

Hadronic Matter at Extreme Energy Density

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THERMODYNAMICS OF NUCLEAR MATTER FROM THE
STATISTICAL BOOTSTRAP MODEL

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ABSTRACT

We study the properties of nuclear matter within the framework of a modified and generalized statistical bootstrap model in which the volume of a fireball grows with its mass. We find that the such described nuclear matter can exist in two phases. In particular we consider in a numerical example the high temperature ($T \lesssim T_0 \approx 150$ MeV) regime of the gaseous phase with a density of less than ~ 0.75 of normal nuclear density.

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1. INTRODUCTION

In order to understand high-energy heavy ion collisions or even perhaps high-energy hadron-nucleus scattering, we must study the equations of state of nuclear matter. From the point of view of a theoretical physicist, the inverse statement is even more natural: our ideas about the properties of nuclear matter at high and low densities and temperatures can be tested in high-energy nucleus-nucleus and hadron-nucleus collisions. Moreover, this knowledge is certainly essential in order to understand the properties of hadronic many-body objects in astrophysics, such as neutron stars, the Universe at early time, stellar collapse and perhaps even quasars.

While we are aware of the possible richness of the nuclear matter properties, in our approach to these problems we will concentrate on the gross features of nuclear matter that follow when we incorporate into the description the following basic properties:

- 1) conservation of baryon number and clustering of nucleons (i.e., attractive forces leading to many-body clusters with well-defined baryon number);
- 2) nucleon (isobar) excitations and internal cluster excitations (i.e., internal degrees of freedom that can absorb part of the energy of the system at finite temperature, thus transforming kinetic energy into mass);
- 3) approximate extensivity of nuclear matter (volume roughly proportional to baryon number, i.e., effectively a short-range repulsion);
- 4) co-existence of a pion gas when the temperature is not equal to zero (and behaving properly even in the absence of nuclear matter);
- 5) baryon-antibaryon pair creation;
- 6) "chemical" equilibrium between all constituents of the system (nucleons, isobars, clusters, pions ...).

Our present work should be most trustworthy in the domain of high temperatures and moderately high density, where details of the interaction, Fermi and Bose statistics, as well as the quark structure of nucleons, are most likely negligible. Also not considered explicitly is the isospin of the nuclei.

In order to derive the physical properties of such a system which are independent of a particular choice of the two-body and multibody interaction we employ a technique ("bootstrap") developed for similar problems in elementary particle physics -- here, however, sufficiently modified to suit the different physical environment. An additional motivation in this direction is the recent recognition that the understanding of nuclear matter at the saturation point depends very sensitively on the character of the two-body potential at short distances which is not well defined by two-body reactions. It is possible to view the bootstrap technique only as a convenient way to introduce some physical properties which cannot be so easily defined by the choice of a specific potential but which globally might even be more important than details of the two-body force.

Let us now explain the general idea of the bootstrap description of the nuclear matter. Consider an assembly of b nucleons. We can view it as an assembly of $(b/2)$ two-particle clusters ... or also as two $(b/2)$ -nucleon clusters -- all possible divisions will contribute to the number of states of the b nucleon system. In turn, each cluster containing b_i nucleons can be viewed in a similar fashion. It is simple to write an equation characteristic of such a system; neglecting for the moment all further degrees of freedom but the baryon number b , the number of states will be described by a function $\rho(b)$ obeying the equation

$$\rho(b) = \rho(1) \delta_K(b-1) + \sum_{j=2}^b \frac{1}{j!} \sum_{\{b_i\}_j} \delta_K\left(\sum_{i=1}^j b_i - b\right) \prod_{i=1}^j \rho(b_i) \quad (1.1)$$

where the sum runs over all the sets with j elements (clusters) with $b_i \in \{1, \dots, b-1\}$, such that the conservation of baryon number is secured by the Kronecker delta function $\delta_K(\cdot)$. The existence of the quantity ρ as defined by the above equation is obvious by recursion, which, however, is not practical for actual calculations. The above equation is already a true "integral" equation in the bootstrap sense. To see this more clearly we consider in the same context a continuous variable m , say "mass", for illustrative purposes

$$\rho(m) = \rho_0(m) + \sum_{j=2}^{\infty} \frac{1}{j!} \int_0^{\infty} dm_1 \int_0^{\infty} dm_2 \dots \int_0^{\infty} dm_j \delta\left(m - \sum_{i=1}^j m_i\right) \prod_{i=1}^j \rho(m_i) \quad (1.2)$$

here ρ_0 is some given function expressing a limiting case in which clusters do not contribute. As the lowest hadronic mass is > 0 , the sum over j runs in practice only over a finite number of contributions with m_i different from zero. This integral equation expresses most clearly the fact that the cluster described by $\rho(m)$ consists of an arbitrary number of clusters of smaller mass, each of which is in turn made of an arbitrary number of smaller clusters, etc., as can be seen iterating the above equation ^{*}).

The quantity ρ (derived from a more realistic equation) may be taken as the level density of nuclear matter, provided that some further statistical factors are considered and that ρ_0 expresses the level density properly when other terms do not contribute. Since the knowledge of the level density is sufficient for a complete description of the physical properties of the system from the point of view of statistical physics, the system is completely defined once the proper bootstrap equation reflecting our physical requirements is written. Clearly, one can write many different bootstrap equations under given general constraints, each with some parameters allowing an adjustment to experimental data. Here we follow the simplest possible line of approach choosing the simplest possible case in a numerical example.

Plan of the paper

- Section 2 : We discuss the bootstrap hypothesis first in the context of a strongly interacting pion gas. The bootstrap equation of the pion gas is solved and discussed.
- Section 3 : The mass spectrum and its Laplace transform are used to obtain a thermodynamic description of the system.
- Section 4 : We write down, discuss and solve the bootstrap equation for nuclear matter. It is much richer than that of the pion gas, which it contains as a special case.
- Section 5 : The general solution of the bootstrap equation for nuclear matter leads to a corresponding statistical thermodynamics. Our postulates result in two main

^{*}) At this point the reader may have the uneasy feeling that we are doubly (in fact more than doubly) counting states. This problem has been discussed in detail in references 1) and 2), to which we refer the reader. Here we must ask him to simply believe that our counting is correct, if clusters exist (see also Subsection 3.2).

properties of the thus described system:

- i) there exists a maximum temperature, which is of the order of that of the pion gas ($T_0 \approx m_\pi$);
- ii) there exists at all temperatures $0 \leq T \leq T_0$ a critical baryon number density separating a low density "gas phase" from a state where a condensate and its vapour exist in equilibrium.

A numerical study is presented in which the simplest non-trivial input spectrum is assumed; the corresponding model is solved explicitly and the results are displayed graphically. This case shows all essential features but it is still too far from reality to be taken as more than a qualitative prediction.

Section 6 : The "liquid phase" requires different mathematical techniques than the "gas phase". The general approach is outlined.

Section 7 : Summary and outlook.

Our notation and units:

$\hbar = c = k$ (Boltzmann constant) = 1

the only dimensional unit is 1 GeV

Metric : $a \cdot b \equiv a_\mu b^\mu \equiv a_0 b_0 - \vec{a} \cdot \vec{b}$.

Remark: Throughout this paper we use only Boltzmann statistics. As the bootstrap approach leads to an extremely rich mass spectrum, it is practically irrelevant whether a particular cluster or particle is a boson or a fermion or a Boltzmannion: it (almost) never happens that two equal clusters occupy the same state.

2. THE STATISTICAL BOOTSTRAP METHOD

2.1 The statistical bootstrap model in particle physics (SBM)

The motivation for a statistical bootstrap model in particle physics comes from two sources:

- i) the abundant production of particles in high-energy p-p collisions and a momentum distribution of these particles which suggests that there might be some analogy to black body radiation emitted from moving sources;
- ii) the apparent existence of intermediate states in which lumps of highly excited hadronic matter ("fireballs") are staying together before decaying.

Thus it was tempting to describe the particle production process as pion black-body radiation emitted from one or several fireballs with a volume $v_0 \approx 4\pi(3m_\pi^3)$. This idea was first proposed by H. Koppe³⁾ and for this reason it is called the Fermi Statistical Model⁴⁾. As for a statistical-thermodynamical description the density of states $\sigma(E)$ is necessary and sufficient, we may express the Koppe-Fermi approach as follows:

$$\frac{v_0 \sigma_0(E)}{(2\pi)^3} = \sum_{n=1}^{\infty} \frac{1}{n!} \times \int \delta(E - \sum_{i=1}^n \sqrt{\vec{p}_i^2 + m_\pi^2}) \delta^3(\sum_{i=1}^n \vec{p}_i) \prod_{i=1}^n \frac{v_0 d^3 p_i}{(2\pi)^3} \quad (2.1)$$

This is nothing else than the phase space density of a pion gas with free particle creation. If one puts $m = 0$ and multiplies by 2 for the two helicity states of a light quantum, one obtains from (2.1) all the usual formulae of the electromagnetic black-body radiation (Planck's law) in the Boltzmann limit.

Before proceeding we shall write this formula, which is meant to be valid in the rest frame of the emitting volume v_0 , in a

manifestly Lorentz-invariant form⁵⁾. We introduce the four volume = (rest volume) × (four velocity)

$$V^\mu := \sigma_0 u^\mu$$

$$u^2 = u_\mu u^\mu = 1; \quad u_0 \geq 1 \quad (2.2)$$

Then with

$$\delta_0(p^2 - m^2) := \delta(p^2 - u^2) \Theta(p_0) \quad (2.3)$$

$$\frac{v_0 d^3 p}{(2\pi)^3} \Rightarrow \frac{2V_\mu p^\mu}{(2\pi)^3} \delta_0(p^2 - m_\pi^2) d^4 p \quad (2.4)^*$$

The right-hand side is the number of states in $\{\vec{p}, d^3 p\}$ of a scalar neutral particle of mass m_π enclosed in v_0 , expressed covariantly. Now we modify the definition of the density of states

$$\sigma(p^2, p \cdot V) d^4 p : \quad = \text{number of states in } \{p, d^4 p\} \quad (2.5)$$

and rewrite (2.1) as

$$\sigma(p^2, p \cdot V) = \frac{2p \cdot V}{(2\pi)^3} \delta_0(p^2 - m_\pi^2) +$$

$$+ \sum_{n=2}^{\infty} \frac{1}{n!} \int \delta^4(p - \sum_{i=1}^n p_i) \prod_{i=1}^n \frac{2V \cdot p_i}{(2\pi)^3} \delta_0(p_i^2 - m_i^2) d^4 p_i \quad (2.6)$$

*) We shall refer to $2V \cdot p / (2\pi)^3 d^4 p$ as the "Touschek measure".

where we write the one-particle term separately for convenience. This is a relativistically invariant equation for the density of states of a gas in which the interaction manifests itself only via creation and absorption of Boltzmann pions.

The next important idea (see also Subsection 3.2) was to admit other particles than just pions, and in particular resonant states of pions, just as if they were stable particles⁶⁾. Not knowing which ones should be admitted and how many there are, we might put them in a mass spectrum of admissible input particles $\rho_{in}(m)$; the pion contributes to $\rho_{in}(m)$ a δ function $\delta(m - m_\pi)$, resonances contribute smeared-out δ functions. For the moment, $\rho_{in}(m)$ is a function which represents our (incomplete) knowledge of the true mass spectrum $\rho(m)$.

Now (2.6) becomes

$$\sigma(p^2, p \cdot V) = \frac{2p \cdot V}{(2\pi)^3} \delta_0(p^2 - m_\pi^2) +$$

$$+ \sum_{n=2}^{\infty} \frac{1}{n!} \int \delta^4(p - \sum_{i=1}^n p_i) \prod_{i=1}^n \frac{2V \cdot p_i}{(2\pi)^3} \rho_{in}(m_i) \delta_0(p_i^2 - m_i^2) d^4 p_i d m_i \quad (2.7)$$

Note that we have restricted the one-particle state to have the pion mass. Higher mass "one-particle states" are already contained in the sum, namely when in any of its terms all $p_i \rightarrow m_i$. Our new equation for $\sigma(p^2, p \cdot V)$ describes the density of states of a many-component gas: each species of particle contained in $\rho_{in}(m)$ is present in the gas. All these components are in "chemical" equilibrium, since all kinds of reactions (e.g., $\pi + \pi \rightarrow \omega + \pi + \pi + \pi$, etc.) are now possible and neither the total particle number nor that of any of the various components is fixed.

Such a system should already exhibit a good deal of the properties of a strongly interacting gas; we argue that our description will indeed contain the essence of strong interactions, once the incomplete $\rho_{in}(m)$ is replaced by the true mass spectrum $\rho(m)$ — which unfortunately is not known to us.

