

satisfies Laplace's equation, and (preferably by inspection) find an analytic function f which has u as its real part.

CHAPTER TWO

Cauchy's Residue Theorem

1. Laurent's Theorem

The main purpose of the next two chapters is to develop methods of calculating contour integrals. If f is analytic in a domain containing a closed Jordan contour γ and the points inside γ, then Cauchy's Theorem states that

$$\int_\gamma f(z)dz = 0.$$

In this chapter we are concerned with calculating $\int_\gamma f(z)dz$ where f is not analytic at a finite number of points inside γ. The solution to this problem is given by Cauchy's Residue Theorem which will then be used in Chapter Three to calculate a number of specific integrals.

We first generalize Taylor's Theorem. This states that if f is analytic for $|z-z_0|<R$, then we have a power series expansion $f(z_0+h) = \sum_{n=0}^{\infty} a_n h^n$, valid for $|h|<R$. Now suppose that f is only assumed analytic for $R_1<|z-z_0|<R_2$. We cannot hope to express $f(z_0+h)$ as a power series $\sum_{n=0}^{\infty} a_n h^n$ valid for $R_1<|h|<R_2$, since by the comparison test this series would converge for $|h|<R_2$, and by extending the domain of definition of $\sum a_n h^n$ to $|h|<R_2$ we may consider f to be analytic for $|z-z_0|<R_2$. We can however express $f(z_0+h)$ as a series involving both positive and negative powers of h.

LAURENT'S THEOREM. If f is analytic in the annulus $R_1<|z-z_0|<R_2$ (where $R_1 \geqslant 0$), then

$$f(z_0+h) = \sum_{n=0}^{\infty} a_n h^n + \sum_{n=1}^{\infty} b_n h^{-n} \text{ for } R_1<|h|<R_2.$$

If C is the circle centre z_0, radius r (given by $z(t) = z_0+re^{it}$ $(0\leqslant t\leqslant 2\pi)$) where $R_1<r<R_2$, then

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz,\ b_n = \frac{1}{2\pi i}\int_C (z-z_0)^{n-1} f(z)dz.$$

Note: If f is analytic for $R_1<|z-z_0|$, we may formally take $R_2 = \infty$.

The proof is by expressing $f(z_0+h)$ in terms of two integrals; one is shown to equal $\sum_{n=0}^{\infty} a_n h^n$ for $|h|<R_2$ and the other $\sum_{n=1}^{\infty} b_n h^{-n}$ for $|h|>R_1$. Finding the two integrals is quite straightforward. To express each integral as a series is a little more technical, but is modelled on the proof of Taylor's Theorem (as in lemma 3.1, Chapter Three of Volume I). We now give the details.

Fix h and choose r_1, r_2 such that $R_1<r_1<|h|<r_2<R_2$. Let C_m be the circular contour $z(t) = z_0+r_m e^{it}$ $(0\leqslant t\leqslant 2\pi)$ for $m = 1, 2$. Note that z_0+h lies between C_1 and C_2. By making

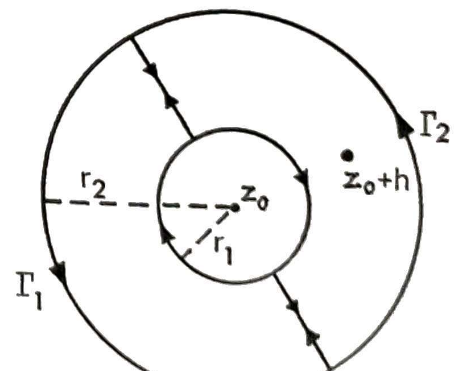

Figure 7

two cross-cuts from C_1 to C_2 avoiding the point z_0+h, let Γ_1, Γ_2 be the two closed Jordan contours as in figure 7.

Then

$$\int_{\Gamma_1} \frac{f(z)}{z-(z_0+h)}\,dz = 0 \text{ by Cauchy's Theorem}$$

and

$$\int_{\Gamma_2} \frac{f(z)}{z-(z_0+h)}\,dz = 2\pi i f(z_0+h) \text{ by Cauchy's integral formula.}$$

Adding these integrals, the contributions along the cross-cuts cancel giving

$$f(z_0+h) = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-z_0-h}\,dz - \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0-h}\,dz.$$

As in the proof of Taylor's Theorem† we find that

$$\frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-z_0-h}\,dz = \sum_{n=0}^{\infty} a_n h^n \qquad |h|<r_2$$

where

$$a_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

But if C is any circle $z(t) = z_0+re^{it}$ $(0 \leqslant t \leqslant 2\pi)$, $R_1<r<R_2$, by making cross-cuts from C_2 to C in the usual way we find

$$a_n = \frac{1}{2\pi i}\int_{C} \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

(Note that we do not have $a_n = \dfrac{f^{(n)}(z_0)}{n!}$ because f may not be analytic throughout the interior of C.)

Also, since $\sum a_n h^n$ converges for $|h|<r_2<R_2$, by choosing

† *Functions of a Complex Variable I*, p. 54.

r_2 as close as we please to R_2, we find that $\sum a_n h^n$ converges for $|h|<R_2$.

Similarly

$$-\frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0-h}\,dz$$

$$= \frac{1}{2\pi i}\int_{C_1} f(z)\left\{\frac{1}{h}+\frac{z-z_0}{h^2}+\ldots+\frac{(z-z_0)^{n-1}}{h^n}-\frac{(z-z_0)^n}{h^n(z-z_0-h)}\right\}dz$$

$$= \sum_{r=1}^{n} b_r h^{-r} - B_n$$

where

$$b_r = \frac{1}{2\pi i}\int_{C_1} f(z)(z-z_0)^{r-1}\,dz,\quad B_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)(z-z_0)^n}{h^n(z-z_0-h)}\,dz.$$

Now for some constant M we have $|f(z)| \leqslant M$ for z on the track of C_1. Moreover for such z we have $|z-z_0| = r_1$ and $|z-z_0-h| \geqslant ||h|-|z-z_0|| = |h|-r_1$. Hence

$$|B_n| \leqslant \frac{1}{2\pi}\,\frac{Mr_1^{\,n}}{|h|^n(|h|-r_1)}.2\pi r_1 = \frac{Mr_1}{|h|-r_1}\left(\frac{r_1}{|h|}\right)^n.$$

Since $r_1<|h|$, we have $B_n \to 0$ as $n \to \infty$, and so

$$-\frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0-h}\,dz = \sum_{n=1}^{\infty} b_n h^{-n} \text{ for } |h|>r_1.$$

Arguing as for a_n, we find $b_n = \dfrac{1}{2\pi i}\displaystyle\int_C f(z)(z-z_0)^{n-1}\,dz$ where C is the circle $z(t) = z_0+re^{it}$ $(0 \leqslant t \leqslant 2\pi)$ for *any* r in $R_1<r<R_2$. Also, by choosing r_1 as close to R_1 as we please, we find the series $\sum_{n=1}^{\infty} b_n h^{-n}$ converges for $|h|>R_1$.

This completes the proof.

Remark. By writing $b_n = a_{-n}$ for $n \geqslant 1$, we can express the result in a more symmetric form as

$$f(z_0+h) = \sum_{-\infty}^{\infty} a_n h^n \text{ where } a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

The integral formula allows us to show that the Laurent expansion is unique. That is to say that if we find $f(z) = \sum_{-\infty}^{\infty} c_n(z-z_0)^n$ by some other method, then $c_n = a_n$ for all n.

First note that $(z-z_0)^n = \frac{d}{dz}\left(\frac{(z-z_0)^{n+1}}{n+1}\right)$ $(n \neq -1)$, and so

$$\int_C (z-z_0)^n dz = 0 \qquad (n \neq -1).$$

If we recall that C is given by $z(t) = z_0 + re^{it}$ $(0 \leqslant t \leqslant 2\pi)$, then by direct calculation

$$\int_C \frac{1}{z-z_0}\,dz = \int_0^{2\pi} \frac{1}{re^{it}}\, ire^{it}\,dt = 2\pi i.$$

Hence, assuming term by term integration is justified in the annulus, we have

$$\int_C (z-z_0)^{-m-1}\left\{\sum_{-\infty}^{\infty} c_n(z-z_0)^n\right\}dz = \sum_{-\infty}^{\infty} c_n \int_C (z-z_0)^{n-m-1}dz$$
$$= 2\pi i c_m. \qquad (1)$$

This gives

$$a_m = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{m+1}}\,dz = c_m.$$

The integration in (1) is easily justified. We write

$$\sum_{-\infty}^{\infty} c_n(z-z_0)^{n-m-1} = \left(\ldots + \frac{c_{m-2}}{(z-z_0)^3} + \frac{c_{m-1}}{(z-z_0)^2}\right)$$
$$+ \frac{c_m}{z-z_0} + (c_{m+1} + c_{m+2}(z-z_0) + \ldots)$$
$$= f_1(z) + \frac{c_m}{z-z_0} + f_2(z).$$

We need only show that f_1, f_2 each have a primitive in the annulus and then

$$\int_C \left\{\sum_{-\infty}^{\infty} c_n(z-z_0)^{n-m-1}\right\}dz = \int_C \left\{f_1(z) + \frac{c_m}{z-z_0} + f_2(z)\right\}dz$$
$$= 0 + 2\pi i c_m + 0$$
$$= 2\pi i c_m \text{ as required.}$$

But $\qquad f_2(z) = \sum_{n=1}^{\infty} c_{m+n}(z-z_0)^{n-1} \qquad |z-z_0| < R_2$

and so if $\qquad F_2(z) = \sum_{n=1}^{\infty} \frac{c_{m+n}}{n}(z-z_0)^n \qquad |z-z_0| < R_2$

then $\qquad \frac{d}{dz}F_2(z) = f_2(z)$ and F_2 is a primitive for f_2.

For f_1, we have $f_1(z) = \sum_{n=1}^{\infty} \frac{c_{m-n}}{(z-z_0)^{n+1}} = \sum_{n=1}^{\infty} c_{m-n}w^{n+1}$ where $w = (z-z_0)^{-1}$. This is valid for $|z-z_0| > R_1$, i.e. for $|w| < 1/R_1$. If we choose

$$G(w) = -\sum_{n=1}^{\infty} \frac{c_{m-n}}{n} w^n \qquad |w| < 1/R_1$$

then $\qquad \frac{d}{dw}G(w) = -\sum_{n=1}^{\infty} c_{m-n}w^{n-1}.$

Hence if $F_1(z) = G((z-z_0)^{-1})$

$$= -\sum_{n=1}^{\infty} \frac{c_{m-n}}{n}(z-z_0)^{-n} \qquad |z-z_0| > R_1,$$

then $\frac{d}{dz}F_1(z) = -(z-z_0)^{-2}G'((z-z_0)^{-1})$

$$= (z-z_0)^{-2}\sum_{n=1}^{\infty} c_{m-n}(z-z_0)^{-n+1} = f_1(z), \text{ as required.}$$

2. Isolated Singularities

If f is analytic in $0 < |z-z_0| < R$ we say that z_0 is an *isolated singularity* of f. For example, if $f(z) = 1/z$, then the origin is an isolated singularity. However if $f(z) = \text{Log } z$ in the cut-plane, then the origin is not an isolated singularity since every annulus $0 < |z| < R$ contains points on the negative real axis where Log z is not analytic.

By Laurent's Theorem (with $R_1 = 0$ and $z = z_0+h$), near an isolated singularity we may write

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n} \text{ for } 0<|z-z_0|<R.$$

The series $\sum_{n=1}^{\infty} b_n(z-z_0)^{-n}$ is called the *principal part* of f at z_0. The behaviour of f near z_0 depends on the nature of the principal part and we distinguish three cases.

CASE 1.

The principal part is zero, i.e. every b_n is zero. Here z_0 is called a *removable singularity*, for we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \qquad 0<|z-z_0|<R$$

and by defining $f(z_0) = a_0$ we can consider f to be analytic at z_0. (This is a trivial example of extension to an analytic function!)

EXAMPLE 1(A). $f(z) = \dfrac{\sin z}{z} \; (z \neq 0)$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots + \frac{(-1)^n z^{2n}}{(2n+1)!} + \ldots$$

If z_0 is an isolated singularity of f and $\lim_{z\to z_0} f(z)$ is finite, then z_0 must be a removable singularity. This is because

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n} \qquad 0<|z-z_0|<R$$

where $b_n = \dfrac{1}{2\pi i}\displaystyle\int_C (z-z_0)^{n-1} f(z)\,dz$, C being the circle centre z_0,

radius r. But $\lim_{z\to z_0} f(z)$ is finite and so in a neighbourhood of z_0 we have $|f(z)| \leqslant M$ for some M. This gives $|b_n| \leqslant \dfrac{1}{2\pi} r^{n-1} M 2\pi r = Mr^n$ and letting $r\to 0$, we see that $b_n = 0$ for $n \geqslant 1$.

EXAMPLE 1(B). $f(z) = \dfrac{z}{e^z-1}$ has a removable singularity at the origin, because as $z\to 0$, we have

$$\frac{e^z-1}{z} = 1 + \frac{z}{2!} + \ldots + \frac{z^{n-1}}{n!} + \ldots \to 1$$

and so $f(z)\to 1$.

CASE 2

The principal part is a finite series, $b_m \neq 0$ but $b_n = 0$ for $n>m$. In this case we call z_0 a *pole of order m*. A pole of order 1, 2, 3, . . . is also termed simple, double, triple, . . . respectively. For a pole of order m we have

$$f(z) = \frac{b_m}{(z-z_0)^m} + \ldots + \frac{b_1}{z-z_0} + \sum_{n=0}^{\infty} a_n(z-z_0)^n \qquad 0<|z-z_0|<R$$

$$= (z-z_0)^{-m} g(z)$$

where

$$g(z) = b_m + b_{m-1}(z-z_0) + \ldots + b_1(z-z_0)^{m-1} + \sum_{n=0}^{\infty} a_n(z-z_0)^{m+n}$$

is analytic for $|z-z_0|<R$ and $g(z_0) = b_m \neq 0$.

