# Duality Between the Weak and Strong Interaction Limits for Randomly Interacting Fermions

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We establish the existence of a duality transformation for generic models of interacting fermions with two-body interactions. The eigenstates at weak and strong interaction $U$ possess similar statistical properties when expressed in the $U = 0$ and $U = \infty$ eigenstates bases, respectively. This implies the existence of a duality point $U_d$ where the eigenstates have the same spreading in both bases. $U_d$ is surrounded by an interval of finite width which is characterized by a non-Lorentzian spreading of the strength function in both bases. Scaling arguments predict the survival of this intermediate regime as the number of particles is increased.

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In noninteracting systems, many-body fermionic states are totally antisymmetrized products of one-body states, so-called Slater determinants. For strongly interacting fermions, on the other hand, the eigenstates expressed in the basis of Slater determinants have a large number of nonzero components, since then the one-orbital occupation is, in general, no longer a good quantum number. As the interaction is turned on, the number of such components increases, and this crossover from the weak to the strong interaction regime has been investigated both from the point of view of the statistical properties of the spectrum and the eigenstates [1–6]. This crossover determines the threshold in excitation energy above which Dyson's random matrix theory applies for atomic levels (as, e.g., in the Ce atom) [7] or nuclear energy levels (as obtained in neutron scattering experiments) [8] and below which, for example, quasiparticle excitations in chaotic quantum dots are well defined [9].

In this article, we investigate the reversed crossover from strong to weak interaction. Relying both on analytical arguments and exact numerical diagonalization of systems with few particles, we show that the spreading of the eigenstates over the strong interaction basis, as the interaction is decreased, is similar to the spreading over the weak noninteracting basis as the interaction is increased. This results in a duality transformation between the strong and weak interaction limits. The fixed (dual) point of this transformation lies inside a finite-width intermediate regime, which is characterized by a non-Lorentzian spreading and a maximal complexity (which we define below through the structural entropy) of the eigenstates in both bases. Scaling arguments predict the survival of this intermediate regime as the number of particles increases. In this regime, eigenstates properties cannot be extracted perturbatively from any of the asymptotic limits, but are well captured by our method. The eigenstate structure suggests the existence of quasiparticle excitations in the strong interaction limit, which we were, however, unable to determine.

We consider the deformed two-body random ensemble (TBRE) for $n$ interacting spinless fermions [10–12]:

$$H = H_0 + UH_1 = \Lambda\Big(\sum \epsilon_\alpha c_\alpha^\dagger c_\alpha + U \sum Q_{\alpha,\beta}^{\gamma,\delta} c_\alpha^\dagger c_\beta^\dagger c_\delta c_\gamma\Big). \quad (1)$$

The $m$ different one-body energies are distributed as $\epsilon_\alpha \in [-m/2; m/2]$ so as to fix the mean level spacing $\Delta \equiv 1$. The interaction matrix elements $Q_{\alpha,\beta}^{\gamma,\delta}$ are independent random variables with a zero-centered Gaussian distribution of unit variance. The parameter $\Lambda = \Delta/(U + \Delta)$ has been introduced to keep the density of states roughly constant as $U$ varies. In the $U = 0$ eigenstate basis, the Hamiltonian is represented by a $N \times N$ matrix of size $N = \binom{m}{n}$ with $K = 1 + n(m-n) + n(n-1)(m-n)(m-n-1)/4$ nonzero matrix elements per row. For a sufficiently large number of particles, the many-body density of states (DOS) is well approximated by a Gaussian of width $B_n \sim \sqrt{K}\Delta$ [11,12]. Below, we will investigate the properties of levels in the middle of the DOS.