The key idea leading to the hadronic bootstrap is the observation that $\sigma(p^2, p \cdot V)$ can be related to the mass spectrum $\rho(m)$. Suppose we could insert the true mass spectrum $\rho(m)$ into Eq. (2.7), then $\sigma(p^2, p \cdot V)$ would be the density of states of a "fireball" of hadronic dimension built up from all strongly interacting particles in statistical equilibrium. Such a fireball is itself a highly excited hadron with mass $m = \sqrt{p^2}$. For reasons of consistency it should then be admitted as a constituent particle in fireballs of larger mass. Hence it should already be present in the true $\rho(m)$. As both, $\sigma(p^2, p \cdot V)$ and $\rho(m)$, are densities of states, it follows that if $\rho(m)$ is the true mass spectrum, $\sigma(p^2, p \cdot V)$ is itself, apart from some minor kinematical differences, the true mass spectrum at $m = \sqrt{p^2}$. This statement establishes a new relation between ρ and σ , leading to an integral equation, the "bootstrap equation" (BE). Physically it is equivalent to the postulate that resonances and fireballs are one and the same and that fireballs consist of fireballs.

In order to find the precise relation between $\rho(m)$ and $\sigma(p^2, p \cdot V)$ we consider the conceptual differences between them as exhibited by Eq. (2.7): while $\rho(m)$ counts all hadrons (as given for example in the Tables of the Particle Data Group⁷⁾) as being at rest in their own confining volume, $\sigma(p^2, p \cdot V)$ is the density of states of an object with mass $m = \sqrt{p^2}$ allowed to freely move in its confining volume instead of being at rest. This fact is also reflected by the dependence of σ on the scalar product of $p \cdot V$. Thus σ counts more states than ρ (and contains more information). In order to relate σ to ρ , we restrict this freedom by

requiring that p^μ and v^μ be parallel four vectors, i.e., have a common rest frame. Then the first term of Eq. (2.7) becomes

$$\begin{aligned} \frac{2V_\mu p^\mu}{(2\pi)^3} \delta_0(p^2 - m_\pi^2) &= \\ &= \frac{v_0}{(2\pi)^3} 2m \delta_0(p^2 - m_\pi^2) \\ &= v_0 / (2\pi)^3 \delta(m - m_\pi) \end{aligned} \quad (2.8)$$

which, apart from the factor $v_0 / (2\pi)^3$, is just the beginning of $\rho(m)$.

Hence we multiply Eq. (2.7) by $(2\pi)^3 / v_0$ and go to the common rest frame of p^μ and v^μ . We obtain

$$\begin{aligned} \sigma(m^2) \frac{(2\pi)^3}{v_0} &= \delta(m - m_\pi) + \\ + \sum_{n=2}^{\infty} \left[\frac{v_0}{(2\pi)^3} \right]^{n-1} \frac{1}{n!} &\int \delta\left(m - \sum_{i=1}^n \sqrt{\vec{p}_i^2 + m_i^2}\right) \times \\ \times \delta^3\left(\sum_{i=1}^n \vec{p}_i\right) \prod_{i=1}^n &\rho(m_i) d^3 p_i dm_i \end{aligned} \quad (2.9)$$

Now the left-hand side has dimension m^{-1} and represents the internal density of states of a system of mass m at rest in its own confining volume v_0 ; this density begins with $\delta(m - m_\pi)$ and has a continuum for $m > 2m_\pi$. It therefore might be considered as an averaged mass spectrum (the true one is not yet continuous at $m \geq 2m_\pi$), which asymptotically becomes physically equivalent to $\rho(m)$. In the first paper on statistical bootstrap⁸⁾, the postulate was accordingly that the logarithms of the densities of states (i.e.,

essentially the entropies of the corresponding systems) should become asymptotically equal: define

$$\sigma(m^2) \frac{(2\pi)^3}{v_0} = : \rho_{av}(m) \quad (2.10)$$

Then the two equations

$$\lim_{m \rightarrow \infty} \frac{\ln \rho_{av}(m)}{\ln \rho_{in}(m)} = 1 \quad (2.11a)$$

and

$$\rho_{av}(m) = \delta(m - m_\pi) + \quad (2.11b)$$

$$+ \sum_{n=2}^{\infty} \left[\frac{v_0}{(2\pi)^3} \right]^{n-1} \frac{1}{n!} \int \delta\left(\epsilon - \sum_{i=1}^n \sqrt{p_i^2 + m_i^2}\right) \delta^3\left(\sum_{i=1}^n \vec{p}_i\right) \prod_{i=1}^n \rho(m_i) d^3 p_i dm_i$$

constitute the bootstrap postulate in its weakest form. In the following development⁹⁾⁻¹⁶⁾ the formulation underwent several changes: Frautschi⁹⁾ required $\rho_{av} = \rho_{in} = \rho$ (= some approximate average over the true complete mass spectrum). Yellin¹³⁾ wrote the equation in invariant momentum space [see Eq. (2.15b)]

$$B\tau(p^2) = B\delta_0(p^2 - m_\pi^2) + \sum_{n=2}^{\infty} \frac{1}{n!} \int \delta^4\left(p - \sum_{i=1}^n p_i\right) \prod_{i=1}^n B\tau(p_i^2) d^4 p_i \quad (2.12)$$

which is almost the same as Frautschi's version, but not quite, since

$$\begin{aligned} d^4 p_i \tau(p_i^2) &= d^3 p_i \int \tau(m_i^2) \delta_0(p_i^2 - m_i^2) dm_i^2 dp_{0,i} \\ &= d^3 p_i \int \tau(m_i^2) \frac{m_i dm_i}{\sqrt{p_i^2 + m_i^2}} \end{aligned} \quad (2.13)$$

Comparing this integration measure with the one in (2.11b) we see that the square root in the denominator of (2.13) makes equation (2.12) dynamically different from Frautschi's. If all interaction is believed to be contained in $\rho(m)$ then (2.12) appears not to be correct. As we do not really know, we might accept (2.12) as well (in fact, it has been widely considered as "the bootstrap equation"). Many applications have been based on both of them. The results in particle physics are, to a large extent, independent of which particular version one chooses. Namely, all equations of this type (there exist various other forms of it which we cannot discuss here) lead to one common characteristic result: asymptotically for $m \rightarrow \infty$ the mass spectrum grows exponentially:

$$\rho(m) \sim \frac{\text{const}}{m^a} e^{\frac{m}{T_0}} \quad (2.14)$$

$$T_0 \approx m_\pi ; \quad \frac{3}{2} < a < \frac{7}{2}$$

where T_0 is a "limiting temperature" and where the values of 'a' and T_0 depend on the version chosen. We will solve a BE and prove Eq. (2.14) further below.

When we decided to apply the above line of approach to infinite nuclear matter, we had to take extensivity (volume \approx proportional to baryon number) into account. By trying to do that relativistically and consistently we became aware of several obscure points in the statistical bootstrap model (SBM) which had to be cleared up; the result was yet another version of SBM, which then could easily be generalized to include the nuclear matter case.

2.2 SBM reconsidered

From the above discussion, it follows that two questions have to be answered:

- i) SBM has much to do with phase space; which is the correct integration measure?
- ii) a mass spectrum deals by definition with particles carrying their confining volume. The co-moving volume should be introduced covariantly and more consistently than by (2.8), (2.9); furthermore, in that equation all fireballs have the same volume v_0 , independently of their mass. While this may or may not be true for pionic fireballs (linear Regge trajectories $J = J_0 + \alpha m^2$ require that some relevant length of the particles should grow proportional to m), it cannot be assumed for nuclear clusters. How can we incorporate covariantly additive co-moving volumes?

We now discuss these two questions in detail.

2.2.1 The integration measure

For n free (Boltzmann) particles of mass m two sorts of integrals are commonly used without much reflection: Invariant Phase Space (IPS)^{*}) and Invariant Momentum Space (IMS):

^{*}) which has Touschek measure

$$IPS := \frac{1}{n!} \int \delta^4(p - \sum_{i=1}^n p_i) \prod_{i=1}^n \frac{2V \cdot p_i}{(2\pi)^3} \delta_0(p_i^2 - m^2) d^4 p_i \quad (2.15a)$$

$$IMS := \frac{1}{n!} \int \delta^4(p - \sum_{i=1}^n p_i) \prod_{i=1}^n B \delta_0(p_i^2 - m^2) d^4 p_i \quad (2.15b)$$

For the light quantum gas ($m = 0$) we obtain the Stefan-Boltzmann law by summing over n and going over to the thermodynamic description (see, e.g., Ref. 2)). The result is, with $V = (V_\mu V^\mu)^{\frac{1}{2}}$

$$IPS : \langle E \rangle = \left(\frac{V \pi^2}{15} \right) T^4 \quad (2.16a)$$

$$IMS : \langle E \rangle = (2\pi B \zeta(3)) T^3 \quad (2.16b)$$

For a dilute gas we know that the first expression is correct, the second wrong. Furthermore, (2.15a) can be derived in a straightforward way from the elementary rule that the available number of states in $\{\vec{p}, d^3 p\}$ of one particle enclosed in a volume V is given by

$$N(V, \vec{p}) = \frac{V}{h^3} d^3 p \quad (2.17)$$

while (2.15b) cannot be justified that way. For more detail see Ref. 17). Since in the absence of interaction $\rho(m) \rightarrow \delta(m - m_\pi)$, the density of states $\sigma(p^2, p \cdot V)$ should reduce to that of a free pion gas. Therefore, we conclude that SBM should be formulated in

IPS (Touschek measure) as was done in the first papers and not in IMS as has become customary since.

2.2.2 The co-moving volume

In deriving Eqs. (2.9)-(2.11) we have treated the volume unsymmetrically by attributing it to the main fireball (on the left-hand side of the BE), while its constituents described by the $\rho(m_i)$ (on the right-hand side of the BE) behave as point-like particles all enclosed in the same common volume; as they, in turn, are described by the BE and then acquire the same volume, they must penetrate each other perfectly.

We now shall treat all particles on the same footing.

We start by defining a new density of states of a system enclosed in a four volume V :

$$\frac{2V \cdot p}{(2\pi)^3} \sigma(p, V) d^4 p d^4 V : = \quad (2.18)$$

$$= \text{number of states in } \{p, d^4 p; V, d^4 V\}$$

According to the conclusions reached in the foregoing subsection, we have used the Touschek measure to express the kinematic part of the level density (cf. Eq. (2.4)); $\sigma(p, V)$ represents the part due to internal degrees of freedom.

In a world consisting only of strongly interacting neutral scalar Boltzmann pions the mass spectrum must start with a δ function representing the pion:

$$\left. \frac{2V \cdot p}{(2\pi)^3} \sigma(p, V) \right|_{p^2 < 4m_\pi^2} = \frac{2V \cdot p}{(2\pi)^3} \delta_0(p^2 - m_\pi^2) \delta^4(V - V_\pi) \quad (2.19)$$

We generalize to arbitrary p by postulating a new BE of the type of Eq. (2.7), in which we give additive volumes to all constituents. This procedure enables us later to generalize straightforwardly to the case of nuclear matter. Whether in the pion case it is better than the old BE's remains to be seen.

$$\frac{2V \cdot p}{(2\pi)^3} \sigma(p, V) = \delta^4(V - V_\pi) \cdot \frac{2V \cdot p}{(2\pi)^3} \delta_0(p^2 - m_\pi^2) + \sum_{m=2}^{\infty} \frac{1}{m!} \int \delta^4(p - \sum_{i=1}^m p_i) \delta^4(V - \sum_{i=1}^m V_i) \prod_{i=1}^m \frac{2V_i \cdot p_i}{(2\pi)^3} \sigma(p_i, V_i) d^4 p_i d^4 V_i$$

(2.20)

This equation is relativistically invariant and is written in Touschek measure. Furthermore we have postulated additive four volumes such as to correspond to "dense packing". Strict additivity is not necessary but it is assumed here on physical grounds. Mathematically, it is possible to introduce in the bootstrap equation some prescribed way of volume addition replacing

$$\delta^4(V - \sum_{i=1}^m V_i) \implies \delta^4(V - \sum_{i=1}^m V_i \frac{A(p_i^2)}{A(p^2)}) \quad (2.21)$$

with arbitrary given $A(p^2)$; we will not pursue this possibility.