This implies that $g(z)\neq 0$ in a small neighbourhood of z_0 and since $\dfrac{1}{f(z)} = \dfrac{(z-z_0)^m}{g(z)}$, we see that $\dfrac{1}{f}$ has a zero of order m at z_0.

Hence as $z\to z_0$ we have $\dfrac{1}{|f(z)|}\to 0$ and so $|f(z)|\to +\infty$.

EXAMPLE 2(A). $f(z) = \dfrac{1}{z^2-1}\ (z \neq 1)$.

Put $z = 1+h$, then

$$f(z) = \frac{1}{h(2+h)}$$

$$= \frac{1}{2h}\{1-\tfrac{1}{2}h+\tfrac{1}{4}h^2-\ldots+(-\tfrac{1}{2})^n h^n+\ldots\} \text{ for } 0<|h|<2$$

$$= \frac{1}{2h}-\tfrac{1}{4}+h-\ldots-(-\tfrac{1}{2})^{n+2}h^n+\ldots$$

Thus f has a simple pole at $z = 1$.

It is possible to show that a point is a pole of order m without actually calculating the Laurent series. If

$$f(z) = b_m(z-z_0)^{-m}+\ldots+b_1(z-z_0)^{-1} + \sum_{n=0}^{\infty} a_n(z-z_0)^n \qquad 0<|z-z_0|<R$$

then $(z-z_0)^m f(z) \to b_m \neq 0$ as $z \to z_0$. Conversely, if $(z-z_0)^m f(z)$ tends to a non-zero limit, then, as we have seen, $(z-z_0)^m f(z)$ has a removable singularity at z_0 and so $f(z)$ has a pole of order m. (It may also be seen that $(z-z_0)^n f(z) \to 0$ for $n>m$ and $(z-z_0)^n f(z)$ does not tend to a finite limit for $n<m$.)

EXAMPLE 2(B). $f(z) = \dfrac{2z+4}{(1-z^2)\sin^3 z}$ has a triple pole at the origin because $z^3 f(z) = \dfrac{2z+4}{1-z^2}\left(\dfrac{z}{\sin z}\right)^3 \to 4$ as $z \to 0$.

CASE 3

The principal part is an infinite series, i.e. an infinite number of the b_n are non-zero. Such a singularity is called an *isolated essential singularity*. The behaviour of f near z_0 is very peculiar.

As $z \to z_0$, we cannot have $|f(z)| \to +\infty$ because this would imply that f has a pole at z_0. (This follows because $|f(z)| \to +\infty$ implies $\dfrac{1}{f(z)} \to 0$, so $\dfrac{1}{f}$ has a removable singularity at z_0 and may be considered analytic there. Since $\lim_{z\to z_0} \dfrac{1}{f(z)} = 0$, $\dfrac{1}{f}$ has a zero of order m for some $m \geq 1$, and f must have a pole of order m.)

If $f(z)$ does not approach infinity, what happens? In fact the behaviour of f is very wild near z_0 in the sense that in any neighbourhood of z_0 (however small) f takes every complex value with perhaps one exception. This is Picard's Theorem; the proof is omitted.

EXAMPLE 3. $\exp(1/z) = 1+\dfrac{1}{z}+\dfrac{1}{2!z^2}+\ldots+\dfrac{1}{n!z^n}+\ldots\ |z|>0.$

In $0<|z|<\varepsilon$ (no matter how small ε), $\exp(1/z)$ takes on every complex value except $w = 0$. To see this, we require to find z such that $w = \exp(1/z)$, $0<|z|<\varepsilon$. This is equivalent to solving the equations:

$$\text{(a)}\quad \frac{1}{z} = \text{Log}|w|+i(\arg w+2\pi k) \qquad \text{(b)}\quad \frac{1}{|z|^2} > \frac{1}{\varepsilon^2}.$$

For $w \neq 0$ and any integer k we can find z from (a), and by choosing k very large, we can make

$$\frac{1}{|z|^2} = (\text{Log } |w|)^2 + (\arg w+2\pi k)^2 > \frac{1}{\varepsilon^2}.$$

Note. If $z_1, z_2, \ldots$ is a sequence of distinct isolated singularities of f which tends to a limit z_0, then z_0 cannot be an isolated singularity of f. This is because every annulus $0<|z-z_0|<\varepsilon$ contains points of the sequence and at these points f is not analytic. In such a case, z_0 is called an *essential singularity* of f.

EXAMPLE 4. $f(z) = \left(\sin\left(\frac{1}{z}\right)\right)^{-1}$ has an essential singularity at the origin because $\frac{1}{\pi}, \frac{1}{2\pi}, \ldots, \frac{1}{n\pi}, \ldots$ is a sequence of singularities of f which tends to zero.

3. The Point at Infinity

In the last section we saw that if z_0 was a pole of f, then $|f(z)| \to +\infty$ as $z \to z_0$. It is possible to adjoin a single point at infinity (denoted by ∞) to the complex plane so that $f(z) \to \infty$ as $z \to z_0$.

Consider a sphere touching the complex plane at the origin and let N (the 'north pole') be the point on the sphere diametrically opposite the point of contact.

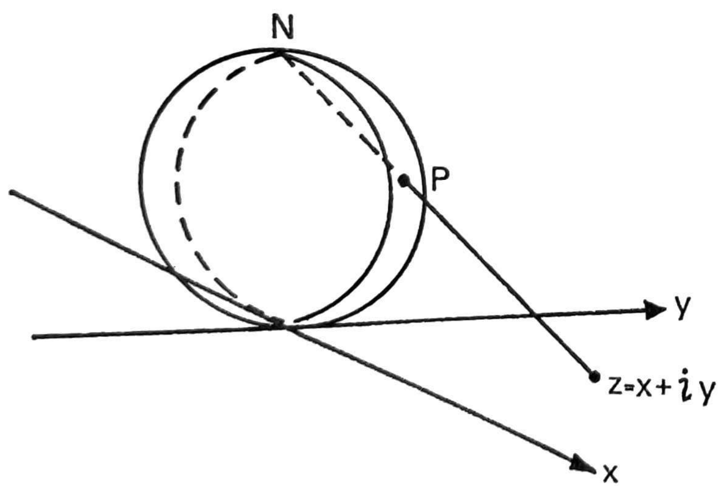

Figure 8

If P is any point on the sphere distinct from N, then the straight line NP meets the plane in a unique point $z = x+iy$ and this sets up a correspondence between all the points of the sphere except N and all the points of the complex plane.

We note that N is omitted in this correspondence and we suppose that it corresponds to the symbol ∞. The complex plane together with ∞ is called the *extended complex plane* and we see that there is a correspondence between the points on the sphere and the points of the extended complex plane.

We remark that 'lines of latitude' on the sphere correspond to circles of the form $|z| = R$ in the plane and the 'polar cap' between a line of latitude and the north pole corresponds to the domain $|z| > R$. As R increases, the corresponding line of latitude approaches N. For this reason we define the domain $|z| > R$ (together with ∞) to be a neighbourhood of ∞ and we write $w \to \infty$ if the real number $|w| \to +\infty$. For example if z_0 is a pole of f, then $\lim_{z \to z_0} f(z) = \infty$.

Suppose that f is analytic for $|z| > R$, then $\frac{d}{dz}\left(f\left(\frac{1}{z}\right)\right) = -\frac{1}{z^2} f'\left(\frac{1}{z}\right)$ and so $f\left(\frac{1}{z}\right)$ is analytic for $\left|\frac{1}{z}\right| > R$, i.e. for $0 < |z| < \frac{1}{R}$.

Thus $f\left(\frac{1}{z}\right)$ has an isolated singularity at the origin. We say that $f(z)$ has a removable singularity at ∞, pole of order m at ∞, or isolated essential singularity at ∞ if $f\left(\frac{1}{z}\right)$ has the corresponding singularity at the origin. In particular, if f has a removable singularity at ∞, we may regard f as being analytic at ∞ and define $f(\infty) = \lim_{z \to \infty} f(z) = \lim_{z \to 0} f\left(\frac{1}{z}\right)$.

EXAMPLE 1. $f(z) = z^{-3} \exp\left(\frac{1}{z}\right)$ has a removable singularity at ∞ since $f\left(\frac{1}{z}\right) = z^3 e^z$ $(z \neq 0)$ has a removable singularity at the origin.

EXAMPLE 2. $f(z) = z^3$ has a triple pole at ∞ since $f\left(\frac{1}{z}\right) = \frac{1}{z^3}$.

EXAMPLE 3. $f(z) = e^z$ has an isolated essential singularity at ∞ since $f\left(\frac{1}{z}\right) = \exp\left(\frac{1}{z}\right)$.

If $z_1, z_2, \ldots$ is a sequence of isolated singularities of f and $\lim_{n\to\infty} z_n = \infty$, then f cannot have an isolated singularity at ∞ since every domain $|z| > R$ contains points of the sequence where f is not analytic. In this case f is said to have an essential singularity at ∞.

EXAMPLE 4. $f(z) = \tan z$ has an essential singularity at ∞ since $(n+\frac{1}{2})\pi$ is a singularity of f for every integer n.

4. Cauchy's Residue Theorem

If z_0 is an isolated singularity of f, then by Laurent's Theorem we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n} \quad 0 < |z-z_0| < R.$$

Also the coefficient b_n is given by

$$b_n = \frac{1}{2\pi i}\int_C (z-z_0)^{n-1} f(z)dz$$

where C is the circle centre z_0, radius r, $z(t) = z_0 + re^{it}$ $(0 \leqslant t \leqslant 2\pi)$. In particular, the case $n = 1$ holds a special place because

$$b_1 = \frac{1}{2\pi i}\int_C f(z)dz.$$

The coefficient b_1 is called the *residue* of f at z_0.

(Note that the importance of b_1 is to be expected, for term

by term integration gives

$$\int_C f(z)dz = \int_C \left\{ \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n} \right\} dz$$

$$= \sum a_n \int_C (z-z_0)^n dz + \sum b_n \int_C (z-z_0)^{-n} dz$$

$$= b_1.2\pi i.$$

The last line follows because $(z-z_0)^n = \frac{d}{dz}\left\{\frac{(z-z_0)^{n+1}}{n+1}\right\}$ for $n \neq -1$, giving $\int_C (z-z_0)^n dz = 0$ $(n \neq -1)$, whereas $\int_C (z-z_0)^{-1} dz = 2\pi i$ by direct calculation.)

Suppose γ is a closed Jordan contour (described anti-clockwise) whose track lies in the domain of definition of f and suppose f is analytic everywhere inside γ except at the isolated singularity z_0.

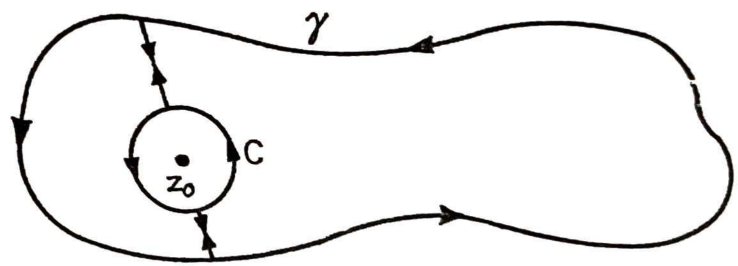

Figure 9

By choosing a small circle around z_0, making cuts from γ to C in the usual fashion, we see that

$$b_1 = \frac{1}{2\pi i}\int_C f(z)dz = \frac{1}{2\pi i}\int_\gamma f(z)dz.$$

Hence if we know the residue b_1 of f at z_0, we can calculate $\int_\gamma f(z)dz$ by the formula

$$\int_\gamma f(z)dz = 2\pi i b_1 \qquad (2)$$

EXAMPLE. $f(z) = \dfrac{1}{z}$ has residue 1 at the origin. Hence if γ is any closed Jordan contour described anti-clockwise round the origin,

$$\int_\gamma \frac{1}{z}\,dz = 2\pi i$$

This generalizes the case where γ is the unit circle which may be calculated directly.

This method of calculating integrals by residues is an extremely useful technique. It generalizes to the case of several singularities inside γ.

CAUCHY'S RESIDUE THEOREM. Let γ be a closed Jordan contour described anti-clockwise. Suppose the function f is analytic in a domain which includes the track and the interior of γ except for a finite number of isolated singularities $z_1, \ldots, z_n$ in the interior. Then if the residues at $z_1, \ldots, z_n$ are $\rho_1, \ldots, \rho_n$ respectively we have

$$\int_\gamma f(z)dz = 2\pi i(\rho_1 + \ldots + \rho_n).$$

Proof. Make cross-cuts dividing the interior of γ into n domains, each of which contains precisely one singularity.

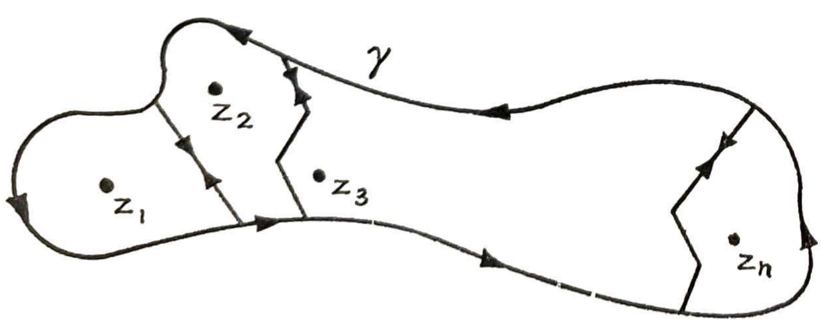

Figure 10

If Γ_r is the boundary contour (described anti-clockwise) of the region containing z_r, by (2) we have

$$\int_{\Gamma_r} f(z)dz = 2\pi i\rho_r.$$

Adding these integrals, the contributions due to the cross-cuts cancel in pairs and we find

$$\int_\gamma f(z)dz = 2\pi i(\rho_1 + \ldots + \rho_n).$$

Note. This is yet another proof which relies on geometric intuition because we have not specified precisely how to make the cuts. Nevertheless, in any particular case that we meet in this text, this would be obvious.