In the $U = 0$ basis [superscripts (0) and ($\infty$) indicate the corresponding basis], the eigenstates structure is conveniently described by the strength function (SF):

$$\rho^{(0)}(E) = \Big\langle \sum_A |\psi_A(I)|^2 \delta(E + E_I - E_A) \Big\rangle_I, \quad (2)$$

and the inverse moments,

$$\xi_p^{(0)} = \Big\langle \Big( \sum_I |\psi_A(I)|^{2p} \Big)^{-1} \Big\rangle_A, \quad (3)$$

where the averages $\langle \ldots \rangle_{I,A}$ are taken either over the eigenstates $\phi_I$ (with eigenvalues $E_I$) of $H_0$ or the eigenstates $\psi_A$ (with eigenvalues $E_A$) of $H$. The moment $p = 2$ is the participation ratio (PR) and gives the typical number of nonzero components $\psi_A(I)$. For model (1), one has $\xi_p^{(0)} \in [1, \xi_p^{\rm GOE} = N^{p-1}/(2p-1)!!]$, where the maximal value corresponding to Dyson's Gaussian orthogonal ensemble (GOE) is achieved for $U = \infty$ [13]. There are, beside the one-body spacing $\Delta$, two important energy scales [1]: the mean spacing between states directly coupled by the two-body interaction $\Delta_c^{(0)} = B_2^{(0)}/K \approx 4\Delta/mn^2$ and the $n$-body spacing $\Delta_n^{(0)} = B_n^{(0)}/N$, where $B_n^{(0)} \approx n(m-n)\Delta$ is the $n$-body band [and, hence, $B_2^{(0)} \approx 2(m-2)\Delta$]. As $U$ increases, it has been found that $\rho^{(0)}(E)$ undergoes two transitions from a delta peak to a Lorentzian shape first, then to a Gaussian shape [1,12,14]. In the Lorentzian regime, the width $\Gamma^{(0)}$ of $\rho^{(0)}(E)$ is given by the golden rule $\Gamma^{(0)} \propto U^2/\Delta_c^{(0)}$ and the PR is $\xi_2^{(0)} \approx \Gamma^{(0)}/\Delta_n^{(0)}$ [1]. The Lorentzian regime is defined by the two conditions $\xi_2^{(0)} \gg 1$ and $\Gamma^{(0)} < B_n^{(0)}$ [14,15]. In the dilute limit $1 \ll n \ll m$, these conditions translate into

$$\Delta/\sqrt{nN} \ll U < \Delta/\sqrt{n}. \qquad (4)$$

When the Gaussian regime is entered, the SF spreads over the full bandwidth so that the eigenstates have a finite fraction of nonzero components $\xi_2^{(0)} = O(N)$.

Having summarized some of the known results for the structure of the eigenstates as the interaction is switched on, we now turn our attention to the reversed problem and decrease $U$ starting from $U = \infty$. Both the SF $\rho^{(\infty)}$ and the moments $\xi_p^{(\infty)}$ can be defined in the same way as above provided the $U = \infty$ basis is fixed individually for each realization of the Hamiltonian (1). Since the occupation operator $n_\alpha = c_\alpha^\dagger c_\alpha$ does not commute with $H_1$, $H_0$ induces one-body transitions between different eigenstates of $H_1$, leading to an increase of $\xi_p^{(\infty)} > 1$ and a broadening of $\rho^{(\infty)}$ as $U$ decreases. The $H_0$-induced transitions also lead to two successive crossovers of the SF $\rho^{(\infty)}$, first from a delta peak to a Lorentzian shape, then to a Gaussian shape, and this is shown on the upper inset of Fig. 1. As is the case for $\Gamma^{(0)}$, the golden rule gives a good estimate of the width $\Gamma^{(\infty)}$ of the SF in the Lorentzian regime, as we now proceed to show. Under the assumption that, for $U = \infty$, the TBRE has random ergodic eigenstates in the middle of its spectrum [13], all eigenstates are directly connected to each other by $H_0$ so that $\Delta_c^{(\infty)} = \Delta_n^{(\infty)} = B_n^{(\infty)}/N$. In the limit of a large number of particles and orbitals, the $U = \infty$ density of states is well approximated by a Gaussian with a width $B_n^{(\infty)} \approx \sqrt{K}\Lambda U \approx n(m-n)\Delta$ [11,12], and one has $\Delta_n^{(\infty)} \approx \Delta_n^{(0)} \propto n(m-n)\Delta/N$. Assuming random ergodic many-body eigenfunctions at $U = \infty$ [11,12], the transition matrix elements have a variance $\langle H_0^2 \rangle = \sum_\alpha (\epsilon_\alpha/U)^2/N \approx [n(m-n)\Delta^2/U]^2/N$. The golden rule then predicts