Next we require that all fireballs carry their four volumes (i.e., $V \parallel p$ and $V_i \parallel p_i$) because only then can we interpret σ as a mass spectrum. We therefore demand for all V, V_i :

$$V^{\mu} = V(m) \frac{p^{\mu}}{m}$$

$$m \equiv \sqrt{p^2} \quad (2.22)$$

Here $V(m) = \sqrt{V^2}$ is a Lorentz scalar which may depend on m . These requirements are imposed on the BE (2.20) by the ansatz

$$\frac{2V \cdot p}{(2\pi)^3} \sigma(p, V) = \delta^4(V - V(m) \frac{p}{m}) \frac{2V \cdot p}{(2\pi)^3} \hat{\sigma}(p, V) \quad (2.23)$$

The δ^4 function implies that its factor is a function of $p^2 = m^2$ alone:

$$\frac{2V \cdot p}{(2\pi)^3} \hat{\sigma}(p, V) = : \tilde{B}(p^2) \tilde{\tau}(p^2)$$

$$\tilde{B}(m^2) := \frac{2V(m)m}{(2\pi)^3} \quad (2.24)$$

Here $\tilde{B}(m^2)$ represents the kinematic factor $2V \cdot p / (2\pi)^3$, while $\tilde{\tau}(p^2)$ counts the internal degrees of freedom. Our BE (2.20) now becomes

$$\delta^4(V - V(m) \frac{p}{m}) \tilde{B}(p^2) \tilde{\tau}(p^2) = \delta^4(V - V(m) \frac{p}{m}) \tilde{B}(p^2) \delta_0(p^2 - m^2)$$

$$+ \sum_{n=2}^{\infty} \frac{1}{n!} \int \delta^4(p - \sum p_i) \delta^4(V - \sum V(m_i) \frac{p_i}{m_i}) \prod_{i=1}^n \tilde{B}(p_i^2) \tilde{\tau}(p_i^2) d^4 p_i$$

The $\prod d^4 v_i$ integration has already been done.

All volume δ^4 functions must have the same argument; furthermore, we have $p = \sum p_i$. Hence

$$\sum_{i=1}^n \left(\frac{V(m_i)}{m_i} - \frac{V(m)}{m} \right) p_i^{\mu} = 0 \quad (2.25)$$

for all $n \geq 2$ and arbitrary p_i . Therefore,

$$V(m) = A m$$

$$\tilde{B}(m^2) = 2A m^2 / (2\pi)^3 \quad (2.26)$$

where A is some fundamental parameter of our model^{*}). We can estimate its order of magnitude in various ways.

i) We may require that $\tilde{B}(m^2) = B_0 = 1 / (4\pi m_{\pi}^2)$, which yielded a reasonable value for T_0 in earlier calculations¹⁴⁾; we then obtain

$$A_0 := \pi^2 / m_{\pi}^4 = 2.57 \cdot 10^4 \text{ [GeV}^{-4}\text{]}$$

$$A_0^{-1} \approx 5 \text{ [MeV} \cdot \text{fm}^{-3}\text{]} \quad (2.27)$$

ii) We might fix A by requiring that for some mass μ ($m_{\pi} \leq \mu \leq 10 m_{\pi}$) $V(\mu) = v_0 = (4\pi/3) m_{\pi}^{-3}$. This gives

$$A_{\mu} := \frac{4\pi}{3\mu m_{\pi}^3} = 1.09 \cdot 10^4 \left(\frac{m_{\pi}}{\mu} \right) \text{ [GeV}^{-4}\text{]}$$

$$A_{\mu}^{-1} \approx 11.5 (\mu / m_{\pi}) \text{ [MeV} \cdot \text{fm}^{-3}\text{]} \quad (2.28)$$

iii) We might choose A to correspond to the MIT bag model¹⁸⁾:

^{*}) A^{-1} is the constant (!) energy density of fireballs.

$$V_{\text{bag}}(m) = \frac{m}{4 \cdot (120 \text{ MeV})^4} = A_{\text{bag}} \cdot m. \quad (2.29)$$

$$\begin{aligned} A_{\text{bag}} &:= 1.2 \cdot 10^3 [\text{GeV}^{-4}] \\ A_{\text{bag}}^{-1} &\approx 103.4 [\text{MeV} \cdot \text{fm}^{-3}] \end{aligned} \quad (2.30)$$

which corresponds to $\mu = 9 m_\pi$ in our previous estimate. We shall consider A as a free parameter limited to 10^3 to 10^4 GeV^{-4} .

With Eq. (2.26) satisfied, the δ^4 functions for the volumes factor out and we obtain the reduced BE

$$\begin{aligned} \tilde{B}(p^2) \tilde{\tau}(p^2) &= \tilde{B}(p^2) \delta_0(p^2 - m_\pi^2) + \\ &+ \sum_{n=2}^{\infty} \frac{1}{n!} \int \delta^4(p - \sum_{i=1}^n p_i) \prod_{i=1}^n \tilde{B}(p_i^2) \tilde{\tau}(p_i^2) d^4 p_i \end{aligned} \quad (2.31)$$

Now we have to identify the physical mass spectrum $\tau(m^2)$. This identification is not at all obvious. Remember that $\tilde{\tau}(p^2)$ originates from $\sigma(p, V)$, which we called above "the part of the level density due to internal degrees of freedom" while $\tilde{B}(p^2)$ (see Eq. (2.24)) comes from the "kinematical factor" $2V \cdot p / (2\pi)^3$. One could therefore argue that $\tilde{\tau}(m^2)$ is the mass spectrum, while $\tilde{B}(m^2)$ is some irrelevant reminiscence of external degrees of freedom. And indeed: the mass spectrum should start with $\delta_0(m^2 - m_\pi^2)$ and not with $\tilde{B}(m_\pi^2) \delta_0(m^2 - m_\pi^2)$. One can, however, also argue that $\tilde{B}(p^2)$ is of dynamical origin and should be absorbed into the mass spectrum. Namely, by imposing co-moving additive volumes on our dynamical equation (the BE) we have made the volume an inseparable,

dynamical part of the fireballs. Furthermore, if the volumes move with the fireballs, one can no longer claim that the factor $2V \cdot p / (2\pi)^3$ represents the kinematical degrees of freedom of a particle confined to some externally given volume V , since now the particle is always at rest: $2V \cdot p / (2\pi)^3 \rightarrow 2Am^2 / (2\pi)^3$. This factor has thus become a dynamical factor.

We now take this latter point of view^{*)}; the spectrum can be defined as to contain all p^2 dependence and the factor can be made constant such that the spectrum $\tau(p^2)$ starts with $\delta_0(p^2 - m_\pi^2)$:

$$\begin{aligned} \tau(p^2) &:= \frac{\tilde{B}(p^2)}{\tilde{B}(m_\pi^2)} \tilde{\tau}(p^2) \\ B &:= \tilde{B}(m_\pi^2) \end{aligned} \quad (2.32)$$

With the identity $B\tau(p^2) = \tilde{B}(p^2)\tilde{\tau}(p^2)$ our new BE is

$$\begin{aligned} B\tau(p^2) &= B \delta_0(p^2 - m_\pi^2) + \\ &+ \sum_{n=2}^{\infty} \frac{1}{n!} \int \delta^4(p - \sum_{i=1}^n p_i) \prod_{i=1}^n B\tau(p_i^2) d^4 p_i \end{aligned} \quad (2.33)$$

*) By adopting this interpretation we minimize the difference with the older BE's. The case of nuclear matter discussed in Section 4 will lead us nearer to the first argument where $\tilde{\tau}(p^2)$ is the mass spectrum.

It is identical to Yellin's BE (2.12). Although we started with an equation written in Touschek measure, the co-moving volumes have led to IMS measure (Eqs. (2.22)-(2.24)). We keep in mind that B might be taken to be proportional to p^2 instead of being constant; we are aware of the important consequences this will have, but we cannot discuss them at length in the present publication.

We stress that we do not claim that our BE (2.33) be "the BE"; any of the other versions may be considered as well; the situation is similar to the one in Lagrangian field theories: which, if any, is "the Lagrangian"? Only experiment will give us the answer.

2.3 Solution of the bootstrap equation

The standard method of solving Eq. (2.33) is by Laplace transformation. We define a four vector

$$\beta^\mu \equiv (\beta^0, \vec{\beta}) ; \beta^0 \geq 0 ; \beta := \sqrt{\beta_\mu \beta^\mu} \quad (2.34)$$

and the two Lorentz invariant functions:

$$\Phi(\beta) := \int B \tau(p^2) e^{-\beta_\mu p^\mu} d^4 p \quad (2.35a)$$

$$\varphi(\beta) := \int B \delta_0(p^2 - m_\pi^2) e^{-\beta_\mu p^\mu} d^4 p \quad (2.35b)$$

Applying the operator

$$\mathcal{L} := \int [\dots] e^{-\beta_\mu p^\mu} d^4 p \quad (2.36)$$

to the whole BE (2.33), we obtain

$$\Phi(\beta) = \varphi(\beta) + e^{\Phi(\beta)} - \Phi(\beta) - 1 \quad (2.37)$$

$\varphi(\beta)$ is a well-known function; Eq. (2.35b) gives

$$\varphi(\beta) = 2\pi B m_\pi^2 \frac{K_1(\beta m_\pi)}{\beta m_\pi} \quad (2.38)$$

Define $G(\varphi)$ by

$$G(\varphi) := \Phi(\beta) \quad (2.39)$$

then Eq. (2.37) can be written

$$\varphi = 2G - e^G + 1 \quad (2.40)$$

and the problem is to invert this equation, that is: to find $G(\varphi) = \Phi(\beta)$. The easiest way to do this is a graphical solution by first plotting $\varphi(G)$ and then consider the curve as $G(\varphi)$. By expanding $\exp(G)$ one sees that $\varphi(G) = G + \dots$; with growing G the exponential function takes the lead and $\varphi(G)$ goes exponentially to $-\infty$. The maximum lies at $G_0 = \ln 2$ and has value $\varphi_0 = \ln 4 - 1$; $\varphi''(G_0) \neq 0$ (Fig. 2.1a). The graphical solution is presented in Fig. 2.1b. From the figure and $\varphi''(G_0) \neq 0$, it follows that $G(\varphi)$ has a square root branch point at φ_0 and is complex for $\varphi > \varphi_0$ ¹⁰⁾. We note for later use that to $\varphi_0 = \ln 4 - 1$ corresponds a value β_0 for which

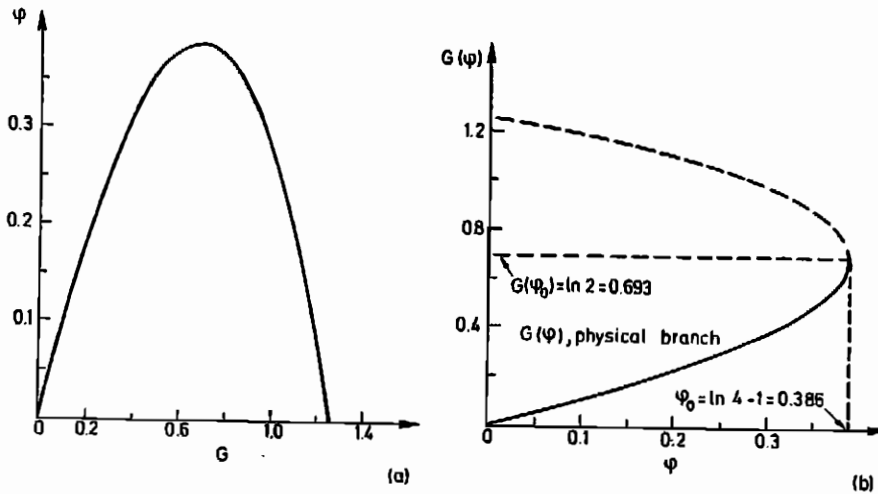


Fig. 2.1 Graphical solution of the bootstrap Eq. (2.40)
(a) $\varphi = \varphi(G)$; (b) $G = G(\varphi)$.

$$2\pi B m_\pi^2 \frac{K_1(\beta_0 m_\pi)}{\beta_0 m_\pi} = \ln 4 - 1 \quad (2.41)$$

$$\beta_0 \approx 1/m_\pi$$

As $\varphi(\beta)$ increases monotonically with $1/\beta$, it follows that β_0 is the lowest value for which a real solution $\phi(\beta)$ of Eq. (2.37) exists. With the knowledge of $\phi(\beta)$ our problem is, in principle, solved, since the wanted mass spectrum follows by inverse Laplace transformation

$$B\tau(p^2) = \mathcal{L}^{-1}[\phi(\beta)] \quad (2.42)$$

We can obtain some information about $\tau(p^2)$ even without explicitly inverting Eq. (2.35a); we rewrite this equation by inserting $1 = \int dm^2 \delta_0(p^2 - m^2)$:

$$\begin{aligned} \phi(\beta) &= \int \tau(m^2) dm^2 \int B \delta_0(p^2 - m^2) e^{-\beta \sqrt{p^2}} d^4 p = \\ &= \frac{2\pi B}{\beta} \int \tau(m^2) m K_1(\beta m) dm^2 \end{aligned} \quad (2.43)$$