In Chapter Three we will use Cauchy's Residue Theorem to calculate specific integrals and will give several examples there. We now use the theorem to obtain some more general results.

5. Number of Zeros and Poles

Cauchy's Residue Theorem may be used to find the number of zeros and poles of an analytic function inside a closed Jordan contour. For this purpose a zero of order m is counted m times and a pole of order n is counted n times.

THEOREM 5.1. Let γ be a closed Jordan contour described anti-clockwise. Suppose that f is analytic in a domain which includes the track and interior of γ except possibly for a finite number of poles inside γ. If f is non-zero on the track of γ, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = N - P$$

where N is the number of zeros and P is the number of poles inside γ.

Proof. First note that the integral is well-defined because f is non-zero on the track of γ and so $\frac{f'}{f}$ is analytic there. In fact $\frac{f'}{f}$ only has poles where f has a zero or f' (and hence f) has a pole.

If z_0 is a zero of order m, we have $f(z) = (z-z_0)^m g(z)$ where g is analytic and non-zero in a neighbourhood of z_0. Thus

$$f'(z) = m(z-z_0)^{m-1}g(z)+(z-z_0)^m g'(z)$$

and $$\frac{f'(z)}{f(z)} = \frac{m}{z-z_0}+\frac{g'(z)}{g(z)}.$$

But $\frac{g'}{g}$ is analytic in the neighbourhood of z_0 and so $\frac{f'}{f}$ has a simple pole of residue m at z_0.

Similarly if f has a pole of order n at z_1, then $f(z) = (z-z_1)^{-n}h(z)$ where h is analytic and non-zero in a neighbourhood of z_1. Thus

$$f'(z) = -n(z-z_1)^{-n-1}h(z)+(z-z_1)^{-n}h'(z)$$

and $$\frac{f'(z)}{f(z)} = \frac{-n}{z-z_1}+\frac{h'(z)}{h(z)}.$$

Again $\frac{h'}{h}$ is analytic in a neighbourhood of z_1 and $\frac{f'}{f}$ has a simple pole of residue $-n$ at z_1. By adding all the residues together, we obtain the required result.

As a corollary of this theorem, we see that if f is actually analytic inside γ, then the number of zeros of f inside γ is $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz$.

ROUCHÉ'S THEOREM. Suppose that f and g are both

analytic in a domain containing the track and interior of a closed Jordan contour γ (described anti-clockwise). If $|g(z)| < |f(z)|$ on the track of γ then f and $f+g$ have the same number of zeros inside γ.

Proof. Suppose that f has m zeros and $f+g$ has n zeros inside γ. Then if $F(z) = \frac{f(z)+g(z)}{f(z)}$, we see that F has n zeros and m poles inside γ. Also $|f(z)| > |g(z)| \geqslant 0$ on the track of γ showing that F is analytic there.

We will show

$$n-m = \frac{1}{2\pi i}\int_\gamma \frac{F'(z)}{F(z)}\,dz = 0.$$

This is done by transforming the integral.

Write $w = F(z)$, then as z describes the contour γ in the z-plane, w describes a contour Γ in the w-plane. Explicitly, if γ is given by $z(t) = x(t)+iy(t)$ $(\alpha \leqslant t \leqslant \beta)$, then Γ is given by $w(t) = F(z(t))$ $(\alpha \leqslant t \leqslant \beta)$.

Hence

$$\int_\gamma \frac{F'(z)}{F(z)}\,dz = \int_\alpha^\beta \frac{F'(z(t))}{F(z(t))}z'(t)dt$$
$$= \int_\alpha^\beta \frac{w'(t)}{w(t)}\,dt$$
$$= \int_\Gamma \frac{1}{w}\,dw.$$

But for w on the track of Γ, the real part satisfies

$$\Re w = \Re(F(z)) = \Re\left(\frac{f(z)+g(z)}{f(z)}\right)$$
$$= 1+\Re\left(\frac{g(z)}{f(z)}\right) \geqslant 1-\left|\frac{g(z)}{f(z)}\right| > 0,$$

by hypothesis.

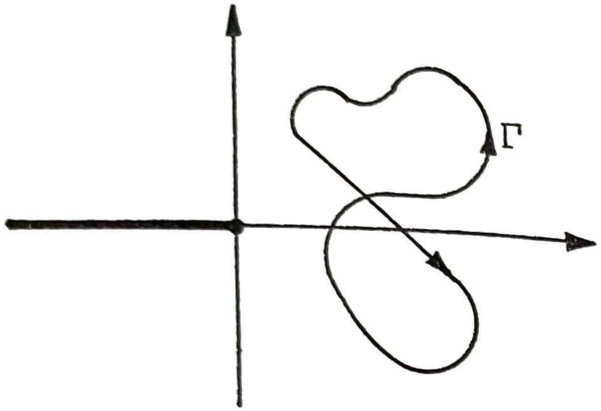

Figure 11

This means that the track of Γ lies in the half-plane $\Re w > 0$ and so must lie in the cut w-plane (cut along the negative real axis).

In the cut-plane we have $\frac{1}{w} = \frac{d}{dw}(\text{Log } w)$, and by the Fundamental Theorem of Contour Integration round a closed contour, $\int_\Gamma \frac{1}{w}\,dw = 0$. This completes the proof.

As a consequence of Rouché's Theorem, we can deduce the *Fundamental Theorem of Algebra*. This states that a polynomial equation

$$z^n + a_1 z^{n-1} + \ldots + a_n = 0$$

has n roots (counted according to multiplicity).

Take $f(z) = z^n$, $g(z) = a_1 z^{n-1} + \ldots + a_n$. Let C be the circle centre the origin, radius $R \geqslant 1$. On C we have $|f(z)| = R^n$ and

$$|g(z)| \leqslant |a_1|R^{n-1} + \ldots + |a^n| \leqslant (|a_1| + \ldots + |a_n|)R^{n-1}.$$

Hence choosing $R > |a_1| + \ldots + |a_n|$, we have $|g(z)| < |f(z)|$ on C.

But f has precisely one zero of order n (at the origin) inside

C and so $f+g$ has n zeros inside C. Thus the polynomial equation has n solutions.

Notice that we have only shown that the polynomial has n zeros; we have also given their approximate location, inside the circle C. In particular cases we can use Rouché's Theorem to give further information of this kind.

EXAMPLE. $z^9 - 6z^2 + 10 = 0$ has all nine zeros between the circles $|z| = 1$ and $|z| = 2$.

Consider the circle $|z| = 1$, $g(z) = z^9 - 6z^2$, $f(z) = 10$. If $|z| = 1$, then $|g(z)| = |z^9 - 6z^2| \leqslant |z|^9 + 6|z|^2 = 7 < |f(z)|$. Since $f(z)$ has no zeros inside $|z| = 1$, $f(z) + g(z) = z^9 - 6z^2 + 10$ also has no zeros there.

Similarly on $|z| = 2$, $f(z) = z^9$, $g(z) = 10 - 6z^2$, we have

$$|g(z)| \leqslant 10 + 6|z|^2 = 10 + 24 < 2^9 = |f(z)|$$

and since $f(z)$ has a zero of order 9 at the origin, $f(z) + g(z) = z^9 - 6z^2 + 10$ has 9 zeros inside $|z| = 2$.

EXERCISES ON CHAPTER TWO

For each of the isolated singularities in exercises 1–6, calculate the Laurent expansion and state what type of singularity is involved:

1. $z^{-5}e^{z^3}$ at $z = 0$
2. $(z^2 - a^2)^{-1}$ at $z = a$ $(a > 0)$
3. $z^{-1}\cos(z^{-1})$ at $z = 0$
4. $\text{Log}\left(\frac{z+z^2}{z-1}\right)$ at $z = 0$
5. $z^{-5}(2\cos z + z^2 - 2)$ at $z = 0$
6. $\{(z-1)(z-2)\}^{-1}$ at $z = 1$.

Classify the singularities of the functions given by the formulae in exercises 7–11 (a) at the origin, (b) at ∞.

7. $\frac{e^z}{z\sin^2 z}$
8. $\frac{z}{1-\cos z}$
9. $z^3\sin(z^{-1})$
10. $\tan(z^{-1})$
11. $z^{-3}e^z$.
12. Use Rouché's Theorem to show that if $|\alpha| > e$, then $\alpha z^n = e^z$ has n solutions inside $|z| = 1$.

CHAPTER THREE

The Calculus of Residues

1. Residues

In this chapter we intend to use Cauchy's Residue Theorem to calculate specific integrals. In order to do this we must be able to calculate the residue at an isolated singularity. The most direct method is to calculate part of the Laurent series of f at the singularity z_0 to find the coefficient of $\dfrac{1}{z-z_0}$. In simple cases this calculation may be avoided.

METHOD 1.

For a simple pole, the residue of f at z_0 is $\lim_{z\to z_0}(z-z_0)f(z)$.

This is because $f(z) = \dfrac{b_1}{z-z_0}+\sum_{n=0}^{\infty}a_n(z-z_0)^n$ and so b_1 is the given limiting value.

EXAMPLE 1. If $f(z) = \dfrac{z}{1-\cos z}$, then the residue of f at zero is

$$\lim_{z\to 0} z\frac{z}{1-\cos z} = \lim_{z\to 0}\frac{4(\frac{1}{2}z)^2}{2\sin^2(\frac{1}{2}z)} = 2.$$

Sometimes we have $f(z) = \dfrac{p(z)}{q(z)}$ and f has a pole at z_0 because $q(z)$ is zero there.

METHOD 2.

If $f(z) = \dfrac{p(z)}{q(z)}$ where $p(z_0)\neq 0$ and z_0 is a simple zero of q, then the residue of f at z_0 is $\dfrac{p(z_0)}{q'(z_0)}$.

This is because $q(z_0) = 0$ and $q'(z_0)\neq 0$, hence

$$\lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to z_0} p(z)\Big/\left\{\frac{q(z)-q(z_0)}{z-z_0}\right\} = \frac{p(z_0)}{q'(z_0)}.$$

EXAMPLE 2. If $f(z) = \dfrac{1}{1-z^4}$ then the residue of f at $z_0 = 1$ is $\dfrac{1}{-4z_0^3} = -\dfrac{1}{4}$.

Methods 1, 2 may be generalized for poles of higher order, but the calculations sometimes become complicated and then the best method is direct calculation from the Laurent series. However, generalizing Method 1 for a pole of order m, we have:

METHOD 3.

If z_0 is a pole of order m of the function f, then the residue of f at z_0 is

$$\lim_{z\to z_0}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-z_0)^m f(z)\}.$$

This is because

$$f(z) = b_m(z-z_0)^{-m}+\dots+b_1(z-z_0)^{-1}+\sum_{n=0}^{\infty}a_n(z-z_0)^n,$$

and so

$$(z-z_0)^m f(z) = b_m+\dots+b_1(z-z_0)^{m-1}+\sum_{n=0}^{\infty}a_n(z-z_0)^{m+n}.$$

This gives

$$\frac{d^{m-1}}{dz^{m-1}}\{(z-z_0)^m f(z)\} = (m-1)!\,b_1+m!\,a_0(z-z_0)+\dots$$

and the result follows.

EXAMPLE 3. If $f(z) = \left(\frac{z+1}{z-1}\right)^2$ then the residue at the double pole $z_0 = 1$ is

$$\lim_{z\to1}\frac{1}{1!}\frac{d}{dz}\left\{(z-1)^2\left(\frac{z+1}{z-1}\right)^2\right\} = \lim_{z\to1}(2z+2) = 4.$$

In cases where methods 1-3 are not applicable, or the calculations become difficult, we must determine the relevant part of the Laurent series. (We only require the coefficient of $(z-z_0)^{-1}$, so the reader who calculates the whole series is wasting a great deal of energy!)

The calculation can often be performed by manipulating Taylor series. We recall that we can add or multiply power series $\sum_{n=0}^{\infty}a_n(z-z_0)^n$, $\sum_{n=0}^{\infty}b_n(z-z_0)^n$ term by term in any disc $|z-z_0|<R$ where both series converge. In particular the product is $\sum_{n=0}^{\infty}c_n(z-z_0)^n$ where $c_n = a_0b_n+a_1b_{n-1}+\dots+a_nb_0$.

To calculate $1/f(z)$ where $f(z) = \sum_{n=0}^{\infty}a_n(z-z_0)^n$ for $|z-z_0|<R$ and $a_0\neq0$, we remark first that $1/f(z)$ certainly has a unique power series expansion $\sum_{n=0}^{\infty}b_n(z-z_0)^n$ in a small disc centre z_0. (Because $f(z_0) = a_0\neq0$ and by continuity $f(z)\neq0$ in $|z-z_0|<\varepsilon$ for some $\varepsilon>0$. Hence the inverse $1/f(z)$ is analytic (with derivative $-f'(z)/(f(z))^2$) in $|z-z_0|<\varepsilon$ and so has a unique Taylor series.†) Since $\sum_{n=0}^{\infty}a_n(z-z_0)^n\sum_{n=0}^{\infty}b_n(z-z_0)^n = 1$, multi-

† *Functions of a Complex Variable I*, p. 55–6.

plying out and comparing coefficients we have $a_0b_0 = 1$, $a_0b_1+a_1b_0 = 0, \dots, a_0b_n+\dots+a_nb_0 = 0, \dots$. But $a_0\neq0$ and so we can use these equations successively to find b_0, $b_1, \dots$. For example if $f(z) = \frac{\sin z}{z} = 1-\frac{z^2}{3!}+\dots$, then $1/f(z) = b_0+b_1z+b_2z^2+\dots$ where

$$(1-\tfrac{1}{6}z^2+\dots)(b_0+b_1z+b_2z^2+\dots) = 1.$$

Hence $b_0 = 1$, $b_1 = 0$, $b_2 = \frac{1}{6}, \dots$, implying

$$z/\sin z = 1+\tfrac{1}{6}z^2+\text{higher order terms.}$$

We now calculate a residue which will later prove useful.