$$\Gamma^{(\infty)} \propto n(m-n)\Delta^3/U^2. \qquad (5)$$

This prediction is confirmed by numerical data presented in Fig. 1. These data, as well as those to be presented below, have been obtained via exact diagonalization of systems of up to $N = 3432$ (corresponding to $n = 7$ and $m = 14$), performing averages over 20 (for the largest systems) to 500 realizations of Hamiltonian (1) for each parameter set and avoiding the tails of the DOS by keeping only 33% of the states in the middle of the spectrum.

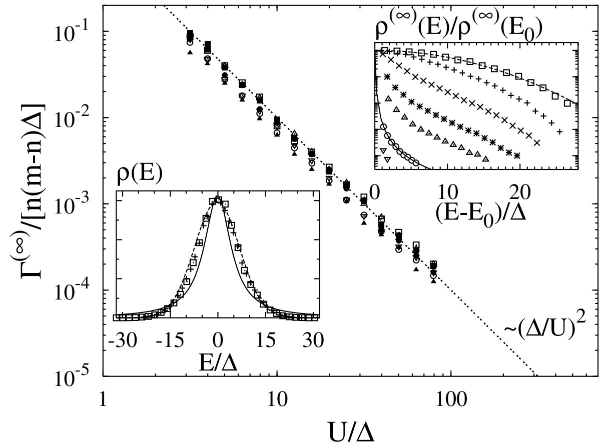


FIG. 1. Width of the strength function $\rho^{(\infty)}$ as a function of $U/\Delta$ for $n \in [3, 6]$ and $m \in [10, 14]$. The dotted line indicates the golden rule predicted behavior (5). Upper inset: crossover of $\rho^{(\infty)}$ from delta peak to Lorentzian to Gaussian shape for $m = 12$, $n = 5$, and $U/\Delta = 0.55$ (□), 1.09 ( + ), 2.15 ( × ), 4.26 ( ∗ ), 8.43 ( △ ), 33 ( ○ ), and 129.15 ( ▽ ). The solid and dashed lines give a Lorentzian (solid) and a Gaussian (dashed) fit, respectively. Lower inset: strength functions $\rho^{(\infty)}$ (□) and $\rho^{(0)}$ ( + ) at the dual point $U_d \approx 1.11$ for $m = 12$ and $n = 5$. Note that the shape is neither Lorentzian (solid line) nor Gaussian (dashed line).

In the golden rule regime, the PR is given by $\xi_2^{(\infty)} = \Gamma^{(\infty)}/\Delta_n^{(\infty)} \approx N(\Delta/U)^2$ and, according to the conditions $\xi_2^{(\infty)} > 1$ and $\Gamma^{(\infty)} < B_n^{(\infty)}$, the Lorentzian regime in the $U = \infty$ basis is bounded by the inequalities,

$$\Delta < U < \sqrt{N}\Delta. \qquad (6)$$

This is confirmed qualitatively by the upper inset of Fig. 1 and by numerical data to be published elsewhere [16].