As we have just seen, $G(\varphi)$ has a square root branch point at φ_0 , and so has $\phi(\beta)$ at β_0 , since φ is monotonous in β . Since $K_1(m\beta)$ behaves like $\exp(-\beta m)$ for $m \rightarrow \infty$, Eq. (2.43) can yield a singularity of $\phi(\beta)$ only if $\tau(m^2)$ grows asymptotically like $\exp(\beta_0 m)$; a square root branch point requires

$$\tau(m^2) \sim \frac{\text{const}}{m^3} e^{\beta_0 m} \quad (2.44)$$

A more familiar expression for the mass spectrum is $\rho(m)$ (cf. the text and the equations from (2.7) to (2.11)), related to $\tau(m^2)$ by $\tau(m^2) dm^2 = \rho(m) dm$:

$$\rho(m) = 2m\tau(m^2) \quad (2.45)$$

$\rho(m)$ starts with $\delta(m - m_\pi)$ and, according to Eq. (2.44) behaves asymptotically as

$$\rho(m) \sim \frac{\text{const}}{m^2} e^{\beta_0 m} \quad (2.46)$$

For later we note that, had we not absorbed the p^2 factor into $\tau(p^2)$, but defined $\tilde{\tau}(p^2)$ to be the mass spectrum, we would have obtained

$$\begin{aligned}\tilde{\tau}(m^2) &\sim \frac{\text{const}}{m^5} e^{\beta_0 m} \\ \tilde{\rho}(m) &\sim \frac{\text{const}}{m^4} e^{\beta_0 m}\end{aligned}\quad (2.47)$$

In Eqs. (2.44), (2.46) and (2.47) the value of β_0 is independent of the definition of the mass spectrum, while the constants and higher order corrections are not. One can calculate the constant as well as higher corrections to $\tau(m^2)$ by expanding $G(\varphi)$ at the branch point; one obtains

$$G(\varphi) = G_0 - (\varphi_0 - \varphi)^{1/2} - \frac{1}{6} (\varphi_0 - \varphi) - \frac{1}{36} (\varphi_0 - \varphi)^{3/2} - \dots \quad (2.48)$$

Rewriting this expansion in terms of β and applying \mathcal{L}^{-1} to it is straightforward but tedious. The result is found in Ref. 14). While this recipe yields still only an asymptotic expression for $\tau(m^2)$ — which is, however, rather good down to almost m_π — an exact expression for small $p^2 = m^2$ has been given by Yellin¹³⁾ in terms of a series expansion of $G(\varphi)$ at $\varphi = 0$:

$$G(\varphi) = \sum_{n=1}^{\infty} g_n \varphi^n \quad (2.49)$$

Therefore $G(\varphi)$ is the generating function for the Yellin coefficients g_n which are very interesting numbers by themselves: $n!g_n$ is the total number of different ways to cluster n elements. The g_n as well as our Eq. (2.40) were already discovered in 1870 by E. Schröder¹⁹⁾:

Using Eq. (2.49) we can invert the Laplace transformation directly, because we know the original function $\mathcal{L}^{-1}\varphi(\beta)$ from Eq. (2.35b). Write

$$\begin{aligned}\varphi(\beta)^m &= \int e^{-\beta \mu \sum_{i=1}^m p_i^\mu} \prod_{i=1}^m B \delta_0(p_i^2 - m_\pi^2) d^4 p_i \\ &= \int [\delta^4(p - \sum_{i=1}^m p_i) \prod_{i=1}^m B \delta_0(p_i^2 - m_\pi^2) d^4 p_i] e^{-\beta \mu p^\mu} d^4 p \\ &= \mathcal{L} \left[\int \delta^4(p - \sum_{i=1}^m p_i) \prod_{i=1}^m B \delta_0(p_i^2 - m_\pi^2) d^4 p_i \right]\end{aligned}\quad (2.50)$$

The expression in brackets is the n -pion IMS integral

$$\Omega_m(p^2, B) := \int \delta^4(p - \sum_{i=1}^m p_i) \prod_{i=1}^m B \delta_0(p_i^2 - m_\pi^2) d^4 p_i \quad (2.51)$$

Applying \mathcal{L}^{-1} to Eq. (2.49) yields therefore

$$\begin{aligned}\mathcal{L}^{-1}\varphi(\beta) &= B\tau(p^2) = \\ &= \sum_{n=1}^{\infty} g_n \mathcal{L}^{-1}[\varphi(\beta)]^n = \sum_{n=1}^{\infty} g_n \Omega_n(p^2, B)\end{aligned}\quad (2.52)$$

The IMS integrals are well-known functions, for which powerful computer programs exist. Therefore, Eq. (2.52) is very useful at not too large p^2 , since the sum has actually only a finite number

of terms -- it is cut off at $n \leq \sqrt{p^2}/m_\pi$ by the momentum δ^4 function and by the condition $p_0 \geq m_\pi$.

Had we used IMS measure in Eq. (2.6), the density of states of the pion gas would have read

$$\sigma_{\text{IMS}}(p^2) = \sum_{n=1}^{\infty} \frac{1}{n!} \Omega_n(p^2, B) \quad (2.53)$$

while now

$$B\tau(p^2) = \sum_{n=1}^{\infty} g_n \Omega_n(p^2, B) \quad (2.54)$$

One sees that the rapidly decreasing $1/n!$ has been replaced by the (exponentially increasing!) g_n . Thus the Ω_n in Eq. (2.53) have been multiplied by $n!g_n$, which is the total number of possible ways to cluster n pions; this factor represents the whole bootstrap dynamics.

With the help of Laplace transforms one can easily prove a recurrence relation for $\Omega_n(p^2, B)$:

$$\Omega_n(p^2, B) = \int \delta^4(p - \sum_{k=1}^{\ell} p_k) \prod_{k=1}^{\ell} \Omega_{n_k}(p_k^2, B) d^4 p_k \quad (2.55)$$

for any choice of n_1, n_2, \dots, n_ℓ such that $\sum_{k=1}^{\ell} n_k = n$. Inserting the expression (2.54) into the BE (2.33) and using the recursion relation (2.55), one obtains a similar recursion relation for the

g_n

$$g_1 = 1$$

$$g_n = \sum_{k=2}^n \frac{1}{k!} \sum_{m_i} \delta(m - \sum_{i=1}^k m_i) \prod_{i=1}^k g_{m_i} \quad (2.56)$$

This equation determines the g_n uniquely. For practical calculations another recursion relation²¹⁾ is more useful:

$$g_n = -\frac{n-1}{n} g_{n-1} + \sum_{k=1}^{n-1} g_k g_{n-k} \quad (2.57)$$

$$g_0 = 0; g_1 = 1$$

Given Eqs. (2.48), (2.49) and (2.57) the bootstrap equation (2.33) can be considered as solved.

3. THERMODYNAMICS

We have solved in Section 2 the BE with the help of the Laplace transformation. The same mathematical procedure is used in statistical thermodynamics to obtain the partition function from the density of states. This coincidence has had the effect that the Laplace transformation $\phi(\beta)$ of the mass spectrum $\tau(p^2)$ and the Laplace transform $Z(\beta, V)$ of the density of states of a thermodynamical system containing particles with the mass spectrum $\tau(m^2)$, have sometimes been confounded. We expect a relation between $\phi(\beta)$ and $Z(\beta, V)$ -- and we will exploit it below -- but conceptually these two quantities are different.

3.1 The partition functions of the one-component ideal gas

Consider an ideal relativistic Boltzmann gas with one sort of particle of mass m enclosed in an arbitrary, macroscopic external volume V_μ^{ex} . From Eq. (2.4) the number of states in $\{p, d^4 p\}$ of one particle in the four volume V_μ^{ex} is

$$\frac{2 V_\mu^{\text{ex}} p^\mu}{(2\pi)^3} \delta_0(p^2 - m^2) d^4 p \quad (3.1)$$

From this, the one-particle partition function $Z_1^{(0)}$ (the superscript denotes "non-interacting") is defined by

$$Z_1^{(0)}(\beta, v^{\text{ex}}) := \int \frac{2V_\mu^{\text{ex}} p^\mu}{(2\pi)^3} \delta_0(p^2 - m^2) e^{-\beta_\mu p^\mu} d^4 p \quad (3.2)$$

Here the four volume V_μ^{ex} is an arbitrary external parameter (a box of arbitrary volume having an arbitrary four velocity) while before, in Section 2, we took the volume to be the dynamically determined proper co-moving volume of the particle.

Thus, in Eq. (3.2) v^{ex} is a constant with respect to the p integration. Furthermore, β_μ has from now on the physical meaning of the inverse temperature four vector:

$$\sqrt{\beta_\mu \beta^\mu} = \frac{1}{T} =: \beta \quad (3.3)$$

where the Lorentz invariant T is the usual temperature (remember: the Boltzmann constant $k = 1$) in the rest frame of the thermometer.

$Z_1^{(0)}$ is by construction a function of the invariants β^2 , v_{ex}^2 and $\beta^\mu v_\mu^{\text{ex}}$. As it seems not very useful to consider a description where the thermometer moves (fast) with respect to the container of a gas, we take here β^μ parallel to v^μ :

$$\beta^\mu = \beta u^\mu \quad ; \quad V^\mu = V u^\mu \quad (3.4)$$

where u^μ is the common four-velocity of the thermometer and of the container (we drop from now on the subscript "ex" of the volume V of the container).

We then obtain in the common rest frame of β^μ and v^μ

$$Z_1^{(0)}(\beta, V) = \frac{Vm^3}{2\pi^2} \frac{K_2(\beta m)}{\beta m} \quad (3.5)$$

(Compare this to Eq. (2.38) which followed from taking v^μ parallel to p^μ , while β^μ was arbitrary.)

From the one-particle partition function the N particle partition function is found

$$Z_N^{(0)}(\beta, V) = \frac{1}{N!} Z_1^{(0)}(\beta, V)^N \quad (3.6)$$

The grand canonical partition function^{*} is

$$Z^{(0)}(\beta, V, \lambda) := \sum_{n=0}^{\infty} \lambda^n Z_n^{(0)}(\beta, V) = e^{\lambda Z_1^{(0)}(\beta, V)} \quad (3.7)$$

with λ being the fugacity. From $Z^{(0)}(\beta, V, \lambda)$ nearly all relevant quantities can be found by logarithmic differentiation, in particular

^{*}) We keep the bad habit of physicists to use the same symbol for mathematically different functions; the information is contained in the arguments.

$$\varepsilon^{(0)}(\beta, V, \lambda) := -\frac{1}{V} \frac{\partial}{\partial \beta} \ln Z^{(0)}(\beta, V, \lambda) = \text{energy density}$$

$$P^{(0)}(\beta, V, \lambda) := \frac{T}{V} \ln Z^{(0)}(\beta, V, \lambda) = \text{pressure} \quad (3.8)$$

$$n^{(0)}(\beta, V, \lambda) := \frac{\lambda}{V} \frac{\partial}{\partial \lambda} \ln Z^{(0)}(\beta, V, \lambda) = \text{particle number density}$$

and so on. We introduce the relativistic chemical potential (equal to $\mu_{\text{non-rel}} + m$) by

$$\lambda = e^{\beta \mu} \quad (3.9)$$

$\mu = 0$ ($\lambda = 1$) corresponds to black-body radiation of quanta with rest mass m .

3.2 The strongly interacting pion gas

A gas of strongly interacting particles (of one kind^{*)}) enclosed in an arbitrary volume at arbitrary temperature and chemical potential may be described

either : as a gas of one single sort of particle with a complicated interaction

or : as a non-interacting gas consisting of an infinity of different species with a mass spectrum appropriate to the interaction in question.

This statement has been discussed in great detail in Refs. 1) and 2).

*) The generalization to several different species is straightforward.

What it claims is that if the mass spectrum of the interaction is known, replacing the interacting one-component gas by an ideal infinite-component gas and weighing the different components according to the mass spectrum, generates the same distortion of phase space as the interaction would do. An example is, for instance, a dilute He gas; usually it is not described as a gas of protons, neutrons, electrons with a Hamiltonian containing QED and strong interactions; instead one uses the mass spectrum (here essentially one state with mass, spin etc. of He⁴) and calculates the properties of an ideal Bose gas of He atoms, considering the latter as elementary.

Accordingly we make here the hypothesis: should the mass spectrum of strong interactions be known, it could be used to arrive at a statistical-thermodynamical description of strongly interacting particles in terms of an ideal gas of infinitely many components. The above hypothesis that the mass spectrum adequately represents the otherwise unmanageable interaction is used, not only in dealing with a strong interaction gas, but the SBM which finally yields the mass spectrum is built itself on this same hypothesis. Taking now the attitude that SBM has provided us with the correct spectrum, the corresponding statistical thermodynamics of the strongly interacting pion gas follows by simply generalizing the formulae of the ideal gas given in Subsection 3.1.