EXAMPLE 4. The residue of $z^{-2}\cot\pi z$ at the origin.
Replacing z by πz in the series for $z/\sin z$, we find

$$\pi z/\sin\pi z = 1+\tfrac{1}{6}\pi^2z^2+\dots$$

Hence
$$z^{-2}\cot\pi z = \frac{1}{\pi z^3}\cos\pi z\frac{\pi z}{\sin\pi z}$$
$$= \frac{1}{\pi z^3}(1-\tfrac{1}{2}\pi^2z^2+\dots)(1+\tfrac{1}{6}\pi^2z^2+\dots).$$

The coefficient of $1/z$ is $\pi(\frac{1}{6}-\frac{1}{2}) = -\frac{1}{3}\pi$, i.e.the residue is $-\frac{1}{3}\pi$.

2. Integrals of the Form $\int_0^{2\pi}f(\cos t, \sin t)\,dt$

If C is the unit circle $z(t) = \cos t+i\sin t$ $(0\leqslant t\leqslant2\pi)$ we may transform $\int_0^{2\pi}f(\cos t, \sin t)\,dt$ into a contour integral of the form $\int_C g(z)\,dz$ and use Cauchy's Residue Theorem to calculate the latter. This is always possible if the function g is analytic in a domain including C and its interior except possibly for a finite number of isolated singularities inside C.

Specifically, if $z = e^{it}$ then $\cos t = \frac{1}{2}\left(z+\frac{1}{z}\right)$,

$$\sin t = \frac{1}{2i}\left(z - \frac{1}{z}\right) \text{and } z'(t) = ie^{it} = iz.$$

Let $g(z) = f\left(\tfrac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\frac{1}{iz}$.

then†

$$\int_C g(z)dz = \int_0^{2\pi} g(z(t))z'(t)dt = \int_0^{2\pi} f(\cos t, \sin t)dt.$$

Thus $\int_0^{2\pi} f(\cos t, \sin t)dt = 2\pi i$(sum of residues of g at isolated singularities inside C).

EXAMPLE. $I = \displaystyle\int_0^{2\pi} \frac{dt}{a+b\cos t} \qquad (a>b>0).$

We find
$$I = \int_C \frac{1}{a+\frac{1}{2}b(z+1/z)}\cdot\frac{1}{iz}dz$$
$$= \frac{2}{i}\int_C \frac{dz}{bz^2+2az+b}$$
$$= \frac{2}{i}\int_C \frac{dz}{q(z)}.$$

Since $\dfrac{1}{q(z)}$ only has poles where $q(z) = bz^2+2az+b = 0$, there are simple poles at $\dfrac{-a\pm\sqrt{(a^2-b^2)}}{b}$. Let $\alpha = \dfrac{-a+\sqrt{(a^2-b^2)}}{b}$, $\beta = \dfrac{-a-\sqrt{(a^2-b^2)}}{b}$, then $\alpha\beta = \dfrac{b}{b} = 1$ and so $|\alpha|\,|\beta| = 1$. Since $|\alpha|<|\beta|$, we must have $|\alpha|<1$, $|\beta|>1$, and the only pole

† The reader may also remember this formula by substituting for cos t, sin t and $dt = \dfrac{dz}{iz}$ in $\displaystyle\int_0^{2\pi} f(\cos t, \sin t)\,dt = \int_C f\left(\tfrac{1}{2}\left(z+\tfrac{1}{z}\right)\right), \tfrac{1}{2i}\left(z+\tfrac{1}{z}\right)\frac{dz}{iz} = \int_C g(z)dz$, but strictly speaking we have not justified the use of the differential dz as a separate entity.

of $\dfrac{1}{q(z)}$ inside C is a simple pole at α with residue $\dfrac{1}{q'(z)} = \dfrac{1}{2b\alpha+2a} = \dfrac{1}{2\sqrt{(a^2-b^2)}}$.

$$\text{Hence } I = \frac{2}{i}\int_C \frac{dz}{q(z)} = 2\pi i\,.\,\frac{2}{i}\cdot\frac{1}{2\sqrt{(a^2-b^2)}} = \frac{2\pi}{\sqrt{(a^2-b^2)}}.$$

3. Integrals of the Form $\int_{-\infty}^{\infty} f(x)dx$

Under suitable conditions we have

$\int_{-\infty}^{\infty} f(x)dx = 2\pi i$ (sum of residues of f at isolated singularities in the upper half-plane).

To obtain this result, we integrate round the contour composed of the semicircle S_R given by $z(t) = Re^{it}$ $(0\leqslant t\leqslant 2\pi)$ and its diameter from $-R$ to R.

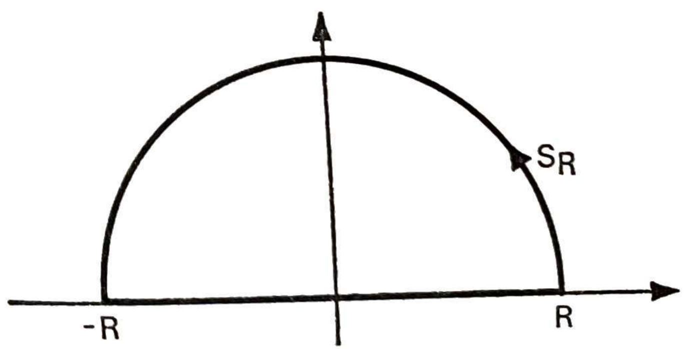

Figure 12

The calculation is possible if:

(i) f is analytic in a domain which includes the upper half-plane ($\mathscr{I}z\geqslant 0$) except for a finite number of isolated singularities which do not lie on the real axis.

(ii) for large R, $|f(z)| \leqslant \dfrac{M}{R^2}$ when z lies on the semicircle S_R.

To see this we choose R so large that (ii) is satisfied and also all the singularities lie inside the closed contour of figure 12.

Then we have

$\int_{-R}^{R} f(x)dx + \int_{S_R} f(z)dz = 2\pi i$ (sum of residues in the upper half-plane).

Now let $R \to \infty$. Since

$$\left|\int_{S_R} f(z)dz\right| \leqslant \frac{M}{R^2}\cdot \pi R = \frac{\pi M}{R}$$

we have $\lim_{R\to\infty} \int_{S_R} f(z)dz = 0$.

Thus

$\lim_{R\to\infty} \int_{-R}^{R} f(x)dx = 2\pi i$ (sum of residues in upper half-plane).

Remark. The symbol $\int_{-\infty}^{\infty} f(x)dx$ actually incorporates two distinct limits,

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{Y\to\infty}\int_{-Y}^{0} f(x)dx + \lim_{X\to\infty}\int_{0}^{X} f(x)dx. \tag{1}$$

Since we have only calculated $\lim_{R\to\infty}\int_{-R}^{R} f(x)dx$, it is theoretically possible for this limit to exist but not the individual limits in (1). For example, if $\phi(x) = \dfrac{2x}{x^2+1}$, then

$$\int_{-Y}^{X} \phi(x)dx = \log\left(\frac{X^2+1}{Y^2+1}\right).$$

Thus we find that $\int_{-R}^{R} \phi(x) = 0$ and $\lim_{R\to\infty}\int_{-R}^{R} \phi(x)dx = 0$, but $\lim_{Y\to\infty}\int_{-Y}^{0} \phi(x)dx = -\infty$, and $\lim_{X\to\infty}\int_{0}^{X} \phi(x)dx = +\infty$. In such a

case, $\lim_{R\to\infty}\int_{-R}^{R}\phi(x)dx$ is called the *Cauchy principal value* and is denoted by $P\int_{-\infty}^{\infty}\phi(x)dx$. Luckily $\phi(x) = \dfrac{2x}{x^2+1}$ does not satisfy condition (ii) and subject to this condition, there is no problem with the limits. This is because there is a comparison test for infinite integrals analogous to the real case.†

If $p(x)$ is a continuous, positive real-valued function such that $|f(x)| \leqslant p(x)$ for $x \geqslant K$ and $\lim_{X\to\infty}\int_K^X p(x)dx$ exists, then $\lim_{X\to\infty}\int_K^X f(x)dx$ exists. (To prove this, note that $|\Re f(x)| \leqslant |f(x)| \leqslant |p(x)|$ and so $\lim_{X\to\infty}\int_K^X \Re f(x)dx$ exists by the comparison test in the real case; similarly for the imaginary part.) Using condition (ii) and comparing $|f(x)|$ with $p(x) = \dfrac{M}{x^2}$, we see $\displaystyle\int_K^X \frac{M}{x^2}dx = \frac{M}{K} - \frac{M}{X}$ tends to $\dfrac{M}{K}$ as $X\to\infty$. Hence $\lim_{X\to\infty}\int_K^X f(x)dx$ exists and similarly for $\lim_{Y\to\infty}\int_{-Y}^{K} f(x)dx$. Thus $\int_{-\infty}^{\infty} f(x)dx$ exists.

A suitable function for this type of calculation is any rational function $\dfrac{N(z)}{D(z)}$ where N, D are polynomials such that

(i) $D(x) \neq 0$ when x is real,
(ii) degree $D \geqslant 2 +$ degree N.

EXAMPLE 1. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)}$ when $a>0$, $b>0$, $a\neq b$.

The only singularities of the integrand in the upper half-plane are simple poles at ia, ib. The residue at ia is

$$\lim_{z\to ia} \frac{z-ia}{(z^2+a^2)(z^2+b^2)} = \frac{1}{2ia(b^2-a^2)}$$

† W. Ledermann, *Integral Calculus*, pp. 21, 22.

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and at ib it is $\dfrac{1}{2ib(a^2-b^2)}$.

Thus $$\int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i\left(\frac{1}{2ia(b^2-a^2)}+\frac{1}{2ib(a^2-b^2)}\right)$$
$$= \frac{\pi(b-a)}{ab(b^2-a^2)}$$
$$= \frac{\pi}{ab(a+b)}.$$

As a further refinement, note that we did not require $f(z)$ to be real on the real axis. The function e^{imz} $(m>0)$ is everywhere analytic and satisfies

$$|e^{imz}| = |e^{imx-my}| = |e^{-my}| \leqslant 1 \text{ for } y \geqslant 0 \text{ (since } m>0).$$

Hence if $f(z)$ satisfies conditions (i), (ii), then so does $e^{imz}f(z)$.

EXAMPLE 2. Consider $f(z) = \dfrac{1}{(z^2+a^2)(z^2+b^2)}$ where $a>0$, $b>0$, $a \neq b$.

The residue of $e^{imz}f(z)$ at ia is

$$\lim_{z\to ia}\frac{(z-ia)e^{imz}}{(z^2+a^2)(z^2+b^2)} = \frac{e^{-ma}}{2ia(b^2-a^2)}$$

and at ib it is $\dfrac{e^{-mb}}{2ib(a^2-b^2)}$. Thus we have

$$\int_{-\infty}^{\infty}\frac{e^{imx}}{(x^2+a^2)(x^2+b^2)}\,dx = 2\pi i\left(\frac{e^{-ma}}{2ia(b^2-a^2)}+\frac{e^{-mb}}{2ib(a^2-b^2)}\right)$$

Equating real and imaginary parts, this gives

$$\int_{-\infty}^{\infty}\frac{\cos mx}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{b^2-a^2}\left(\frac{e^{-ma}}{a}-\frac{e^{-mb}}{b}\right)$$

$$\int_{-\infty}^{\infty}\frac{\sin mx}{(x^2+a^2)(x^2+b^2)}\,dx = 0.$$

Notice that if $g(x)$ is an odd function $(g(-x) = -g(x))$, as in the second case, then we must have $\int_{-\infty}^{\infty}g(x)dx = 0$. Also if $g(x)$ is even $(g(-x) = g(x))$, then $\int_{-\infty}^{\infty}g(x)dx = 2\int_0^{\infty}g(x)dx$. Thus from example 1,

$$\int_0^{\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

and from example 2,

$$\int_0^{\infty}\frac{\cos mx}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{2(b^2-a^2)}\left(\frac{e^{-ma}}{a}-\frac{e^{-mb}}{b}\right).$$

4. Integrals of the Form $\int_{-\infty}^{\infty} e^{imx} f(x)dx$

Integrals of this form are substantially covered by the conditions of the last section. However we can make a slight improvement in condition (ii) below.

For $m>0$, we have $\int_{-\infty}^{\infty}e^{imx}f(x)dx = 2\pi i$ (sum of residues of $e^{imz}f(z)$ at isolated singularities in the upper half-plane) provided that

(i) f is analytic in a domain containing the upper half-plane except for a finite number of isolated singularities, none of which lie on the real axis.

(ii) for large R, $|f(z)| \leqslant \dfrac{M}{R}$ when $|z| = R$, $\mathscr{I}z \geqslant 0$.

We may use a semicircular contour† as in the last section and prove that $\int_{S_R} e^{imz}f(z)dz \to 0$ as $R\to\infty$. However this

† This method is used in E. G. Phillips, *Functions of a Complex Variable*, Oliver & Boyd, p. 123.

method has a basic drawback: it only calculates

$$\lim_{R\to\infty}\int_{-R}^{R} e^{imx}f(x)dx$$

and we still have to show that

$$\int_{-\infty}^{\infty} e^{imx}f(x)dx$$

exists. This would require a delicate argument. The comparison test is of no use because we only have $|e^{imx}f(x)|\leqslant\frac{M}{X}$ and $\int_{K}^{\infty}\frac{M}{x}dx$ diverges.