The remarkable fact that $\Delta_n^{(\infty)}$ and $\Delta_n^{(0)}$ have the same parametric dependence in $n$ and $m$ implies that $\Gamma^{(0)} = \Gamma^{(\infty)}$ and $\xi_2^{(0)} = \xi_2^{(\infty)}$ can both be satisfied for $U_d \sim \Delta/n^{1/4}$. This is illustrated on the lower inset of Fig. 1 and the inset of Fig. 2. Remarkably enough, we have found that, at $U_d$, the PRs take a universal value $\xi_2^{(0)} = \xi_2^{(\infty)} \approx 0.8\xi_2^{\mathrm{GOE}}$ independently on $n$ and $m$. This calls for a rescaling $U \to U/U_d$ for $\xi_2^{(0)}/\xi_2^{\mathrm{GOE}}$ and $U \to U_d/U$ for $\xi_2^{(\infty)}/\xi_2^{\mathrm{GOE}}$ after which all data for the PR fall on top of each other, as can be seen in Fig. 2.

The inset of Fig. 2 shows that, at $U_d$, the higher moments $\xi_p$ with $p > 2$ also cross. This behavior has been found to hold for all $m$ and $n$. It implies that the eigenstates have exactly the same spreading over both bases, and that the SFs have not only the same width, but also exactly the same shape at $U_d$ ($\xi_p^{-1}$ are moments of the SF), as shown in the lower inset of Fig. 1. Moreover, we found that at the dual point the SF satisfies the scaling $\Gamma(U_d) \propto mn^{3/2}\Delta$ as follows from (5) and $U_d \propto \Delta/n^{1/4}$.

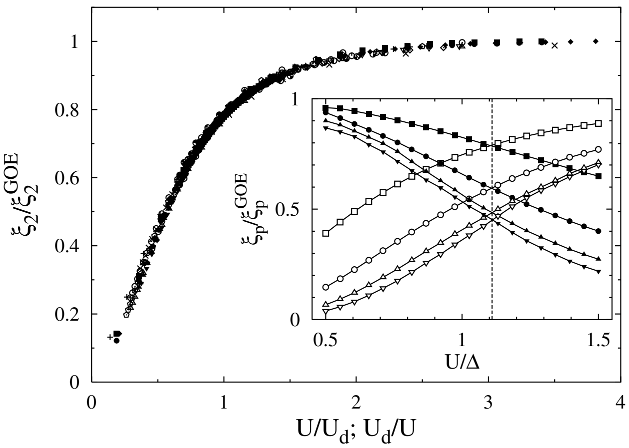


FIG. 2. $\xi_2^{(0)}/\xi_2^{\rm GOE}$ (empty symbols) and $\xi_2^{(\infty)}/\xi_2^{\rm GOE}$ (full symbols) vs the rescaled interactions $U/U_d$ and $U_d/U$, respectively. Different symbols correspond to different $n \in [2, 7]$ and $m \in [10, 16]$. Inset: $U$ dependence of $\xi_p^{(0)}$ (empty symbols) and $\xi_p^{(\infty)}$ (full symbols) for $p = 2$ (□), 3 (○), 4 (△), and 5 (▽), for $m = 12$ and $n = 5$. The dashed line indicates the ($p$-independent) intersection point $U_d/\Delta \approx 1.11$.

We then calculate the PR $\xi_2^{(d)}$ in the $U = U_d$ (dual) basis, and numerical results are shown in Fig. 3. The symmetry around $U = U_d$ is evident and, for both $U \to 0$ and $U \to \infty$ limits, there is saturation at $\xi_2^{(d)} \approx 0.8\xi_2^{\rm GOE}$. These results establish the existence of a duality transformation $U^* = U_d^2/U$ connecting the strong and weak interaction regimes, so that the PRs and the SF satisfy $\xi_2^{(0)}(U) = \xi_2^{(\infty)}(U^*)$, $\xi_2^{(d)}(U) = \xi_2^{(d)}(U^*)$, $\rho^{(0)}(U) = \rho^{(\infty)}(U^*)$, and $\rho^{(d)}(U) = \rho^{(d)}(U^*)$.