The one-particle phase space measure (3.1) becomes now the "one-fireball" phase space measure

$$\begin{aligned} d^4p \sigma_1(p, V) &= \frac{2V^M p^M}{(2\pi)^3} d^4p \int dm^2 \tau(m^2) \delta_0(p^2 - m^2) \\ &= \frac{2V^M p^M}{(2\pi)^3} \tau(p^2) d^4p \end{aligned} \quad (3.10)$$

Accordingly we find the "one-fireball" partition function

$$Z_1(\beta, V) = \int \frac{2V_\mu p^\mu}{(2\pi)^3} \tau(p^2) e^{-\beta_\mu p^\mu} d^4p \quad (3.11)$$

We do not need to find $\tau(p^2)$ by an explicit inverse Laplace transform from $\phi(\beta)$ as given by Eqs. (2.42) and (2.43), because Z_1 can be computed directly from $\phi(\beta)$. Recall Eq. (2.35a):

$$\phi(\beta) = \int B\tau(p^2) e^{-\beta_\mu p^\mu} d^4p \quad (3.12)$$

Obviously

$$Z_1(\beta, V) = -\frac{2V_\mu}{B(2\pi)^3} \frac{\partial}{\partial \beta_\mu} \phi(\beta) \quad (3.13)$$

Our postulate that V^μ and β^μ should be parallel allows us to go to their common rest frame and rewrite Eq. (3.13) as

$$Z_1(\beta, V) = -\frac{2V}{B(2\pi)^3} \frac{\partial \phi(\beta)}{\partial \beta} \quad (3.14)$$

We know from Section 2 that no real $\phi(\beta)$ exists for $\beta < \beta_0 \approx m_\pi^{-1}$ (see Eq. (2.41)) and therefore no real $Z_1(\beta, V)$ exists for $\beta < \beta_0$. Thus

$$T_0 : = 1/\beta_0 \approx m_\pi \quad (3.15)$$

is a critical temperature.

We do not specify now what happens at T_0 , because that will depend on the power of m in front of the exponential of $\tau(m^2)$ (see Eqs (2.44)-(2.47) and Table 3.1). We shall come back to this in Subsection 3.3.

In Section 2 we also mentioned another possible definition of the mass spectrum (text following Eq. (2.31)), where the quantity

$$B\tau(p^2) \equiv \tilde{B}(p^2) \tilde{\tau}(p^2) = \frac{2A}{(2\pi)^3} p^2 \tilde{\tau}(p^2) \quad (3.16)$$

obeys, as before, the BE, but where $\tilde{\tau}(m^2)$ is taken to be the mass spectrum. Because of the identity (3.16), $\phi(\beta)$ is the same function as before:

$$\phi(\beta) = \int B\tau(p^2) e^{-\beta_\mu p^\mu} d^4p \equiv \frac{2A}{(2\pi)^3} \int p^2 \tilde{\tau}(p^2) e^{-\beta_\mu p^\mu} d^4p \quad (3.17)$$

but now the partition function

$$\tilde{Z}_1(\beta, V) := \int \frac{2V_\mu p^\mu}{(2\pi)^3} \tilde{\tau}(p^2) e^{-\beta_\mu p^\mu} d^4p \quad (3.18)$$

obeys a different relation with $\phi(\beta)$. We find it by inserting

$$1 \equiv \int \delta_0(p^2 - m^2) dm^2$$

in the integrals of Eqs (3.17) and (3.18) and integrating over p :

$$\phi(\beta) = \frac{A}{\beta 2\pi^2} \int m^3 \tilde{\tau}(m^2) K_1(m\beta) dm^2 \quad (3.19)$$

$$\tilde{Z}_1(\beta, V) = \frac{V}{\beta^2 \pi^2} \int m^2 \tilde{\epsilon}(m^2) K_2(\beta m) dm^2 \quad (3.20)$$

Inserting the integral of the formula²²⁾

$$\frac{d}{d\beta} [\beta^2 K_2(m\beta)] = -m\beta^2 K_1(m\beta) \quad (3.21)$$

into Eq. (3.20), we find by comparing with (3.19)

$$\tilde{Z}_1(\beta, V) = \frac{V}{A} \frac{1}{\beta^3} \int_{\beta}^{\infty} x^3 \phi(x) dx \quad (3.22)$$

Comparison of Eqs. (3.22) and (3.14) shows how important the proper definition of $\tau(m^2)$ is (cf. discussion after Eq. (2.31)). Indeed, Eqs. (3.22) and (3.14) lead to different physical behaviour near the critical temperature T_0 . We recall that $G(\varphi) \equiv \phi(\beta)$ has a square root singularity: $\phi(\beta) = \phi_0 - \text{const} \cdot (\beta - \beta_0)^{\frac{1}{2}}$. Therefore, $Z_1(\beta) \sim \text{const} \cdot (\beta - \beta_0)^{-\frac{1}{2}}$, $\tilde{Z}_1 \sim \text{const} - \text{const} \cdot (\beta - \beta_0)^{3/2}$. The corresponding energy diverges in the first case: $\epsilon(\beta) \sim \text{const} \cdot (\beta - \beta_0)^{-3/2}$ and remains finite in the second: $\tilde{\epsilon}(\beta) = \tilde{\epsilon}_0 - \text{const} \cdot (\beta - \beta_0)^{\frac{1}{2}}$.

We now return to the discussion of the thermodynamics following from the choice of $\tau(m)$ made in Section 2, Eq. (2.32): from $Z_1(\beta, V)$ we find the N fireball partition function

$$Z_N(\beta, V) = \frac{1}{N!} Z_1(\beta, V)^N \quad (3.23)$$

and the grand canonical partition function

$$\begin{aligned} Z(\beta, V, \lambda) &= \sum_{n=0}^{\infty} \frac{1}{n!} [\lambda Z_1(\beta, V)]^n = \\ &= \exp [\lambda Z_1(\beta, V)] \end{aligned} \quad (3.24)$$

The relations (3.8) now become

$$\begin{aligned} \epsilon(\beta, V, \lambda) &= -\frac{1}{V} \frac{\partial}{\partial \beta} \ln Z(\beta, V, \lambda) \\ P(\beta, V, \lambda) &= \frac{T}{V} \ln Z(\beta, V, \lambda) \end{aligned} \quad (3.25)$$

$$n(\beta, V, \lambda) = \frac{\lambda}{V} \frac{\partial}{\partial \lambda} \ln Z(\beta, V, \lambda)$$

Here, however, $n(\beta, V, \lambda)$ is the average number of fireballs present. For this $n(\beta, V, \lambda)$ we have the ideal gas equation (due to the linearity of $\ln Z$ in λ)

$$P = n \cdot T \quad (3.26)$$

while the corresponding equation in terms of the average number of pions (contained in all these fireballs together) would look horribly complicated. This result (3.26), which in the framework of this model is exact, shows once more how simple things become once the interaction is hidden in the mass spectrum.

At this point we can generalize from the pure pion Boltzmann gas to a hadron gas with correct statistics: we have to replace in Eq. (3.11) $\tau(p^2)$ by the full hadronic mass spectrum and build in

Bose and Fermi statistics. We first rewrite Eq. (3.11) in the β, V rest frame inserting $1 = \int \delta_0(p^2 - m^2) dm^2$

$$Z_1(\beta, V) = \frac{V}{\beta 2\pi^2} \int m^2 \tau(m^2) K_2(m\beta) dm^2 \quad (3.27)$$

Assume now we had solved a BE for Bosons and Fermions, then we would have obtained two changes:

i) the mass spectrum is split into a Bose part and a Fermi part

$$2m\tau(m^2) =: \rho(m) = \left\{ \rho_B(m); \rho_F(m) \right\} \quad (3.28)$$

ii) the logarithm of the grand canonical partition function Eq. (3.24) will then become⁸⁾

$$\ln Z(\beta, V, \lambda) = \frac{VT}{2\pi^2} \int \sum_{n=1}^{\infty} \frac{\lambda^n}{m^2} \left[\rho_B(m) - (-)^n \rho_F(m) \right] m^2 K_2(m\beta) dm \quad (3.29)$$

for details see Refs. 8), 10) and 17).

This equation can serve to illustrate the influence of Bose-Einstein and Fermi-Dirac statistics on SBM: in the above version it was shown⁸⁾ that

$$\rho(m) = \rho_B(m) + \rho_F(m) \sim \frac{\text{const}}{m^a} e^{\beta_0 m} \quad (3.30)$$

Furthermore, with the asymptotic behaviour of $K_2(x)$ it follows that⁸⁾

i) only the first term of (3.29) has a singularity at β_0 . This is the first singularity encountered if β decreases from $+\infty$;

ii) the integral over the rest of the sum ($n = 2 \dots \infty$) is finite even if the first term becomes infinite at β_0 .

Now it depends entirely on the power of m in front of the exponential in Eq. (3.30), how the first term of the integral (3.29) behaves for $\beta \rightarrow \beta_0$: it is finite at β_0^+ if the power "a" of m in the spectrum (3.30) is larger than $5/2$.

Thus it follows that for $a \leq 5/2$ the first term alone need be considered when $T \rightarrow T_0^-$. As this first term represents Boltzmann statistics, this means that for $a \leq 5/2$ statistics can be ignored at $T \rightarrow T_0$, while for $a > 5/2$ it may not. In contrast to the Boltzmann partition function (3.24), for which $\ln Z(\beta, V, \lambda)$ is linear in λ , the Bose + Fermi partition function (3.29) has a more complicated λ dependence. Therefore in this case the ideal gas equation (3.26) is no longer valid (except for $a \leq 5/2$ and $T \rightarrow T_0$). If $\tilde{\rho}$ is taken as the mass spectrum (Eq. (2.47)), we obtain $a = 4$ and statistics becomes relevant at T_0 (though perhaps not important numerically).

3.3 Physics near T_0

We have seen how the BE provides us with the function $\phi(\beta)$ from which $Z_1(\beta, V, \lambda) = \ln Z(\beta, V, \lambda)$ can be calculated; $Z_1(\beta, V, \lambda)$ then serves as the generating function for physical quantities. In all versions of SBM we find an exponential mass spectrum

$$\rho(m) \sim \frac{\text{const}}{m^a} e^{m/T_0} \quad (3.31)$$

with T_0 of order m_π . While the small variation of $T_0 \approx m_\pi$ from version to version is of no physical importance, the nature of the system, when $T \rightarrow T_0$, depends critically on the power of m in Eq. (3.31). We study this now by using Eq. (3.27) to determine how the type of singularity at T_0 depends on the power "a" of the mass.

We replace in (3.27) $\tau(m^2) dm^2$ by $\rho(m) dm$:

$$Z_1(\beta, V) = \frac{VT}{2\pi^2} \int m^2 \rho(m) K_2(m\beta) dm \quad (3.32)$$

As we are interested in the behaviour at $T \rightarrow T_0$ ($\beta \rightarrow \beta_0$) we denote all quantities which are constant in this limit by the symbol C (at each place where it occurs, C may have a different value and/or dimension). Using Eq. (3.31) and the asymptotic formula

$$K_2(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \quad (3.33)$$

we obtain

$$Z_1(\beta, V) \underset{T \rightarrow T_0}{\sim} C \int_M^\infty m^{3/2-a} e^{-(\beta-\beta_0)m} dm + C \quad (3.34)$$

M is a mass large enough to justify the use of the asymptotic formulae (3.31) and (3.33), the $+C$ stands for the non-singular integral from m_π to M . With

$$\beta - \beta_0 = \frac{T_0 - T}{TT_0} \rightarrow C (T_0 - T) = C \Delta T \quad (3.35)$$

Eq. (3.34) can be written as

$$Z_1(\beta, V) \underset{T \rightarrow T_0}{\sim} \frac{C}{\Delta T^{5/2-a}} \int_{\Delta M}^\infty x^{3/2-a} e^{-x} dx + C \quad (3.36)$$

$$\Delta M := C \Delta T M$$

As we know that the integral converges at ∞ , we expand the exponential and obtain

$$\int_{\Delta M}^\infty x^{3/2-a} e^{-x} dx = C - \frac{\Delta M^{5/2-a}}{5/2-a} [1 + O(\Delta M)] \quad (3.37)$$

Thus

$$Z_1(\beta, V) \underset{T \rightarrow T_0}{\sim} C + C \Delta T^{a-5/2}; \quad (a \neq 5/2) \quad (3.38)$$

The case $a = 5/2$ yields (see Eq. (3.36))

$$Z_1(\beta, V) \underset{T \rightarrow T_0}{\sim} C - \ln \frac{\Delta T}{T_0}; \quad (a = 5/2) \quad (3.39)$$

In the following table we list the most interesting quantities for $a = \frac{1}{2}, 2/2 \dots 8/2$, namely P = pressure, n = fireball number density, ϵ = energy density, $\delta\epsilon/\epsilon$ mean relative fluctuations of ϵ , $C_V = d\epsilon/dT$ = specific heat and v^2 = square of velocity of sound.