A much better method is to replace the semicircular contour by the rectangular contour in figure 13:

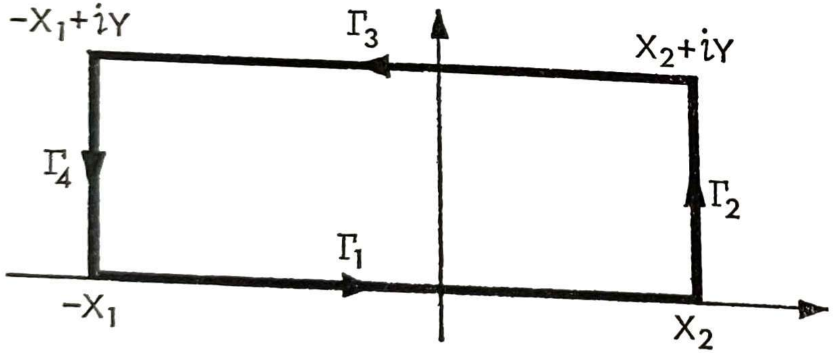

Figure 13

Initially we choose the rectangle large enough to contain all the singularities and such that $|f(z)|\leqslant\frac{M}{|z|}$ on Γ_2, Γ_3, Γ_4. If we show that $\int_{\Gamma_2}$, $\int_{\Gamma_3}$, $\int_{\Gamma_4}$ tend to zero, then

$\lim_{X_1, X_2\to\infty}\int_{-X_1}^{X_2} e^{imx}f(x)dx = 2\pi i$ (sum of residues of $e^{imz}f(z)$ in upper half plane).

In particular, allowing X_1 and X_2 tend to ∞ independently, we know that $\int_{-\infty}^{\infty} e^{imx}f(x)dx$ exists.

$$\left|\int_{\Gamma_2} e^{imz}f(z)dz\right| = \left|\int_0^Y e^{imX_2-mt}f(X_2+it)idt\right| \leqslant \int_0^Y e^{-mt}\frac{M}{X_2}dt \leqslant \frac{M}{X_2}$$

and similarly $\left|\int_{\Gamma_4} e^{imz}f(z)dz\right|\leqslant\frac{M}{X_1}$.

$$\left|\int_{\Gamma_3} e^{imz}f(z)dz\right| = \left|-\int_{-X_1}^{X_2} e^{imt-mY}f(t+iY)dt\right| \leqslant \int_{-X_1}^{X_2} e^{-mY}\frac{M}{Y}dt$$

$$\leqslant\frac{e^{-mY}}{Y}M(X_1+X_2).$$

For fixed X_1, X_2, let $Y\to\infty$, then $\frac{e^{-mY}}{Y}\to 0$ and so $\int_{\Gamma_3}\to 0$. Now let $X_1, X_2\to\infty$ then $\int_{\Gamma_2}, \int_{\Gamma_4}\to 0$, giving the required result.

EXAMPLE. $I = \int_{-\infty}^{\infty}\frac{xe^{imx}}{x^2+a^2}dx \qquad (a>0, m>0)$

The only singularity of the integrand in the upper half-plane is a simple pole at ia with residue

$$\lim_{z\to ia}\frac{(z-ia)ze^{imz}}{z^2+a^2} = \frac{iae^{-ma}}{2ia} = \frac{1}{2}e^{-ma}.$$

Hence $I = 2\pi i.\frac{1}{2}e^{-ma} = \pi i e^{-ma}$.
Taking real and imaginary parts

$$\int_{-\infty}^{\infty}\frac{x\cos mx}{x^2+a^2}dx = 0.$$

$$\int_{-\infty}^{\infty}\frac{x\sin mx}{x^2+a^2}dx = \pi e^{-ma}.$$

Since the second integrand is even, we have

$$\int_0^{\infty} \frac{x \sin mx}{x^2+a^2}\, dx = \tfrac{1}{2}\pi e^{-ma} \text{ (where } a>0, m>0 \text{ in each integral).}$$

5. Poles on the Real Axis

The methods of sections 3, 4 may be extended to the case where f has poles on the real axis. To accommodate these poles, we draw a small semicircle bypassing each of them and let the radius of each semicircle tend to zero. For example, if f has a pole at the origin, we integrate around one of the contours in figure 14.

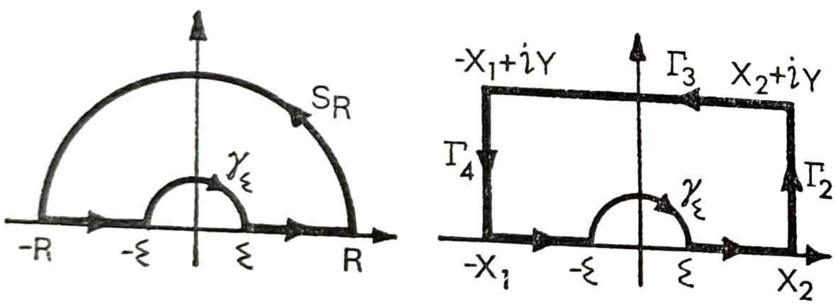

Figure 14

Letting $\varepsilon \to 0$ leads to the same problem as letting $R \to \infty$ in the previous sections. If f has a pole at x_0 where $a \leqslant x_0 \leqslant b$, define the Cauchy principal value of $\int_a^b f(x)dx$ to be

$$P\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\left\{\int_a^{x_0-\varepsilon} f(x)dx + \int_{x_0+\varepsilon}^b f(x)dx\right\}.$$

It may happen that $P\int_a^b f(x)dx$ exists but $\int_a^b f(x)dx$ does not. For example $P\int_{-1}^{1} \frac{1}{x}\, dx = 0.$

The above method of contour integration gives the Cauchy

principal value; we must then discuss the convergence of the integral.

EXAMPLE. $\int_{-\infty}^{\infty} \frac{e^{imx}}{x}\, dx \ (m>0).$

Using the second contour of figure 14, we find $\int_{\Gamma_2}, \int_{\Gamma_3}, \int_{\Gamma_4} \to 0$ and the integral along the real axis converges at infinity as in section 4. There are no poles of $\frac{e^{imz}}{z}$ inside the contour and so

$$\int_{-\infty}^{\varepsilon} \frac{e^{imx}}{x}\, dx + \int_{\varepsilon}^{\infty} \frac{e^{imx}}{x}\, dz + \int_{\gamma_\varepsilon} \frac{e^{mz}}{z}\, dz = 0 \qquad (1)$$

where γ_ε is the opposite contour to $z(t) = e^{it}$ $(0 \leqslant t \leqslant \pi)$ (i.e. γ_ε is the semicircle radius ε, described in the clockwise sense).

But $\frac{e^{imz}}{z} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{i^n m^n z^{n-1}}{n!} = \frac{1}{z} + g(z)$ where g is analytic and hence $g(z)$ is bounded by M, say, in a neighbourhood of zero. This gives $|\int_{\gamma_\varepsilon} g(z)dz| \leqslant M\pi\varepsilon$ and so

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} \frac{e^{imz}}{z}\, dz = \lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} \frac{1}{z}\, dz + \lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} g(z)dz$$

$$= \lim_{\varepsilon\to 0}\left\{-\int_0^{\pi} \frac{1}{\varepsilon e^{it}}\, i\varepsilon e^{it}dt\right\} + 0$$

$$= -i\pi.$$

Thus from equation (1)

$$P\int_{-\infty}^{\infty} \frac{e^{imx}}{x}\, dx = i\pi.$$

Equating real and imaginary parts,

$$P\int_{-\infty}^{\infty} \frac{\cos mx}{x}\, dx = 0, \quad P\int_{-\infty}^{\infty} \frac{\sin mx}{x}\, dx = \pi.$$

The first integral only exists as a Cauchy principal value because near zero $\frac{\cos mx}{x}$ behaves like $\frac{1}{x}$.

$$\text{But } P\int_{-\infty}^{\infty}\frac{\sin mx}{x}dx = \lim_{\varepsilon\to 0}\left\{\int_{-\infty}^{-\varepsilon}\frac{\sin mx}{x}dx+\int_{\varepsilon}^{\infty}\frac{\sin mx}{x}dx\right\}$$

$$= 2\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty}\frac{\sin mx}{x}dx.$$

Thus $\int_0^{\infty}\frac{\sin mx}{x}dx$ exists and equals† $\frac{\pi}{2}$. This also implies that $\int_{-\infty}^{\infty}\frac{\sin mx}{x}dx$ exists and equals π.

Note that the value of $\int_0^{\infty}\frac{\sin mx}{x}dx$ is independent of the value of m, provided that m is positive. (Compare this result with the example of the previous section as $a\to 0$.)

Clearly we have

$$\int_0^{\infty}\frac{\sin mx}{x}dx = \begin{cases}\frac{\pi}{2} & (m>0)\\ 0 & (m=0)\\ -\frac{\pi}{2} & (m<0)\end{cases}$$

This result is sometimes called *Dirichlet's discontinuous factor.*

6. Integrals using Periodic Functions

We can use the fact that e^z is periodic, satisfying $e^z = e^{z+2\pi i}$, to calculate certain integrals. We illustrate this with a particular case.

† By comparing this proof with one avoiding contour integration, the reader may see the power and elegance of this method. See W. Ledermann, *Integral Calculus*, p. 22, Example 6; p. 37, Example 5.

EXAMPLE. $\int_{-\infty}^{\infty}\frac{e^{ax}}{e^x+1}dx = \frac{\pi}{\sin \pi a}$ $(0<a<1)$.

Let $f(z) = \frac{e^{az}}{e^z+1}$ and integrate f around the contour in figure 15:

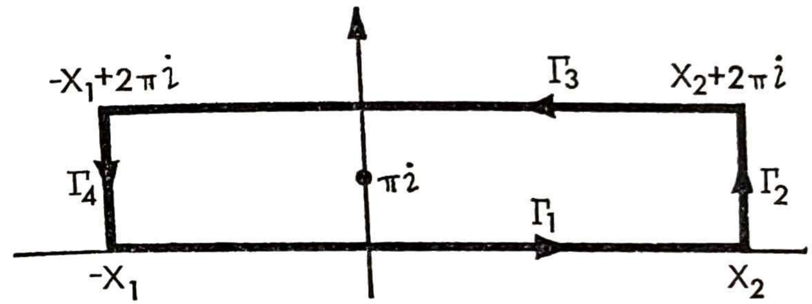

Figure 15

Note that

$$\int_{\Gamma_1} f(z)dz = \int_{-X_1}^{X_2}\frac{e^{ax}}{e^x+1}dx \tag{1}$$

and since Γ_3 is the opposite contour to $z(t) = t+2\pi i$ $(-X_1\leqslant t\leqslant X_2)$, we have

$$\int_{\Gamma_3} f(z)dz = -\int_{-X_1}^{X_2}\frac{e^{a(t+2\pi i)}}{e^{t+2\pi i}+1}dt = -e^{2\pi ai}\int_{-X_1}^{X_2}\frac{e^{ax}}{e^x+1}dx \tag{2}$$

Since f has only one singularity inside the rectangular contour, a simple pole at πi with residue $\frac{e^{a\pi i}}{e^{\pi i}} = -e^{i\pi a}$, we have

$$\int_{\Gamma_1} f(z)dz+\int_{\Gamma_2} f(z)dz+\int_{\Gamma_3} f(z)dz+\int_{\Gamma_4} f(z)dz = -2\pi i e^{i\pi a}.$$

Let $X_1\to\infty$, $X_2\to\infty$, then assuming $\int_{\Gamma_2}\to 0$, $\int_{\Gamma_4}\to 0$, we have from (1), (2)

$$(1-e^{2\pi a i})\int_{-\infty}^{\infty}\frac{e^{ax}}{e^x+1}\,dx = -2\pi i e^{i\pi a}$$

i.e.

$$\int_{-\infty}^{\infty}\frac{e^{ax}}{e^x+1}\,dx = \frac{-2\pi i e^{i\pi a}}{1-e^{2i\pi a}}$$

$$= \frac{2\pi i}{e^{i\pi a}-e^{-i\pi a}}$$

$$= \frac{\pi}{\sin \pi a}\,.$$

Thus to obtain the required result, it only remains to show that $\int_{\Gamma_2}, \int_{\Gamma_4} \to 0$. But on Γ_2 we have $z = X_2+it$ $(0 \leqslant t \leqslant 2\pi)$ and so

$$|f(z)| = \frac{|e^{a(X_2+it)}|}{|e^{X_2+it}+1|} = \frac{e^{aX_2}}{|e^{X_2+it}+1|} \leqslant \frac{e^{aX_2}}{e^{X_2}-1}$$

(since $|e^{X_2+it}+1| \geqslant |e^{X_2+it}|-1 = e^{X_2}-1$).

This gives $\left|\int_{\Gamma_2} f(z)dz\right| \leqslant \frac{e^{aX_2}}{e^{X_2}-1}.2\pi$

and this tends to zero as $X_2 \to \infty$ since $a<1$.

On Γ_4 we have $z = -X_1+it$ $(0 \leqslant t \leqslant 2\pi)$ and so

$$|f(z)| = \frac{|e^{a(-X_1+it)}|}{|e^{-X_1+it}+1|} \leqslant \frac{e^{-aX_1}}{1-e^{-X_1}}\,.$$

This gives $\left|\int_{\Gamma_4} f(z)dz\right| \leqslant \frac{e^{-aX_1}}{1-e^{-X_1}}\,.\,2\pi$

which tends to zero as $X_1 \to \infty$ because $a>0$.

Thus the value of the infinite integral is proved.

By substituting $t = e^x$, we find

$$\int_{-\infty}^{\infty}\frac{e^{ax}}{e^x+1}\,dx = \int_0^{\infty}\frac{t^a}{t+1}\,\frac{dt}{t}\,.$$

This gives†

$$\int_0^{\infty}\frac{t^{a-1}}{t+1}\,dt = \frac{\pi}{\sin \pi a} \quad (0<a<1).$$

7. Summation of Certain Series

The functions cot πz, cosec πz both have poles at $0, \pm 1, \pm 2, \ldots$ and so prove useful for summing series. If f is a function which is analytic at $z = n$, then $f(z)$ cosec πz has a simple pole there with residue

$$\lim_{z\to 0}(z-n)f(z)\operatorname{cosec}\pi z = \lim_{h\to 0}\frac{hf(n+h)}{\sin \pi(n+h)}$$

$$= \lim_{h\to 0}\frac{1}{\pi}\frac{\pi h}{(-1)^n\sin \pi h}f(n+h)$$

$$= \frac{(-1)^n f(n)}{\pi}\,.$$

Also $f(z)\cot \pi z = [f(z)\cos \pi z]\operatorname{cosec}\pi z$ has a simple pole at $z = n$ with residue $\frac{f(n)}{\pi}$.