The above estimate for $U_d \sim \Delta/n^{1/4}$ relies on the assumptions that the golden rule correctly estimates the width of the SF in both bases, and that $\xi_2^{(0,\infty)} = \Gamma^{(0,\infty)}/\Delta_n^{(0,\infty)}$. Equivalently, this requires to be in the Lorentzian regime, which is, however, impossible as the two conditions (4) and (6) are mutually exclusive, and the predicted $U_d \sim \Delta/n^{1/4}$ lies outside both Lorentzian regimes. One may thus wonder if its predicted parametric dependence in $m$ and $n$ makes any sense at all. $U_d$ as a function of both $m$ and $n$ is shown in the insets of Fig. 3. First, it is seen that the $m$ dependence of $U_d$ is very weak (we found $U_d \sim \Delta m^{-\alpha}$ with $\alpha < 0.08$ for a range $m \in [8, 20]$). Even though quite weak, this $m$ dependence presumably indicates that we are not deep enough in the dilute limit. Second, we extracted the $n$ dependence of $U_d$ for $m = 14$, where we had the largest range $n \in [2, 7]$. We got $U_d \sim \Delta n^{-\beta}$, with an exponent $\beta \in [0.3; 0.5]$. These bounds on $\beta$ are compatible with the golden rule estimates which exclude $U_d$ from both Lorentzian regimes, i.e.,

$$\Delta/\sqrt{n} < U_d < \Delta, \qquad (7)$$

giving $\beta \in [0, 0.5]$. We conclude that $U_d \sim \Delta/n^{1/4}$ is in good qualitative agreement with our numerical results, and that the inequalities (7) define an intermediate regime which is nonperturbative in the two bases, and whose width increases parametrically with $n$.

Additional structures in the wave functions can be captured by the structural entropy $S_{\rm str}$, which is defined by subtracting from the Shannon entropy $S = \langle\sum_I |\psi_A(I)|^2 \ln|\psi_A(I)|^2\rangle_A$ the log of the PR: $S_{\rm str}^{(0,\infty)} = S^{(0,\infty)} - \ln\xi_2^{(0,\infty)}$ [17]. It measures the contribution to the Shannon entropy which is not contained in the PR, and thus not in the bulk of the SF. Numerical results are presented in Fig. 4. Expectedly, the crossing of $S_{\rm str}^{(0,\infty)}$ occurs at $U_d$ (since all the moments $\xi_p$ determine $S_{\rm str}$) but the crucial feature of Fig. 4 is that, almost immediately after the crossing, $S_{\rm str}$ takes on its asymptotic GOE value $S_{\rm str}^{\rm GOE} = 0.3689\ldots$—and therefore $U_d$ is the crossover point to the regime with GOE-like behavior *in both bases.* The second crucial feature of Fig. 4 is the peaks in $S_{\rm str}$ located on both sides of $U_d$. Such peaks have already

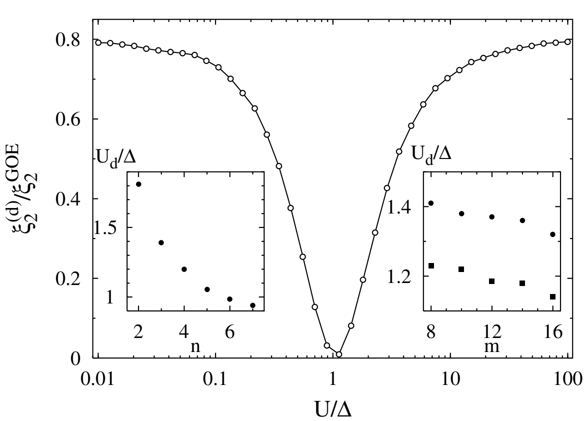


FIG. 3. $\xi_2^{(d)}/\xi_2^{\rm GOE}$ calculated in the $U = U_d$ basis for $m = 12$ and $n = 5$. Left inset: $n$ dependence of $U_d$ for $m = 14$. Right inset: $m$ dependence of $U_d$ for $n = 3$ (○) and 4 (□).