Our new hypothetical interpretation of BE's has added the cases $a = 2$ and $a = 4$ to the previously studied interval $5/2 \leq a \leq 7/2$. In fact, the discussion following Eq. (2.31) suggests that it is possible to split the quantity $\tilde{B}(m^2)\tilde{\tau}(m^2)$ in any way we like into two functions of m^2 , one of which we call "mass spectrum". $\tau(m)$ and $\tilde{\tau}(m)$ corresponding to $a = 2$ and $a = 4$ respectively, are extreme choices; hence $2 \leq a \leq 4$ seems now a reasonable range of physically possible a values. With the exception of the velocity of sound, which is independent of "a" when $T \rightarrow T_0$, and the specific heat, for which we always find $C_V \rightarrow \infty$ when $T \rightarrow T_0$, all other quantities cover the whole range from non-singular to singular when "a" changes in the interval $\{2, 4\}$. The lesson is that much depends on the seemingly unimportant power of m in front of the exponential mass spectrum.

$$\begin{aligned}
 & \frac{2V_b p}{(2\pi)^3} \sigma(p, V, b) = \delta^4(V - V_b) C_b \frac{2V_b p}{(2\pi)^3} \delta_0(p^2 - M_b^2) + \\
 & + \sum_{k=2}^{\infty} \frac{1}{k!} \left\{ \sum_{\{b_i\}} \delta(b - \sum_{i=1}^k b_i) \delta^4(p - \sum_{i=1}^k p_i) \delta^4(V - \sum_{i=1}^k V_i) \right\} \times \\
 & \times \prod_{i=1}^k \frac{2V_i p_i}{(2\pi)^3} \sigma(p_i, V_i, b_i) d^4 p_i d^4 V_i
 \end{aligned}
 \tag{4.1}$$

Equation (4.1) is not a single BE, but a member (with baryon number b) of an infinite set of coupled integral equations, each having its own input term. The $(k!)^{-1}$ is necessary for correct counting. The non-vanishing pion and nucleon mass ensure that for any finite p^2 the set (4.1) has only a finite number of equations: $|b_{\max}| \leq \sqrt{p^2}/m_p$. Therefore the solutions for any finite p^2 can (in principle) be built up iteratively by starting with $4m_\pi^2 \leq p^2 \leq m_p^2$ and by increasing stepwise this interval to include higher and higher $|b|$. This, incidentally, also allows to prove that for any p, V, b , Equations (4.1) have a physical solution.

This equation fulfils the requirements set up in the Introduction:

1) Conservation of baryon number b and clustering of nucleons: the baryon number (number of baryons minus number of antibaryons) is conserved with the help of the Kronecker $\delta_K(b - \sum b_i)$ function. The infinite set of density functions $\sigma(p, V, b)$ corresponds to the admission of nucleon clusters with any baryon number b , four momentum p and four volume V .

2) Nucleon (isobar) excitation and internal cluster excitation: internal cluster excitation is contained in the $p^2 = m^2$ dependence of $\sigma(p, V, b)$ and single nucleon (isobar) excitation is contained in the same way in $\sigma(p, V, 1)$.

3) Extensivity of nuclear matter is ensured by the volume δ^4 -function.

4) Co-existence of a pion gas is contained in the equation with $b = 0$ and in all others by the presence of factors $\sigma(p_i, V_i, b_i = 0)$ on the right-hand side.

5) Baryon-antibaryon pair creation (and annihilation) is built in by allowing $-\infty < b_i, b < \infty$. Then on the right-hand side an arbitrary number of clusters $(\sum b_i)$ and anticlusters $(-\sum \bar{b}_i)$ may occur.

6) "Chemical equilibrium" between all constituents: this is expressed by the infinite set of coupled integral equations (4.1) which allows all multibody reactions between clusters Q_i :



compatible with b and p conservation.

The input terms, except that for $b = 0$ (pion) and for $b = 1$ (nucleon) specify particular features of the model, namely,

i) details of nuclear interaction may be represented by giving clusters (e.g., alpha particles) a special weight.

ii) The equations (4.1) deal with Boltzmann particles without charge and spin. Introducing spin, isospin and statistics would be possible but complicated. We can obtain a similar physical effect by assigning to an input nucleus of baryon number b and volume V_b a mass M_b which is greater than $(b \cdot m_p)$.

4.2 Separation of the volume dependence

We follow closely subsection 2.2.2. We introduce

$$\frac{2Vp}{(2\pi)^3} \sigma(p, V, b) = \delta^4(V - V(m, b) \frac{p}{m}) \tilde{B}(p^2, b) \tilde{\tau}(p^2, b) \quad (4.2)$$

where the function \tilde{B} describes the p^2, b dependence of the $V \cdot p$ term, while $\tilde{\tau}$ that of σ (cf. Eqs. (2.22) to (2.24)). Now we rewrite Eq. (4.1), integrate over the $\mathbb{R}^4 V$; and require that all volume δ^4 functions have the same argument. We find in the present case the condition

$$\sum_{i=1}^m \left(\frac{V(m_i, b_i)}{m_i} - \frac{V(m)}{m} \right) p_i^\mu = 0 \quad (4.3)$$

from which it follows as before that, for all i ,

$$V(m_i, b_i) = A m_i \quad (4.4)$$

The constant A is independent of i and therefore equal to that defined earlier (Eqs (2.26) to (2.30)). It therefore follows that

$$\tilde{B}(m_i^2, b) = \frac{2V(m_i, b) m_i}{(2\pi)^3} = \frac{2A m_i^2}{(2\pi)^3} = \tilde{B}(m_i^2) \quad (4.5)$$

is, in fact, independent of b and the same as the $\tilde{B}(m^2)$ defined in Eq. (2.26).

The volume δ^4 function can now be factored out on both sides of Eq. (4.1) and what remains is a new bootstrap equation for the function $\tilde{\tau}(p^2, b)$:

$$\tilde{B}(p^2) \tilde{\tau}(p^2, b) = C_b \tilde{B}(M_b^2) \delta_0(p^2 - M_b^2) + \quad (4.6)$$

$$+ \sum_{k=2}^{\infty} \frac{1}{k!} \int \sum_{\{t_i\}} \delta^4(b - \sum_{i=1}^k t_i) \delta^4(p - \sum_{i=1}^k p_i) \prod_{i=1}^k \tilde{B}(p_i^2) \tilde{\tau}(p_i^2, t_i) d^4 p_i$$

The essential step now consists in the proper extraction of the mass spectrum $\tau(p^2, b)$ from the function $\tilde{\tau}$. We follow here consequently the recipe developed in Section 2, Eq. (2.32). Thus for each b we now write

$$\tau(p^2, b) = \frac{\tilde{B}(p^2)}{\tilde{B}(M_b^2)} \tilde{\tau}(p^2, b) \quad (4.7a)$$

Introducing for convenience the coefficient B_b through

$$B_b = \tilde{B}(M_b^2) = \frac{2A M_b^2}{(2\pi)^3} \quad (4.7b)$$

we can write the bootstrap equation for the mass spectrum as

$$B_b \tau(p^2, b) = C_b B_b \delta_0(p^2 - M_b^2) + \quad (4.8)$$

$$+ \sum_{k=2}^{\infty} \frac{1}{k!} \int \sum_{\{t_i\}} \delta^4(b - \sum_{i=1}^k t_i) \delta^4(p - \sum_{i=1}^k p_i) \prod_{i=1}^k B_{t_i} \tau(p_i^2, t_i) d^4 p_i$$

The bootstrap equation (4.8) is much richer than that for the pion gas; we have allowed the presence of arbitrarily complicated clusters characterized by the baryonic number b_i . For $b = 0$ we have a description of meson - fireballs - but in order to understand these fireballs properly, especially when baryon-antibaryon clusters are among their constituents we have to obtain a solution for the function τ for all values of b .

We remark that our definition (4.7b) makes B_b proportional to M_b^2 and in this way leads us nearer to the case $B \sim m^2$, discussed in Section 2 after Eq. (2.31), although not exactly to it, since M_b is the mass of the b cluster in the ground state.

4.3.1 Laplace and L transforms of the mass spectrum

In order to solve the nuclear bootstrap equation we apply the same methods and techniques as in Section 2. The new task is the treatment of the b dependence. This is done by defining the "L transform":

$$L[f(b)] := \sum_{b=-\infty}^{\infty} \lambda^b f(b) =: f_\lambda(\lambda) \quad (4.9)$$

Hence $f_\lambda(\lambda) = L[f(b)]$ is the generating function of $f(b)$ *). We multiply the entire bootstrap equation with λ^b and sum over b . Defining the L transforms of $\tau(p^2, b)$ and of the input term, respectively:

$$B_N \tau_\lambda(p_i^2, \lambda) := \sum_{b=-\infty}^{\infty} \lambda^b B_b \tau(p_i^2, b) \quad (4.10a)$$

*) We use the expression "L transform" to stress the formal analogy with the Laplace transform: L is the discrete counterpart of \mathcal{L} .

$$B_N \tau_{0\lambda}(p_i^2, \lambda) := B_\pi \delta_0(p^2 - m_\pi^2) + \sum_{b=1}^{\infty} C_b B_b (\lambda^b + \lambda^{-b}) \delta_0(p^2 - M_b^2) \quad (4.10b)$$

where

$$B_1 \equiv B_N ; M_1 \equiv m_N ,$$

we find that the bootstrap equation takes the form of the pion bootstrap equation, however, with a much more involved input function $\tau_{0\lambda}$:

$$\tau_\lambda(p_i^2, \lambda) = \tau_{0\lambda}(p_i^2, \lambda) + \sum_{k=2}^{\infty} \frac{1}{k!} \int \delta^4(p - \sum p_i) \prod_{i=1}^k \tau_\lambda(p_i^2, \lambda) d^4 p_i \quad (4.11)$$

En passant we note that this confirms the general bootstrap philosophy that the input function characterizes the "raw material" while the integral equation imposes the dynamics onto it. The dynamics should be more or less independent of what the raw material is (but it will depend on kinematics, statistics, etc.)

Once we solve the bootstrap equation for the function τ_λ it is, in principle, possible to invert the L transform by integrating along a circle C around the origin in the λ plane:

$$B_b \tau(p_i^2, b) = \frac{1}{2\pi i} \oint_C \frac{d\lambda}{\lambda^{b+1}} B_N \tau_\lambda(p_i^2, \lambda) \quad (4.12)$$

We need to consider the Laplace transform in p of the function τ in order to proceed further (compare Eqs. (2.35a), (2.35b)):

$$\phi(\beta, \lambda) := \int B_N \tau_\lambda(p^2, \lambda) e^{-\beta p} d^4 p \quad (4.13)$$

$$\varphi(\beta, \lambda) := \int B_N \tau_{0\lambda}(p^2, \lambda) e^{-\beta p} d^4 p \quad (4.14)$$

These two equations define two basic quantities we will often refer to further below. In complete analogy to the case of pionic bootstrap we find now the usual bootstrap equation (see Eq. (2.37)):

$$\phi(\beta, \lambda) = \varphi(\beta, \lambda) + \exp[\phi(\beta, \lambda)] - \phi(\beta, \lambda) - 1 \quad (4.15)$$

For later convenience we introduce the Laplace transform of $\tau(p^2, b)$

$$\phi_b(\beta, b) := \int B_b \tau(p^2, b) e^{-\beta p} d^4 p \quad (4.16a)$$

$$\varphi_b(\beta, b) := \int B_b \tau_0(p^2, b) e^{-\beta p} d^4 p \quad (4.16b)$$

The quantity $B_b \tau_0$ in Eq. (4.16b) is the input term of Eq. (4.6). The relation between ϕ and ϕ_b is given by the inverse L transform:

$$\phi_b(\beta, b) = \frac{1}{2\pi i} \oint_C \frac{d\lambda}{\lambda^{b+1}} \phi(\beta, \lambda) \quad (4.17a)$$

and similarly

$$\varphi_b(\beta, b) = \frac{1}{2\pi i} \oint_C \frac{d\lambda}{\lambda^{b+1}} \varphi(\beta, \lambda) \quad (4.17b)$$

Here $\varphi_b(\beta, b)$ is the term in the Laplace-transformed input function that is associated with the b nucleon cluster, cf. Eq. (2.38). Thus we have explicitly

$$\varphi(\beta, \lambda) = \sum \lambda^b \varphi_b(\beta, b) \quad (4.18a)$$

$$\varphi_b(\beta, b) = C_b B_b 2\pi M_b^2 \frac{K_1(\beta M_b)}{\beta M_b} \quad (4.18b)$$

In Fig. 4.1 we give a short summary of the relations between the four functions introduced above: τ , τ_λ , ϕ_b , ϕ .

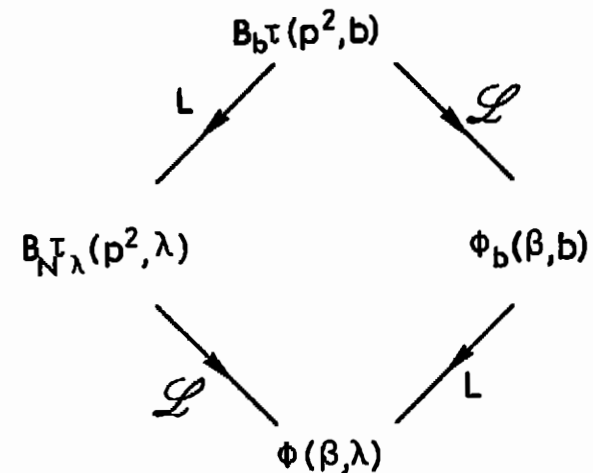


Fig. 4.1 Logical connection between the mass spectrum $\tau(p^2, b)$ and its Laplace (\mathcal{L}) and L transforms.