Let S_N be the square with vertices $(N+\frac{1}{2})(\pm 1 \pm i)$ parametrized in the anticlockwise direction as in figure 16.

The contour S_N is chosen specifically because both cot πz and cosec πz are bounded on S_N. This requires some rather cumbersome calculations. First note that on the sides of S_N parallel to the real axis $z = x+iy$ where $|y| \geqslant \frac{1}{2}$, and on the other sides, $z = n+\frac{1}{2}+it$ where $n = \pm N$. If $z = n+\frac{1}{2}+it$ where $|y| \geqslant \frac{1}{2}$, then

† cf. W. Ledermann, *Integral Calculus*, pp. 64–67, where a proof of this result is given by real variable methods. It is of necessity very technical and this again illustrates the power of the theory of residues in those cases where it is applicable.

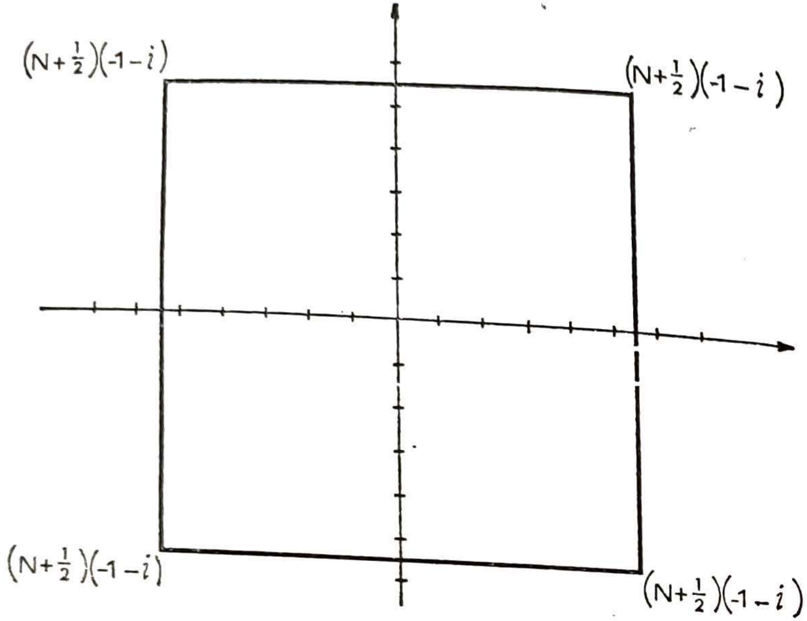

Figure 16

$$|\text{cosec } \pi z| = (\tfrac{1}{2}|e^{i\pi z}-e^{-i\pi z}|)^{-1} \leqslant (\tfrac{1}{2}||e^{i\pi z}|-|e^{-i\pi z}||)^{-1}$$

$$= (\tfrac{1}{2}|e^{-\pi y}-e^{\pi y}|)^{-1} = (\sinh|\pi y|)^{-1} \leqslant \left(\sinh\frac{\pi}{2}\right)^{-1}.$$

Also $$|\cot \pi z| = \left|\frac{\cos \pi z}{\sin \pi z}\right| = \left|\frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}}\right| \leqslant \left|\frac{|e^{i\pi z}|+|e^{-i\pi z}|}{|e^{i\pi z}|-|e^{-i\pi z}|}\right|$$

$$= \left|\frac{e^{-\pi y}+e^{\pi y}}{e^{-\pi y}-e^{\pi y}}\right| = \coth|\pi y| \leqslant \coth\frac{\pi}{2}.$$

If $z = n+\tfrac{1}{2}+it$, then

$$|\text{cosec } \pi z| = |\sin \pi z|^{-1} = |\cos i\pi t|^{-1} = (\cosh|\pi t|)^{-1} \leqslant 1,$$

and

$$|\cot \pi z| = |\tan it| = \left|\frac{1-e^{-2t}}{1+e^{-2t}}\right| \leqslant 1.$$

By Cauchy's Residue Theorem
$$\int_{S_N} f(z) \cot \pi z \, dz$$
$$= 2\pi i\{\text{sum of residues of } f(z) \cot \pi z \text{ inside } S_N\}.$$

If $|f(z)| \leqslant \dfrac{A}{|z|^2}$ for $|z| \geqslant R$ where A, R are positive constants, then $\int_{S_N} f(z) \cot \pi z \, dz \to 0$ as $N \to \infty$. This follows because $|\cot \pi z|$ is bounded on S_N i.e. $|\cot \pi z| \leqslant M$ and so

$$\left|\int_{S_N} f(z) \cot \pi z \, dz\right| \leqslant \frac{A}{N^2} M(8N+4)$$

which tends to zero as $N \to \infty$.

Hence if $|f(z)| \leqslant \dfrac{A}{|z|^2}$ for $|z| \geqslant R$, then as $N \to \infty$, the sum of the residues of $f(z) \cot \pi z$ inside S_N tends to zero. Using the fact that if f is analytic at $z = n$ then $f(z) \cot \pi z$ has residue $\dfrac{f(n)}{\pi}$ there, this allows us to sum a series involving $f(n)$.

EXAMPLE. $f(z) = \dfrac{1}{z^2}$.

At an integer $n \neq 0$, $z^{-2}\cot \pi z$ has a simple pole with residue $1/(n^2\pi)$. At the origin, as calculated on page 45, $z^{-2}\cot \pi z$ has a triple pole with residue $-\tfrac{1}{3}\pi$. Hence the sum of the residues of $f(z) \cot \pi z$ inside S_N is

$$\frac{1}{(-N)^2\pi}+\ldots+\frac{1}{(-1)^2\pi}+\left(-\frac{1}{3}\pi\right)+\frac{1}{1^2\pi}+\ldots+\frac{1}{N^2\pi}$$

$$= \frac{2}{\pi}\sum_{n=1}^{N}\frac{1}{n^2}-\frac{1}{3}\pi.$$

As $N \to \infty$, this tends to zero and so

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{1}{3}\pi = 0$$

$$\text{i.e.}\quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{6}\pi^2.$$

A similar calculation with cot πz replaced by cosec πz gives

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{1}{12}\pi^2.$$

EXERCISES ON CHAPTER THREE

1. Calculate the residues in the following cases:
   (i) $z^{-3}\sin^2 z$ $(z\neq 0)$, residue at $z = 0$.
   (ii) $\exp(1/z)$ $(z\neq 0)$, residue at $z = 0$.
   (iii) $e^z z^{-n-1}$ (n a positive integer, $z\neq 0$), residue at $z = 0$.
   (iv) $z^2(z^2+a^2)^{-3}$ $(a>0,\ z\neq \pm ia)$, residue at $z = ia$.
   (v) $(1+z^2+z^4)^{-1}\left(z\neq \exp\left(\frac{r\pi i}{3}\right)\ r = 1, 2, 4, 5\right)$, residue at $\exp\left(\frac{\pi i}{3}\right)$.

2. Show that $\displaystyle\int_0^{2\pi}\frac{\cos\theta}{2-\cos\theta}\,d\theta = 2\pi.$

3. Show that $\displaystyle\int_0^{\pi}\frac{a}{a^2+\sin^2 t}\,dt = \frac{\pi}{\sqrt{(1+a^2)}}$ $(a>0)$. (Hint: substitute $\theta = 2t$.)

4. If C is the unit circle $z(t) = e^{it}$ $(0\leqslant t\leqslant 2\pi)$, calculate by residues $\displaystyle\int_C e^z z^{-n-1}\,dz$ where n is a positive integer. Hence show that

$$\int_0^{2\pi}\exp(\cos t)\cos(nt-\sin t)\,dt = \frac{2\pi}{n!}$$

$$\int_0^{2\pi}\exp(\cos t)\sin(nt-\sin t)\,dt = 0.$$

5. Evaluate $\displaystyle\int_0^{\infty}\frac{dx}{1+x^2+x^4}$.

6. Evaluate $\displaystyle\int_0^{\infty}\frac{\cos mx}{x^2+a^2}\,dx$ $\quad(a>0,\ m>0)$.

7. Prove that $\displaystyle\int_0^{\infty}\frac{x^2}{(x^2+a^2)^3}\,dx = \frac{\pi}{16a^3}$ $\quad(a>0)$.

8. If $a>b>0$, $m>0$, prove that

$$\int_{-\infty}^{\infty}\frac{x^3\sin mx}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}(a^2e^{-ma}-b^2e^{-mb}).$$

9. Use the rectangle with vertices $-X_1$, X_2, $X_2+\pi i$, $-X_1+\pi i$ to show that $\displaystyle\int_{-\infty}^{\infty}\frac{e^{ax}}{\cosh x}\,dx = \frac{\pi}{\cos\frac{1}{2}\pi a}$ $\quad(-1<a<1)$.

10. Prove that $\displaystyle P\int_{-\infty}^{\infty}\frac{\cos x}{a^2-x^2}\,dx = \frac{\pi\sin a}{a}$ $\quad(a>0)$.

11. Show that $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{1}{12}\pi^2$.

12. Show that $\displaystyle\left(\frac{1}{\xi-z}+\frac{1}{z}\right)\cot\pi z$ has poles at every integer and at ξ. Find the residues at these points when ξ is not an integer and in this case show that $\displaystyle\pi\cot\pi\xi = \frac{1}{\xi}+\sum_{n=1}^{\infty}\frac{2\xi}{\xi^2-n^2}$.

CHAPTER FOUR

Analytic Continuation and Riemann Surfaces

1. Analytic Continuation

We now return to a topic discussed at the end of *Functions of a Complex Variable I* which will allow us to describe 'many-valued functions' in terms of (single-valued) functions. This is of interest when discussing contour integration because if f is analytic in a domain D and z_0 is a fixed point in D, then the integral of f along a contour γ from z_0 to an arbitrary point z in D depends in general on the choice of γ and so is in a sense 'many-valued'.

Recall† that if f and g are analytic functions defined in the same domain D and $f(z) = g(z)$ for all z in some non-empty open subset of D, then $f(z) = g(z)$ throughout the whole of D. It is this constraint on analytic functions, which forces two analytic functions to be equal everywhere in their joint domain of definition when they are only assumed equal on a small part, which leads to the results which we now explain.

Suppose that f_1 is defined in a domain D_1 and f_2 is defined in a domain D_2 where D_1 and D_2 overlap.

Under these conditions we say that f_2 is a *direct analytic continuation* of f_1 from D_1 to D_2. Of course if f_1 is analytic in D_1 and we are simply given the overlapping domain D_2, then we cannot be certain that a direct analytic continuation to D_2 exists. However if f_2 exists, then it is unique, for suppose that g is an alternative direct analytic continuation of f_1 to the domain D_2, then $g(z) = f_1(z) = f_2(z)$ for every point common to D_1 and D_2. But this set of points is a non-empty subset of

† *Functions of a Complex Variable I*, p. 62.

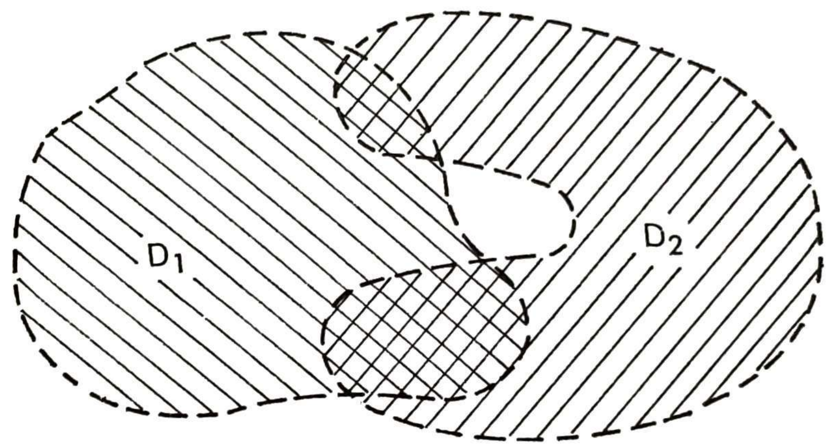

Figure 17

D_2 and is also open. (For if z lies in both D_1, D_2 then since D_1 is open there is an ε_1-neighbourhood of z lying completely in D_1. Similarly there is an ε_2-neighbourhood of z contained in D_2 and if ε is the smaller of ε_1, ε_2 then the ε-neighbourhood of z lies in the overlap of D_1 and D_2 which shows that this overlap is open.) Hence $g(z) = f_2(z)$ throughout D_2.

The notion of direct analytic continuation is most often used when D_2 contains D_1. Here we begin with an analytic function f_1 in D_1 and try to find an analytic function f_2 defined on the larger domain D_2 which equals f_1 on D_1. This idea of extending the domain on which an analytic function is defined was discussed in *Functions of a Complex Variable I*, pages 60-62.

EXAMPLE 1. $f(z) = \sum_{n=0}^{\infty}(-1)^n z^{2n} \quad |z|<1.$

The function $(1+z^2)^{-1}$ is defined and analytic for $z \neq \pm i$ and equals $\sum_{n=0}^{\infty}(-1)^n z^{2n}$ for $|z|<1$. Hence $(1+z^2)^{-1}$ is a direct analytic continuation to the domain consisting of all points

E

except $\pm i$. Evidently there is no direct analytic continuation to the whole plane because $(1+z^2)^{-1}$ has poles at $\pm i$ and so cannot be analytic there.

Sometimes, given an analytic function f defined in a domain D, we cannot continue f analytically outside D. In this case the boundary of D is called a *natural boundary*.