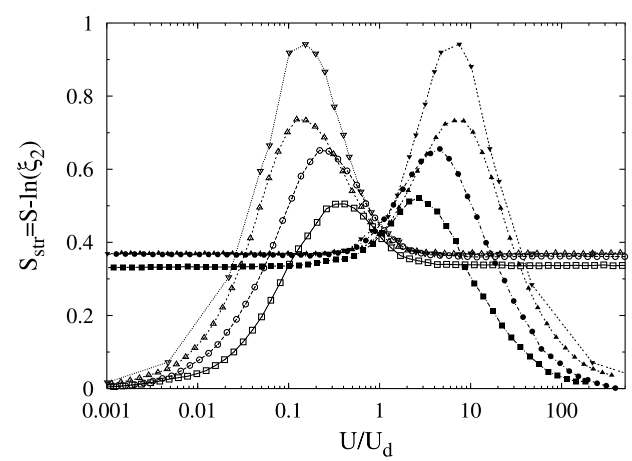


FIG. 4. Structural entropy $S_{\rm str} = S - \ln(\xi_2)$ in the $U = 0$ (empty symbols) and $U = \infty$ (full symbols) basis for $m = 14$, and $n = 2$ (□), 3 (○), 4 (△), and 5 (▽).

been observed in various models where they indicate unusually large fluctuations of eigenstates [18]. It is expectable that the width of these peaks is determined by the intermediate, non-Lorentzian regime $\Delta/\sqrt{n} < U < \Delta$, which is borne out by the data plotted in Fig. 4, where once $S_{\rm str}$ is plotted against $U/U_d \simeq U n^{1/4}/\Delta$, the peaks broaden symmetrically as more particles are added.

Examples of models where such a duality between localized (with $\xi_2 = 1$) and delocalized [with $\xi_2 = O(N)$] asymptotic regimes is related to a sharp metal-insulator transitions are provided by one-dimensional lattices with quasiperiodic potential (where the duality connects momentum and spatial eigenfunction coordinates) [19] and fermionic tight-binding models [4,20]. Our model is, however, fundamentally different in that the duality point $U_d$ is protected by a finite-sized interval characterized by a non-Lorentzian spreading of the eigenstates over a finite fraction of both the strong and weak interaction eigenstate. According to (7), this intermediate regime survives as $m$ and $n$ increase, and this results in a smooth crossover (and not a sharp transition) to fully developed chaos, characterized by the collapse of all the data points over one single curve shown on Fig. 2 (and not a single crossing of the curves at $U/U_d = 1$). This is to be put in perspective with the recent controversy over whether or not particle-hole excitations undergo a delocalization transition in Fock space similar to the Anderson transition as their excitation energy increases [1,5,9] (increasing the excitation energy reduces the level spacing and is thus equivalent to increasing the interaction strength $U$). The results we presented corroborate the conclusions drawn in [1] of the existence of a smooth crossover and not a sharp transition.

In summary, we have established the existence of a duality transformation between the weak and the strong interaction regimes of deformed TBRE. At the duality point $U_d$, the eigenstates have GOE fluctuations [for $U > (<) U_d$ in the $U = 0$ ($\infty$) basis] and $U_d$ is surrounded by a finite-sized, intermediate regime where both limiting bases fail to describe the eigenstates. Together with the Lorentzian form of $\rho^{(\infty)}$ at large $U \gg \Delta$, this duality suggests the existence of quasiparticle excitations at $U_d \ll U \leq \infty$, which we were, however, not able to identify. The existence of this duality transformation is related to the two-body nature of the interaction. For $k$-body interaction, $n \gg k > 2$, the $U = \infty$ bandwidth is given by $B_n^{(\infty)} \sim \Delta n^{k/2} m^{k/2} > B_n^{(0)}$ [11,12], and the PR in both bases are equal to each other for $U_d \propto \Delta m^{1-k/2} n^{3/4-k/2}$. We then have $\Gamma^{(0)}(U_d) \propto \Delta m n^{3/2}$ while $\Gamma^{(\infty)}(U_d) \propto \Delta m^{k/2} n^{1/2+k/2}$ so that the SFs should differ at the intersection point of the PRs for $k \neq 2$.

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