4.3.2 Properties of the transforms

The bootstrap equation (4.15) for the doubly transformed function $\phi(\beta, \lambda)$ has a real solution wherever in the (β, λ) plane the input function φ satisfies the condition (cf. Eq. (2.41) and Figs. 2.1a and b)

$$\varphi(\beta, \lambda) \leq \ln 4 - 1 \quad (4.19)$$

Thus along a curve $\beta_c = f(\lambda_c)$ in the (β, λ) plane defined as the boundary of this domain

$$\varphi(\beta_c, \lambda_c) = \ln 4 - 1 \quad (4.20)$$

a qualitative change in the behaviour of the properties of nuclear matter may occur. Quite aside from the physical questions, we have to ask for a mathematical solution of the bootstrap equation beyond this boundary line. As we have previously argued by a recursive argument, a physical solution for $\tau(p^2, b)$ exists for any p^2 . Our $\phi(\beta, \lambda)$ is the Laplace-L-transform of $B_b \tau(p^2, b)$:

$$\phi(\beta, \lambda) = \sum_{b=-\infty}^{\infty} \lambda^b B_b \int \tau(p^2, b) e^{-\beta p} d^4 p$$

which does not exist in this form everywhere in $(0 \leq \beta < \infty) \otimes (1 \leq \lambda < \infty)$. However, once defined in a domain where it does exist, it fulfils Eq. (4.15) which then permits analytical continuation of $\phi(\beta, \lambda)$ beyond the limit (4.19) into the whole (complex β) \otimes (complex λ) domain (see Section 6).

We remark here that the analytical continuation beyond φ_0 has never been considered in the case of pionic bootstrap, since there this limit on φ led to a limiting temperature;

the energy of fireballs diverged at this point and made a transition from our world to the new domain impossible. Now the presence of baryons changes this -- the introduction of λ leads to the existence of a new region with $T < T_0$ but $\varphi > \ln 4 - 1$ as we will see in the next section. We will find in our present model again a boundary $T = T_0$, at which the energy density diverges -- but this limit is not at $\varphi = \ln 4 - 1$, except when $\lambda = 1$.

As long as φ satisfies the condition of Eq. (4.19) we can use the Yellin expansion (see Eqs. (2.49), (2.52) and (2.54)) to obtain ϕ explicitly as a function of φ . With

$$\phi(\beta, \lambda) = \sum_{l=1}^{\infty} g_l \varphi(\beta, \lambda)^l \quad (4.21)$$

we can find the explicit form of $\phi_b(\beta, b)$ by computing the coefficient of λ^b in (4.21) with the help of Eq. 4.18a)²³⁾

$$\phi_b(\beta, b) = \sum_{l=1}^{\infty} g_l \sum_{\{m_j\}}^{(l, b)} \left\{ l! \prod_{j=-\infty}^{\infty} \frac{\varphi_b(\beta, j)^{m_j}}{m_j!} \right\} \quad (4.22)$$

Here the sum over all sets of n_j is restricted by the conditions

$$n_j \in \left\{ n_j \geq 0 ; \sum_j n_j = l ; \sum_j j n_j = b \right\} \quad (4.23)$$

This expansion considers in each term "l" objects consisting of groups of n_j "elementary" clusters with baryon number j , such that the total baryon number is conserved. This power series expansion in "non-interacting elementary clusters" $\varphi_b(\beta, j)$ has the same physical meaning as the expansion of an interacting field in free-particles -- it is a series of products of free objects

and only the expansion coefficients contain the deeper physical information. The behaviour of this power expansion illustrates the various physical regimes contained in the bootstrap equation, which describes a loosely interacting system at relatively low temperatures and densities where the expansion (4.21) works well, further a strongly interacting system with dominant clustering, where the series (4.21) converges badly and finally a qualitatively different regime, where the Yellin expansion (4.21) does not exist.

With a particular choice of the input function in Eq. (4.6), namely,

$$C_b = 0 \quad [|b| > 1] ; \quad C_b = 1 \quad [|b| \leq 1] \quad (4.24)$$

which neglects all detailed nucleon structure and statistics effects we obtain for the transformed input function $\varphi(\beta, \lambda)$ the expression

$$\varphi(\beta, \lambda) = \varphi_0(\beta) + (\lambda + \frac{1}{2}\lambda) \varphi_1(\beta)$$

$$\varphi_0(\beta) = B_\pi \frac{2\pi m_\pi^2}{\beta m_\pi} K_1(\beta m_\pi) / \beta m_\pi \quad (4.25)$$

$$\varphi_1(\beta) = B_N \frac{2\pi m_N^2}{\beta m_N} K_1(\beta m_N) / \beta m_N$$

Neglecting for the moment the possible presence of antinucleons ($\lambda \gg 1/\lambda$), we can make Eq. (4.22) more explicit, since only $j = 0, +1$ will contribute

$$\varphi_b(\beta, b) = \sum_{l=b}^{\infty} g_l \frac{l!}{b!(l-b)!} \varphi_0^{l-b} \varphi_1^b \quad (4.26)$$

This expression describes the expansion of a b -nucleon cluster into free nucleons and an arbitrary number of pions.

Upon a further inverse Laplace transform we can obtain from Eqs (4.22) and (4.26) the mass spectrum (cf. Eq. (2.54) and text below it). As will be seen in the next section, this will not be necessary -- we will be able to determine the thermodynamic properties of nuclear matter alone from the study of the function ϕ and its functional dependence on φ as described by Eq. (4.15).

5. THERMODYNAMICS OF CLUSTERED MATTER

5.1 The partition function

Let us consider a cluster with baryonic number "b" enclosed in an "external" four volume V_μ^{ex} . Then the one-cluster partition function is given by Eq. (3.11):

$$Z_{1,b}(\beta, V_\mu^{\text{ex}}, b) = \int \frac{2V_\mu^{\text{ex}} p^\mu}{(2\pi)^3} \tau(p^2, b) e^{-\beta p^\mu p_\mu} d^4p \quad (5.1)$$

the only change being the dependence of the mass spectrum on the baryonic number b ^{*}).

When n such clusters are present, but each with the same b , we find for the n cluster function the usual expression, cf. Eq. (3.23):

$$Z_{n,b}(\beta, V, b) = \frac{1}{n!} Z_{1,b}(\beta, V, b)^n \quad (5.2)$$

^{*}) As in Section 3 we drop henceforth the superscript "ex" on the four volume.

When clusters with different "b" are present, then we have to compute the product of the different contributions in Eq. (5.2). Let us consider the case in which "l" clusters are present. The sum over all possible partitions of b nucleons into l clusters gives us the partition function of b baryons assembled into l clusters:

$$Z_b(\beta, V, \lambda, l) = \sum_{\{m_j\}} \prod_{j=-\infty}^{(l; b)} \frac{1}{m_j!} Z_{1, b}(\beta, V, j)^{m_j} \quad (5.3)$$

The sum is over all partitions of b baryons into l clusters with n_j being the number of clusters having baryon number j:

$$m_j \in \left\{ m_j \geq 0; \sum_j m_j = l; \sum_j j m_j = b \right\} \quad (5.4)$$

In order to obtain the partition function of an arbitrary number of clusters having together b baryons, we have to compute in Eq. (5.3) the sum over all possible numbers of clusters "l", since each such configuration is possible. This has the net effect in Eqs. (5.3) and (5.4) that the restriction $\sum_j m_j = l$ is removed:

$$Z_b(\beta, V, \lambda) = \sum_{\{m_j\}} \delta\left(b - \sum_j j m_j\right) \prod_{j=-\infty}^{\infty} \frac{1}{m_j!} Z_{1, b}(\beta, V, j)^{m_j} \quad (5.5)$$

We have made the constraint on baryonic number explicit.

The grand canonical partition function Z is the L transform of Eq. (5.5):

$$Z(\beta, V, \lambda) = \sum_{b=-\infty}^{\infty} \lambda^b Z_b(\beta, V, \lambda) \quad (5.6)$$

It is straightforward to carry out the sum over b when Eq. (5.5) is inserted into Eq. (5.6) and we obtain

$$Z(\beta, V, \lambda) = \sum_{\{m_j\}} \prod_{j=-\infty}^{\infty} \frac{1}{m_j!} \left[\lambda^j Z_{1, b}(\beta, V, j) \right]^{m_j} \quad (5.7)$$

All values of n_j are allowed and the set $\{n_j \geq 0\}$ depends on j only through the fact that there are j members of the set. Since all j are permitted, the order in which the infinite sum and product are evaluated is irrelevant, provided that the sum converges. Under this assumption we obtain

$$\begin{aligned} Z(\beta, V, \lambda) &= \prod_{j=-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} \left[\lambda^j Z_{1, b}(\beta, V, j) \right]^m \\ &= \exp \left\{ \sum_{j=-\infty}^{\infty} \lambda^j Z_{1, b}(\beta, V, j) \right\} \end{aligned} \quad (5.8)$$

or

$$\ln Z(\beta, V, \lambda) = Z_1(\beta, V, \lambda) \quad (5.9)$$

$$Z_1(\beta, V, \lambda) = \sum_{j=-\infty}^{\infty} \lambda^j Z_{1, b}(\beta, V, j) \quad (5.10)$$

Note that the existence of $Z_1(\beta, V, \lambda)$, the one-cluster grand canonical partition function, is not assured. In fact, often only the canonical partition function Z_b exists, Eq. (5.5). When an analytical expression for Z can be found then we can recover the physically relevant quantity Z_b by the inverse L transform

$$\begin{aligned} Z_b(\beta, V, b) &= \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda^{b+1}} Z(\beta, V, \lambda) = \\ &= \frac{1}{2\pi} \oint \frac{d\lambda}{\lambda^{b+1}} \exp \left[\sum_{j=-\infty}^{\infty} \lambda^j Z_{1,b}(\beta, V, j) \right] \end{aligned} \quad (5.11)$$

We will return to this point in Section 6.

5.2 Partition function of nuclear matter

Thus we see that we need only to compute the one-cluster grand canonical partition function Z_1 to determine the grand canonical partition function Z , Eq. (5.9). This is an easy task — we recall the definition of the function ϕ_b in Eq. (4.16) and find from Eq. (5.1)

$$Z_{1,b} = - \frac{2V}{B_b (2\pi)^3} \frac{\partial}{\partial \beta} \phi_b(\beta, b) \quad (5.12)$$

in the common rest frame of the volume and the "thermometer". We recall that β is related to the physical temperature by Eqs (3.3) and (3.4)

$$T = (\beta_\mu \beta^H)^{-1/2} =: \beta^{-1} \quad (5.13)$$

Inserting Eq. (5.12) in (5.10) we find

$$Z_1(\beta, V, \lambda) = - \frac{V}{(2\pi)^3} \frac{\partial}{\partial \beta} \sum_{b=-\infty}^{\infty} \frac{\lambda^b}{B_b} \phi_b(\beta, b) \quad (5.14)$$

Would it not be for the b dependence of the function B_b , Eq. (4.7b):

$$B_b = \frac{2AM_b^2}{(2\pi)^3} \quad (5.15)$$

we would have already the analogue of Eq. (3.14), since according to Eqs. (4.10), (4.13) and (4.16)

$$\phi(\beta, \lambda) = \sum_{b=-\infty}^{\infty} \lambda^b \phi_b(\beta, b) \quad (5.16)$$

In order to proceed further we have to make an assumption about the b dependence of the cluster mass M_b . For the present we choose to consider the case

$$M_b = \begin{cases} b m_N & ; \quad |b| \geq 1 \\ m_\pi & ; \quad b = 0 \end{cases} \quad (5.17)$$

where m_π , m_N are the pion and nucleon masses, respectively. Through Eq. (5.17) we have implemented explicitly the assumption that the mass of a ground state cluster is proportional to the baryonic number (this assumption might be given up in more refined models). We now find for the grand canonical partition function

$$\ln Z(\beta, V, \lambda) = Z_1(\beta, V, \lambda) = -\frac{V}{A} \frac{1}{m_\pi^2} \frac{\partial}{\partial \beta} \phi_b(\beta, 0) - \frac{V}{A} \frac{1}{m_\pi^2} \frac{\partial}{\partial \beta} \sum_{b \neq 0} \frac{\lambda^b}{b^2} \phi_b(\beta, b) \quad (5.18)$$