EXAMPLE 2. The series $f(z) = 1+z+z^2+z^4+\ldots+z^{2^n}+\ldots$ is convergent for $|z|>1$. The unit circle $|z| = 1$ is a natural boundary. If $w^{2^m} = 1$, then we can show that $f(z)$ does not tend to a finite limit as z approaches w from inside the unit circle. Let $z = rw$ where $0<r<1$, then

$$f(z) = 1+z+z^2+\ldots+z^{2^{m-1}}+\sum_{n=m}^{\infty} z^{2^n}$$
$$= f_1(z)+f_2(z).$$

We have $\lim_{r\to 1} f_1(rw) = 1+w+w^2+\ldots+w^{2^{m-1}}$. But since $w^{2^m} = 1$, the series $f_2(rw) = \sum_{n=m}^{\infty} r^{2^n}$ is a series of real, positive terms for $0<r<1$. Hence $f_2(rw) > \sum_{n=m}^{m+N} r^{2^n}$. But $\sum_{n=m}^{m+N} r^{2^n} \to N+1$ and so for some $\varepsilon>0$, if $1-\varepsilon<r<1$ then $\sum_{n=m}^{m+N} r^{2^n} > \frac{1}{2}N$. This gives $f_2(rw) > \frac{1}{2}N$ and since N is arbitrary, $f_2(rw) \to +\infty$ as $r \to 1$.

Thus f cannot be analytically continued into any domain containing w where $w^{2^m} = 1$. But if a domain D_2 crosses the circle $|z| = 1$, then it includes a segment of the circle. The roots of $z^{2^m} = 1$ are $\exp\left(\frac{2\pi iq}{2^m}\right)$ where $q = 1, \ldots, 2^m$. These are spaced at equal intervals around the unit circle. By choosing m large enough, some point w where $w^{2^m} = 1$ lies in the segment of the circle inside D_2. Thus f cannot be analytically continued across $|z| = 1$.

In some cases the process of direct analytic continuation may be repeated. Given a function f_1 defined in a domain D_1, we may find a direct analytic continuation f_2 to a domain D_2 where D_1 and D_2 overlap. Then we may find a direct analytic continuation f_3 of the function f_2 to a domain D_3 where D_2 and D_3 overlap. After a finite number of steps we find a direct analytic continuation f_n of f_{n-1} from D_{n-1} to D_n. In this case, f_n is called an *indirect* analytic continuation to the domain D_n of the function f_1 defined in D_1. We refer to both direct and indirect analytic continuations simply as 'analytic continuations'. Any two analytic continuations of a given function are evidently analytic continuations of each other.

The theory of indirect analytic continuation is much more complicated than direct continuation. The main problem is that it need no longer be unique. This is because we might use a different sequence of domains linking D_1 to D_n.

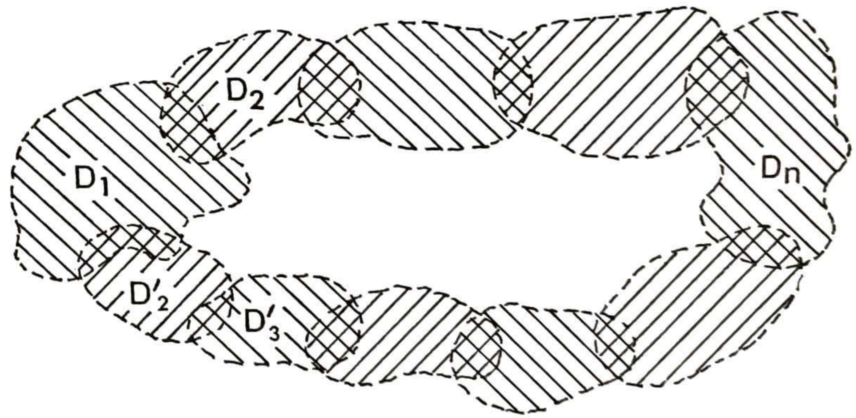

Figure 18

For example we might eventually return to the original domain and have $D_1 = D_n$, but find the indirect continuation f_n different from the original function f_1. We define the *complete analytic function* to consist of the original function and all its possible analytic continuations. In the case where we have different analytic continuations to some domain, the complete analytic function is called *multiform*, otherwise it is called *uniform*. Examples 1, 2 are uniform.

If no analytic continuation can be defined at a point z_0, then z_0 is said to be a *singularity* of the complete analytic function. In example 1, the points $z = \pm i$ are singularities and in example 2 all the points $|z| \geqslant 1$ are singularities.

Note that a multiform complete analytic function is in a sense 'many-valued', but we have formulated it as a collection of (single-valued) functions. Two functions in the collection may have different values in the same domain, but they are analytic continuations of each other.

EXAMPLE 3. The logarithm is multiform. For any integer k we can define $\log_k z$ in the cut-plane by

$$\log_k z = \log |z| + i\,(\arg z + 2\pi k)$$

where $-\pi < \arg z < \pi$. In particular, for $k = 0$, we have the principal value $\operatorname{Log} z = \log_0 z$. We will show $\log_k$ is an analytic continuation of Log.

Let D_n be the half-plane given by $z = re^{i\theta}$ where $r > 0$, $(n-2)\frac{\pi}{2} < \theta < \frac{n\pi}{2}$. Note that $D_{n+4} = D_n$ for every integer n and D_1, D_2 are as in figure 19.

If z is in D_n, write $z = re^{i\theta}$ where $(n-2)\frac{\pi}{2} < \theta < \frac{n\pi}{2}$ and define

$$f_n(z) = \log r + i\theta.$$

Arguing as for $\operatorname{Log} z$ in the cut-plane, $f_n(z)$ may be seen to be analytic in the domain D_n. If $z = re^{i\theta}$ is in D_n and D_{n+1}, then $f_n(z) = f_{n+1}(z)$ and so f_{n+1} is the direct analytic continuation of f_n from D_n to D_{n+1}. By induction, f_m is an analytic continuation of f_n from D_n to D_m for any m and n. In particular, in the domain $D_{n+4} = D_n$, we see that $f_{n+4}(z) = f_n(z) + 2\pi i$ is an analytic continuation of $f_n(z)$. The function $\log_k$ defined in the

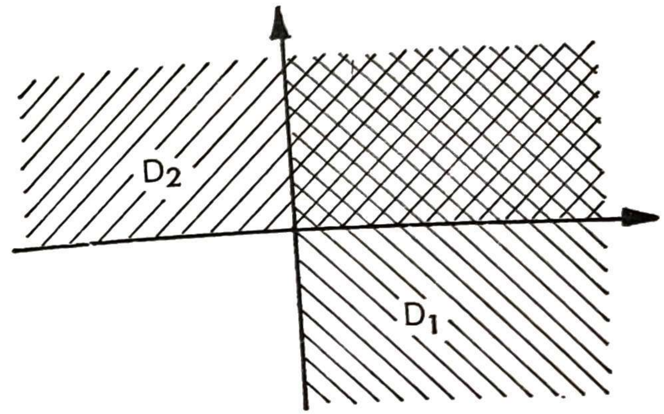

Figure 19

cut-plane coincides in D_1 with the function f_{4k+1}. Thus f_{4k+1} is trivially a direct analytic continuation of $\log_k$. If we start with $\operatorname{Log} = \log_0$ in the cut-plane, we find a chain of analytic continuations, f_1 in D_1, f_2 in D_2, . . . , f_{4k+1} in $D_{4k+1} = D_1$ and finally $\log_k$ in the cut-plane, showing that $\log_k$ is an analytic continuation of Log in the cut-plane.

Note that f_{k+3} is defined in $D_3\left(\text{the half-plane } z = re^{i\theta}, r > 0, \frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$ and D_3 includes all the points on the negative real axis except the origin. Hence the analytic continuation f_{k+3} of Log is defined on the negative real axis except the origin. Thus the only singularity of the complete analytic function can be at the origin. By analytically continuing via a set of domains round the origin we obtain different analytic continuations. In general a singularity with this property is called a *branch point*.

We now consider multiform examples which appear naturally in contour integration.

If f is analytic in a domain D, fix a point z_0 in D and consider the integral of f along a contour γ in D from z_0 to an arbitrary point z. We know that if f has a primitive F in D (i.e. $F' = f$), then the value of this integral is $F(z) - F(z_0)$. In general such a primitive does not exist. However we may subdivide into subcontours $\gamma_1, \ldots, \gamma_n$ such that each subcontour γ_r lies in an open disc D_r which is itself contained in D. (The proof of this in the general case requires a technique which we have not developed, but in particular cases its truth should be fairly evident.)

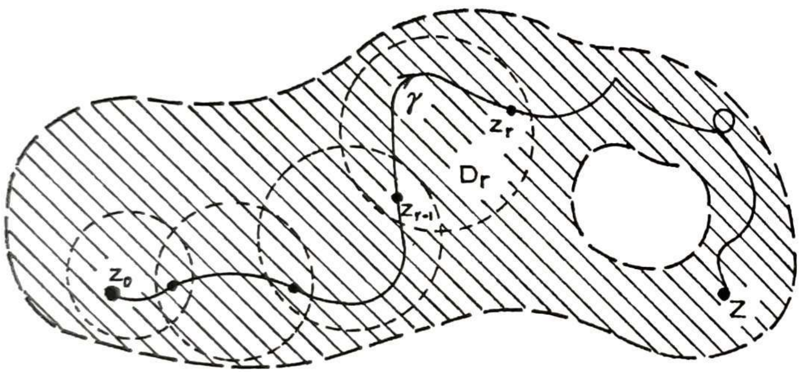

Figure 20

Now in a disc an analytic function *does*† have a primitive which is unique up to an additive constant. Let F_r be a primitive for f in D_r ($r = 1, \ldots, n$). By definition $F_r' = f$ in D_r and $F_{r+1}' = f$ in D_{r+1} and so $F_r' - F_{r+1}' = 0$ in the overlap. But the overlap of two circles is a domain and so

$$F_{r+1}(z) = F_r(z) + \text{constant}$$

for all z in both D_r and D_{r+1}. By adding a suitable constant to each of $F_2, F_3, \ldots, F_n$ in turn, we may suppose that $F_{r+1} = F_r$ in the overlap of D_r and D_{r+1} for $r = 1, 2, \ldots, n-1$. This yields an example of analytic continuation.

† *Functions of a Complex Variable I*, p. 47.

Let the initial and final points of γ_r be z_{r-1}, z_r then by the Fundamental Theorem of Contour Integration,

$$\int_{\gamma_r} f(z)dz = F_r(z_r) - F_r(z_{r-1}). \tag{1}$$

Since z_{r-1} lies in the overlap of D_{r-1} and D_r, we have $F_{r-1}(z_{r-1}) = F_r(z_{r-1})$. Adding up the integrals along the subcontours and cancelling $F_{r-1}(z_{r-1}) - F_r(z_{r-1})$ for $r = 2, \ldots, n$, we find

$$\int_\gamma f(z)dz = F_n(z_n) - F_1(z_0). \tag{2}$$

Of course if f had a primitive F throughout D then in particular $F_1' = F'$ in D_1. Adding a constant if necessary, we may suppose that $F_1 = F$ restricted to D_1. By successive direct analytic continuations, we then find that $F_r = F$ restricted to D_r for $r = 1, \ldots, n$ and so (2) reduces to the Fundamental Theorem $\int_\gamma f(z)dz = F(z_n) - F(z_0)$.

However, if f has an isolated singularity in D with non-zero residue ρ, then selecting a closed Jordan contour γ in D winding once anticlockwise round this singularity, we find

$$\int_\gamma f(z)dz = 2\pi i\rho. \tag{3}$$

Since γ is closed, $z_0 = z_n$ and from (2), (3), $F_n(z_0) = F_1(z_0) + 2\pi i\rho$. Hence F_1, F_2 are *not* equal and we have an example which is multiform. The isolated singularity of f gives a branch point of the complete analytic function found by continuing the primitive F_1.

2. Riemann Surfaces

The notion of analytic continuation explained in the last section is quite difficult for the beginner to grasp. In particular it is difficult to visualize an overall picture of what is going on. This total view of the situation is best described by using the idea of a 'Riemann surface'. We will illustrate this concept by two particular examples, first considering the case of the logarithm.

If $z = e^w$, then all the solutions for w in terms of z (where $z \neq 0$) are given by $w = \log|z| + i(\arg z + 2\pi k)$ where $-\pi < \arg z \leqslant \pi$, and k is an integer. Restricting ourselves to the principal value given by $k = 0$ in the cut-plane, we have an analytic function and in the last section we saw that we could recover all the other values by analytic continuation. Each time we pass round the origin in the anti-clockwise direction the value of $w = \log z$ is increased by $2\pi i$.

We now describe another method of looking at this phenomenon by introducing a Riemann surface. It will have the advantage that we obtain a single-valued function which takes all the values of the logarithm but this function will now be defined on the Riemann surface and not on the complex plane.

Consider the complex plane to be covered by an infinite number of superimposed transparent sheets (each sheet covers the whole plane). From every sheet remove the origin and imagine a cut being made along the negative real axis in such a way that this axis is considered to be affixed to the upper part of the cut. Now smoothly join the negative real axis of the upper part of the cut on each sheet to the lower part of the cut on the sheet above. If we mark a point on one of the sheets and imagine it to move over the cut in the anti-clockwise direction then, because of the smooth join, we suppose that it moves on to the next sheet above. This means that if the superimposed sheets were pulled apart and viewed from the side, then the system would look rather like an infinite winding staircase. This system of sheets is called the *Riemann surface* of the logarithm.

Looking at the Riemann surface from above, since the sheets are transparent, marking a point on one of them represents a non-zero complex number. However, given two real numbers r, θ where $r > 0$ and $(2k-1)\pi < \theta \leqslant (2k+1)\pi$, then by numbering

the sheets in ascending order we can suppose that the pair of numbers r, θ gives the point on the k^{th} sheet which represents the complex number $re^{i\theta}$. Thus the Riemann surface may be considered to have the advantage of distinguishing between the symbols $re^{i(\theta+2\pi k)}$, $k = 0, \pm 1, \pm 2, \ldots$, which are equal as complex numbers, but lie vertically above one another on different sheets of the Riemann surface.†

Define the logarithm on the Riemann surface by

$$\log P = \log|z| + i(\arg z + 2\pi k)$$

where P is the point on the k^{th} sheet representing the complex number z. Alternatively, if $z = re^{i\theta}$ where $r > 0$, $(2k-1)\pi < \theta \leqslant (2k+1)\pi$, then

$$\log P = \log r + i\theta.$$

Note that the logarithm is a single-valued function on the Riemann surface. It is also continuous, in the intuitive sense that as a point P tends to P_0, then $\log P$ tends to $\log P_0$ (even when P moves over the cut from one sheet to the next).

We can now begin to see what happens when we analytically continue some analytic, single-valued choice of the logarithm. To do this we just look at the corresponding situation on the Riemann surface.

We first remark that if we are given an analytic function f defined in a domain D where $f(z)$ is always a logarithm of z, then this gives us a rule to choose a 'domain' on the Riemann surface which corresponds to the domain D in the complex plane. This is because f is a choice of logarithm and so the imaginary part of $f(z)$ is a particular choice θ for an argument of z. For each point z in D we then select the point P on the Riemann surface which represents $z = re^{i\theta}$ on the k^{th} sheet where $(2k-1)\pi < \theta \leqslant (2k+1)\pi$. (This construction does no

† This may be considered in three-dimensional space as the surface given parametrically by $(r \cos \theta, r \sin \theta, \theta)$ where $r > 0$. The k^{th} sheet is given by $(2k-1)\pi < \theta \leqslant (2k+1)\pi$.

more than choose the point P on the appropriate sheet according to the actual value of θ as previously described.) Since f is analytic, its imaginary part is continuous i.e. θ depends continuously on z and if we imagine z moving continuously about in D, the corresponding point P moves continuously about in the corresponding 'domain'.

Now suppose that we analytically continue f outside the domain D. As we move successively from one domain to an overlapping one, we imagine the corresponding movement on the Riemann surface. If the set of domains used wanders round the origin then the corresponding set of domains on the Riemann surface passes round the 'winding staircase', possibly moving up or down to a different level. This possibility of arriving at another level gives a clear geometrical picture of why we can get different analytic continuations into a given domain by choosing alternative routes. The notion of analytic continuation described in section 1 is just a 'flattened version' in the complex plane of what is happening on the Riemann surface.

We can represent other 'many-valued functions' as single-valued functions on Riemann surfaces. In general, an 'n-valued function' requires n sheets. We illustrate this by considering $z^{\frac{1}{2}}$. This requires two sheets each with the origin removed and cut along the negative real axis. If $z = re^{i\theta}$ where $r>0$, $-\pi<\theta\leqslant\pi$, then choose $z^{\frac{1}{2}} = r^{\frac{1}{2}}e^{\frac{1}{2}i\theta}$ on the first sheet and $z^{\frac{1}{2}} = r^{\frac{1}{2}}e^{\frac{1}{2}i(\theta+2\pi)} = -r^{\frac{1}{2}}e^{\frac{1}{2}i\theta}$ on the second. As a point moves over the cut in the anti-clockwise direction, it passes from the first sheet to the second and after a complete circuit round the origin again, when it crosses the cut again, it passes from the second sheet back to the first. The Riemann surface is found by taking the two sheets in figure 21 and joining together the sides of the negative real axis marked '+' and those marked '−'.

This construction can only be performed in an idealized situation since it is not possible to physically cut two actual

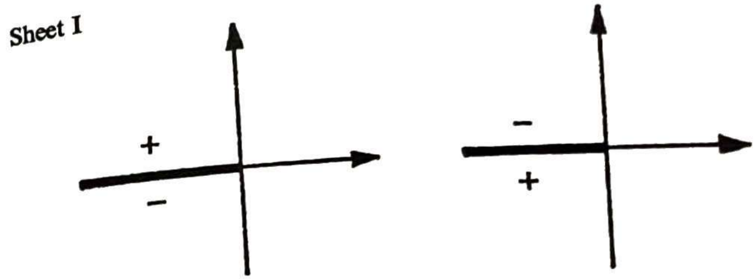

Figure 21

sheets and glue them together in the manner described without unwanted self-intersections. (After glueing '+' to '+', for example, it is not possible to fix '−' to '−' without passing through the glued sheets.) However by a stretch of the imagination using figure 21 it should be possible to visualize the idealized concept. This brings us to a fitting point to end the discussion as the mind grapples with an idea beyond the confines of three dimensional existence.

EXERCISES ON CHAPTER FOUR

Find analytic continuations of the following power series:

1. $\sum_{n=1}^{\infty} (-1)^n z^n \quad |z|<1$
2. $\sum_{n=0}^{\infty} z^{3n} \quad |z|<1.$
3. $\sum_{n=0}^{\infty} 3nz^{3n-1} \quad |z|<1.$
4. What happens when we look for the analytic continuations of $\sum_{n=1}^{\infty} (-1)^{n-1}(z^n/n)$ outside the disc $|z|<1$?
5. Show that $|z| = 1$ is a natural boundary for $\sum_{n=0}^{\infty} z^{n!} \quad |z|<1.$

6. Suppose that γ_1 is a contour from $-i$ to i which does not meet the negative real axis or the origin and γ_2 is a contour from i to $-i$ which does not meet the positive real axis or the origin. Let γ be the closed contour composed of γ_1 followed by γ_2. Show that $\int_\gamma 1/z\, dz = 2\pi i$.

7. For $z \neq 0$, write $z = re^{i\theta}$. Let D be the cut-plane given by $r > 0$, $-\pi < \theta < \pi$ and let D_n be the half-plane $(n-2)\frac{\pi}{2} < \theta < n\frac{\pi}{2}$. By successive direct analytic continuations from D to D_1, from D_1 to D_2, from D_2 to D_3, from D_3 to D_4, from D_4 to $D_5 = D_1$, and from D_1 back to D, show that $-z^{\frac{1}{2}}$ is an indirect analytic continuation of $z^{\frac{1}{2}}$ in D (where $z^{\frac{1}{2}} = r^{\frac{1}{2}}e^{\frac{1}{2}}$, $r > 0$, $-\pi < \theta < \pi$).

Describe the Riemann surfaces for the following 'many-valued functions':

8. $z^{\frac{1}{3}}$. 9. $z^{\frac{1}{n}}$ for a positive integer n. 10. $(z-1)^{-\frac{1}{2}}$.

Solutions to Exercises

Chapter One

1. (i) $w_1(t) = e^t$ $(-1 \leqslant t \leqslant 1)$, $w_2(t) = e^{t(1+i)}$ $(-1 \leqslant t \leqslant 1)$, angle between curves is $\arg w_1'(0) - \arg w_2'(0) = \arg 1 - \arg(1+i) = \frac{\pi}{4}$, similarly for (ii), (iii).
2. $w_1(t) = t^n$ $(0 \leqslant t \leqslant 1)$, $w_2(t) = t^n e^{in\alpha}$ $(0 \leqslant t \leqslant 1)$. The first has track given by $y = 0$, $0 \leqslant x \leqslant 1$, the second is the line segment $y = x \tan n\alpha$ from $(0, 0)$ to $(\cos n\alpha, \sin n\alpha)$. These two lines are at an angle $n\alpha$.
3. $c(x^2+y^2)+x = 0$, circles touching imaginary axis at the origin, $k(x^2+y^2)+y = 0$, circles touching real axis at the origin.
5. $ax^3-3dx^2y-3axy^2+dy^3$. $f(z) = (a+id)z^3+ik$, (k real).
6. $(2+i)\sin z+(1+2i)z^2+ik$, (k real).

Chapter Two

1. $z^{-5}+z^{-2}+\dfrac{z}{2!}+\ldots+\dfrac{z^{3n-5}}{n!}+\ldots$ $(z \neq 0)$ pole of order 5.
2. $\dfrac{1}{2a(z-a)}-\dfrac{1}{4a^2}+\ldots+\dfrac{(-1)^{n+1}(z-a)^n}{(2a)^{n+2}}+\ldots$ $(0 < |z-a| < 2a)$ simple pole.
3. $\dfrac{1}{z}-\dfrac{1}{2!z^3}+\ldots+\dfrac{(-1)^n}{(2n)!z^{2n+1}}+\ldots$ $(z \neq 0)$ essential singularity.
4. $\operatorname{Log}\left(\dfrac{z+z^2}{z-1}\right) = \operatorname{Log}(1+z)-\operatorname{Log}\left(1-\dfrac{1}{z}\right)$
$$= \left(z-\frac{z^2}{2}+\ldots+(-1)^{n+1}\frac{z^n}{n}+\ldots\right)+\left(\frac{1}{z}+\frac{1}{2z^2}+\ldots+\frac{1}{nz^n}+\ldots\right)$$
$(0 < |z| < 1)$ essential singularity.

5. $\frac{1}{12z}-\frac{2z}{6!}+\dots+\frac{(-1)^n 2z^{2n-5}}{(2n)!}+\dots$ $(z\neq 0)$ simple pole.

6. $-(z-1)^{-1}-1-(z-1)-\dots-(z-1)^n-\dots$ $(0<|z-1|<1)$ simple pole.

7. (a) pole of order 3 $\left(\text{since } \lim_{z\to 0} z^3 \frac{e^z}{z\sin^2 z}=1\right)$

(b) essential singularity (since $n\pi$ is a singularity for every integer n).

8. (a) simple pole $\left(\text{since } \lim_{z\to 0} z\frac{z}{1-\cos z}=\lim_{z\to 0}\frac{4(\frac{1}{2}z)^2}{2\sin^2(\frac{1}{2}z)}=2\right)$.

(b) essential singularity (since $(2n+\frac{1}{2})\pi$ is a singularity for every integer n).

9. (a) isolated essential singularity. (b) pole of order 2.

10. (a) essential singularity. (b) removable singularity.

11. (a) pole of order 3. (b) essential singularity.

12. Use $g(z)=e^z$, $f(z)=-\alpha z^n$ in Rouché's Theorem.

Chapter Three

1. (i) 1 (ii) 1 (iii) $\frac{1}{n!}$ (iv) $\frac{-i}{16a^3}$ (v) $(-3+i\sqrt{3})^{-1}$.

5. $\frac{\pi\sqrt{3}}{6}$. 6. $\frac{\pi e^{-ma}}{2a}$.

12. residue of $\left(\frac{1}{\xi-z}+\frac{1}{z}\right)\cot\pi z$ at $n\neq 0$ is $\left(\frac{1}{\xi-n}+\frac{1}{n}\right)/\pi$, at the origin it is $1/\pi\xi$ and at ξ it is $\cot\pi\xi$.

Chapter Four

1. $(1+z)^{-1}$ $z\neq -1$ 2. $(1-z^3)^{-1}$ $z\neq 1, e^{2\pi i/3}, e^{4\pi i/3}$

3. $3z^2(1-z^3)^{-2}$ $z\neq 1, e^{2\pi i/3}, e^{4\pi i/3}$ (hint: differentiate $(1-z^3)^{-1}$)

4. $\sum(-1)^{n-1}(z/n)=\text{Log}(1+z)$ $|z|<1$. Indirect analytic continuation gives all the values of the logarithm of $1+z$ (where $z\neq -1$).

5. If $z_0^m=1$, then $\lim_{z\to z_0}\sum z^{n!}$ does not exist by a proof analogous to that for $\sum z^{2^n}$ given in the text. Any domain crossing $|z|=1$ contains such a point.

6. $\text{Log } z=\log|z|+i\arg z$ $(-\pi<\arg z<\pi)$ is analytic in the cut-plane with the negative real axis including the origin removed. $\frac{d}{dz}\left(\text{Log } z\right)=1/z$. $\int_{\gamma_1}1/z\,dz=\text{Log } i-\text{Log}(-i)=\pi i$. Similarly $\log_* z=\log|z|+i\arg_* z$ $(0<\arg_* z<2\pi)$ is analytic in the cut-plane with the positive real axis and the origin removed. Here $\frac{d}{dz}(\log_* z)=1/z$ and $\int_{\gamma_2}1/z\,dz=\log_*(-i)-\log_* i=\pi i$. Hence $\int_\gamma 1/z\,dz=\pi i+\pi i=2\pi i$. (Remark: Any closed contour γ not passing through the origin satisfies $\int_\gamma 1/z\,dz=2n\pi i$ where n is an integer. The integer n is the number of times γ winds round the origin. Try to visualize this by considering the situation on the Riemann surface for the logarithm.)

7. $z^{\frac{1}{2}}=r^{\frac{1}{2}}e^{\frac{1}{2}i\theta}$ $\left(-\frac{\pi}{2}<\theta<\frac{\pi}{2}\right)$ in D_1, continuation $r^{\frac{1}{2}}e^{\frac{1}{2}i\theta}$ $(0<\theta<\pi)$ in D_2, $r^{\frac{1}{2}}e^{\frac{1}{2}i\theta}$ $\left(\frac{\pi}{2}<\theta<\frac{3\pi}{2}\right)$ in D_3, $r^{\frac{1}{2}}e^{\frac{1}{2}i\theta}$ $(\pi<\theta<2\pi)$ in D_4, $r^{\frac{1}{2}}e^{\frac{1}{2}i\theta}$ $\left(\frac{3\pi}{2}<\theta<\frac{5\pi}{2}\right)$ in $D_5=D_1$. Replacing θ by $\theta+2\pi$, this may be re-written as $r^{\frac{1}{2}}e^{\frac{1}{2}i\theta+i\pi}$ $\left(-\frac{\pi}{2}<\theta<\frac{\pi}{2}\right)$ in D_1. But $r^{\frac{1}{2}}e^{\frac{1}{2}i\theta+i\pi}$ $=-r^{\frac{1}{2}}e^{\frac{1}{2}i\theta}$ which has continuation $-z^{\frac{1}{2}}=-r^{\frac{1}{2}}e^{\frac{1}{2}i\theta}$ $(-\pi<\theta<\pi)$ in D.

8, 9. Take n sheets cut along the negative real axis and join the upper part of the cut on sheet r to the lower part of the cut on sheet $r+1$ for $r=1, 2, \dots, n-1$ and join the upper part on sheet n to the lower part on sheet 1.

10. As for the Riemann surface of $z^{\frac{1}{2}}$ but with the cut on each sheet along the negative real axis through the origin as far as $z=1$.

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