In order to sum the expression (5.18) using Eq. (5.16) we generate b^{-2} in the sum by a double integral over λ :

$$\int_1^\lambda \frac{d\lambda'}{\lambda'} \int_1^{\lambda'} \frac{d\lambda''}{\lambda''} \sum_{b \neq 0} \lambda''^b \phi_b(\beta, b) = \sum_{b \neq 0} \frac{\lambda^b - 1}{b^2} \phi_b(\beta, b) \quad (5.19)$$

where the particle-antiparticle symmetry

$$\phi_b(\beta, b) = \phi_b(\beta, -b) \quad (5.20)$$

was used. For the sum in Eq. (5.18) we obtain

$$\begin{aligned} \sum_{b \neq 0} \frac{\lambda^b}{b^2} \phi_b(\beta, b) &= \int_1^\lambda \frac{d\lambda'}{\lambda'} \int_1^{\lambda'} \frac{d\lambda''}{\lambda''} \sum_{b \neq 0} \lambda''^b \phi_b(\beta, b) + \sum_{b \neq 0} \frac{\phi_b(\beta, b)}{b^2} \\ &= \int_1^\lambda \frac{d\lambda'}{\lambda'} \int_1^{\lambda'} \frac{d\lambda''}{\lambda''} [\phi(\beta, \lambda) - \phi_b(\beta, 0)] + \sum_{b \neq 0} \frac{\phi_b(\beta, b)}{b^2} \end{aligned}$$

The second sum can be written by observing that a contour integral of Eq. (5.19) yields

$$\frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda} \int_1^\lambda \frac{d\lambda'}{\lambda'} \int_1^{\lambda'} \frac{d\lambda''}{\lambda''} [\phi(\beta, \lambda) - \phi_b(\beta, 0)] = - \sum_{b \neq 0} \frac{\phi_b(\beta, b)}{b^2} \quad (5.21)$$

In the same way we find an expression for $\phi_b(\beta, 0)$:

$$\phi_b(\beta, 0) = \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda} \phi(\beta, \lambda) \quad (5.22)$$

We insert Eqs. (5.22), (5.21) (5.20) into Eq. (5.18) to obtain the final result

$$\begin{aligned} \frac{A}{V} \ln Z(\beta, V, \lambda) &= -\frac{1}{m_\pi^2} \frac{\partial}{\partial \beta} \phi_b(\beta, 0) - \frac{1}{m_\pi^2} \frac{\partial}{\partial \beta} \left\{ \int_1^\lambda \frac{d\lambda'}{\lambda'} \int_1^{\lambda'} \frac{d\lambda''}{\lambda''} \times \right. \\ &\quad \left. \times [\phi(\beta, \lambda') - \phi_b(\beta, 0)] + \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda} \int_1^\lambda \frac{d\lambda'}{\lambda'} \int_1^{\lambda'} \frac{d\lambda''}{\lambda''} [\phi(\beta, \lambda'') - \phi_b(\beta, 0)] \right\} \end{aligned} \quad (5.23)$$

On first sight it might seem that the pion and nuclear contributions to $\ln Z$ are additive -- however, when we recall Eq. (4.22) we realize that $\phi_b(\beta, 0)$ contains already a great deal of nuclear contributions; the curly brackets describe that part of $\ln Z$ which is not generated via baryon-antibaryon creation but enforced onto the system by a given fugacity $\lambda \neq 1$. We note that the multiple integrals in Eq. (5.23) may be reduced using:

$$\int_{x_0}^x dx_m \dots \int_{x_0}^{x_2} dx_1 f(x_1) = \frac{1}{m!} \int_{x_0}^x (x-x')^m f(x') dx' \quad (5.24)$$

5.3 Physical Properties of Nuclear Matter

Given the grand canonical partition function $Z(\beta, V, \lambda)$ we want to obtain the quantities of physical interest for nuclear matter. The energy density, pressure and baryon number density are respectively

$$\varepsilon(\beta, V, \lambda) = -\frac{1}{V} \frac{\partial}{\partial \beta} \ln Z(\beta, V, \lambda) \quad (5.25)$$

$$P(\beta, V, \lambda) = \frac{1}{V} \ln Z(\beta, V, \lambda) \quad (5.26)$$

$$\langle \frac{b}{V} \rangle =: \nu(\beta, V, \lambda) = \frac{\lambda}{V} \frac{\partial}{\partial \lambda} \ln Z(\beta, V, \lambda) \quad (5.27)$$

Of further physical interest are the energy per baryon

$$\varepsilon_b(\beta, V, \lambda) = \frac{\varepsilon(\beta, V, \lambda)}{\nu(\beta, V, \lambda)} \quad (5.28)$$

and the relative fluctuations of the energy density and the baryon number density

$$\frac{\delta \varepsilon}{\varepsilon}(\beta, V, \lambda) = \left(\frac{\overline{\varepsilon^2} - \bar{\varepsilon}^2}{\bar{\varepsilon}^2} \right)^{1/2} = - \frac{[\frac{\partial^2}{\partial \beta^2} \ln Z]^{1/2}}{\frac{\partial}{\partial \beta} \ln Z} \quad (5.29)$$

$$\frac{\delta \nu}{\nu}(\beta, V, \lambda) = \left(\frac{\overline{\nu^2} - \bar{\nu}^2}{\bar{\nu}^2} \right)^{1/2} = \frac{[(\lambda \frac{\partial}{\partial \lambda})^2 \ln Z]^{1/2}}{\lambda \frac{\partial}{\partial \lambda} \ln Z} \quad (5.30)$$

All these quantities are directly related to $\phi(\beta, \lambda)$ through Eq. (5.23). As we know that $\phi(\beta, \lambda)$ is singular along the curve $\beta_c = f(\lambda_c)$ given implicitly by Eq. (4.20), we expect that this curve separates two regions of different behaviour of nuclear matter. In fact, $\phi(\beta, \lambda)$ and $\ln Z(\beta, V, \lambda)$ are complex beyond this curve and Eqs. (5.25-30) make no sense there. On the other hand, it will turn out that in approaching this curve (except at $\lambda = 1$) the energy density remains finite so that it is physically

possible to cross this border line (in contrast to the border line $\beta = \beta_0$ whose crossing would require infinite energy). We therefore expect that the line $\beta_c = f(\lambda_c)$ is a critical curve separating two different phases of nuclear matter.

5.4 A Speculation about the Early Universe

Of particular interest is the case in which no baryons are brought in ($\lambda = 1$); then the total baryon number (number of baryons minus number of antibaryons) is zero.

Only in the canonical description (5.5) can this be achieved rigorously by putting there $b = 0$. In the grand canonical description $\langle b \rangle = 0$ is obtained by setting $\lambda = 1$ (i.e., $\mu = 0$). Indeed, from Eq. (5.18) it follows that

$$\lambda \frac{\partial}{\partial \lambda} \ln Z(\beta, V, \lambda) \Big|_{\lambda=1} = -\frac{V}{A} \frac{1}{m_N} \frac{\partial}{\partial \beta} \sum_{b \neq 0} \frac{1}{b} \phi_b(\beta, b) = 0 \quad (5.31)$$

since corresponding terms with b and $-b$ cancel each other. In this case one cannot use Eq. (5.30), since $\langle b \rangle = 0$. However, we may evaluate $\overline{b^2} - \bar{b}^2 = \overline{b^2}$; from (5.18) we have

$$\overline{b^2} = (\lambda \frac{\partial}{\partial \lambda})^2 \ln Z(\beta, V, \lambda) \Big|_{\lambda=1} = -\frac{V}{A} \frac{1}{m_N} \frac{\partial}{\partial \beta} \sum_{b \neq 0} \phi_b(\beta, b) > 0 \quad (5.32)$$

For $\beta \rightarrow \beta_0$ the derivative of the sum diverges, as can be seen from Eq. (4.22), because for all b , $-\partial \phi_b(\beta, b) / \partial \beta$ is positive for $\beta \rightarrow \beta_0$ and the derivative $-\partial / \partial \beta$ of each single contribution to it in the Yellin expansion is positive (see Eq. (4.18b)). Therefore, if we can show that the derivative of some contribution to $\phi_b(\beta, 1)$ diverges, we have proved that $\overline{b^2}$ diverges, all contributions having the same sign. We pick out the contribution to

$\phi_b(\beta, 1)$ coming from the subset $\{n_j\} = \{n_0 = l - 1; n_1 = 1; \text{ all other } n_j = 0\}$ and we call this contribution $\phi_b^{(1)}(\beta, 1)$. Hence

$$\phi_b(\beta, 1) = \phi_b^{(1)}(\beta, 1) + \text{contributions of remaining subsets} \quad (5.33)$$

One easily sees that the above subset of the sum (4.22) yields

$$\phi_b^{(1)}(\beta, 1) = \varphi_b(\beta, 1) \sum_{l=1}^{\infty} l \cdot g_l \varphi_l(\beta, 0)^{l-1} = \varphi_b(\beta, 1) \frac{dG(\varphi)}{d\varphi} \quad (5.34)$$

Here $G(\varphi)$ is the bootstrap function (2.40) which has, for $\beta \rightarrow \beta_0$, infinite derivative (see Fig. 2.1b). Hence $-\partial \phi_b^{(1)}(\beta, 1) / \partial \beta$ and therefore the whole sum (5.32) diverges. Thus

$$\overline{b^2}(\beta \rightarrow \beta_0) \rightarrow \infty \quad (5.35)$$

$$\overline{b} = 0 \quad (5.36)$$

We now apply our model to describe a charge symmetric early Universe. We then have $\langle b \rangle = 0$ at all times and temperatures, but just after the first $\sim 10^{-23}$ sec the temperature will be near T_0 and then, in spite of $\langle b \rangle = 0$, we will have $\langle b^2 \rangle \rightarrow \infty$ in any finite volume. Therefore there will exist regions with large $\langle b \rangle \approx \pm \sqrt{\langle b^2 \rangle}$ which will never completely disappear upon expansion and cooling down. The mechanisms for preserving such fluctuations, once they are there, have been discussed in the beautiful and immense work of Omnès and co-workers²⁴⁾; we present here a non-perturbative mechanism for generating such large initial fluctuations in a charge symmetric big bang.

5.5 Nuclear Matter Without Baryon-Antibaryon Pairs

While the general formula (5.23) describes all possible cases of cold and hot matter, low and high baryon number density, we will be interested from here on in properties of bulk nuclear matter: that is, the case when a certain number of nucleons is already present in a given volume. Unless $T \sim T_0$, we expect only moderate contributions from baryon-antibaryon pair production, since $m_N \gg T_0$. Therefore we further simplify our result and neglect antibaryon production. We can implement this by restricting b to be positive in all previous formulae. We note that in doing so we allow uncompensated baryon production which is, for $T \leq T_0$, a small effect, since $m_N/T_0 \geq 7$.

The bootstrap equation is then as before

$$\phi(\beta, \lambda) = \varphi(\beta, \lambda) + e^{\phi(\beta, \lambda)} - \phi(\beta, \lambda) - 1 \quad (5.37)$$

but the input term that describes only "raw" pions and nucleons takes the form

$$\varphi(\beta, \lambda) = \varphi_\pi(\beta) + \lambda \varphi_N(\beta) \quad (5.38)$$

The relation between the grand canonical partition function and ϕ is given by Eq. (5.18)

$$-\frac{A}{V} \ln Z(\beta, V, \lambda) = \frac{1}{m_N^2} \frac{\partial}{\partial \beta} \phi(\beta, 0) + \frac{1}{m_N^2} \frac{\partial}{\partial \beta} \sum_{b>0} \frac{\lambda^b}{b^2} \phi_b(\beta, b) \quad (5.39)$$

The sum in Eq. (5.39) can now be obtained by integrating from zero to λ :

$$-\frac{A}{V} \ln Z(\beta, V, \lambda) = \frac{1}{m_N^2} \frac{\partial}{\partial \beta} \phi(\beta, 0) + \frac{1}{m_N^2} \frac{\partial}{\partial \beta} \int_0^\lambda \frac{d\lambda'}{\lambda'} \int_0^{\lambda'} \frac{d\lambda''}{\lambda''} [\phi(\beta, \lambda'') - \phi_\beta(\beta, 0)] \quad (5.40)$$

In the next sub-section we illustrate our model by some numerical results obtained studying this equation.

5.6 The Different Phases

In the remainder of this section we will study the physical properties of the model defined above. We begin by considering in more detail the point $\varphi_0 = \ln 4 - 1$, where the function $G(\varphi)$ has a square root singularity. This point corresponds to a curve $\lambda_c = f(\beta_c)$ in the (λ, β) plane, defined implicitly by the equation

$$\varphi_0 = \ln 4 - 1 = \varphi_\pi(\beta_c) + \lambda_c \varphi_N(\beta_c)$$

Thus (with $i = \pi, N$)

$$\lambda_c = \frac{\ln 4 - 1 - \varphi_\pi(\beta_c)}{\varphi_N(\beta_c)}; \quad \varphi_i(\beta) = \frac{A m_i^4}{2\pi^2} \frac{K_i(\beta m_i)}{\beta m_i} \quad (5.41)$$

As in Section 3, Eq. (3.9), we introduce the chemical potential μ by

$$\lambda = e^{\beta \mu} \quad (5.42)$$

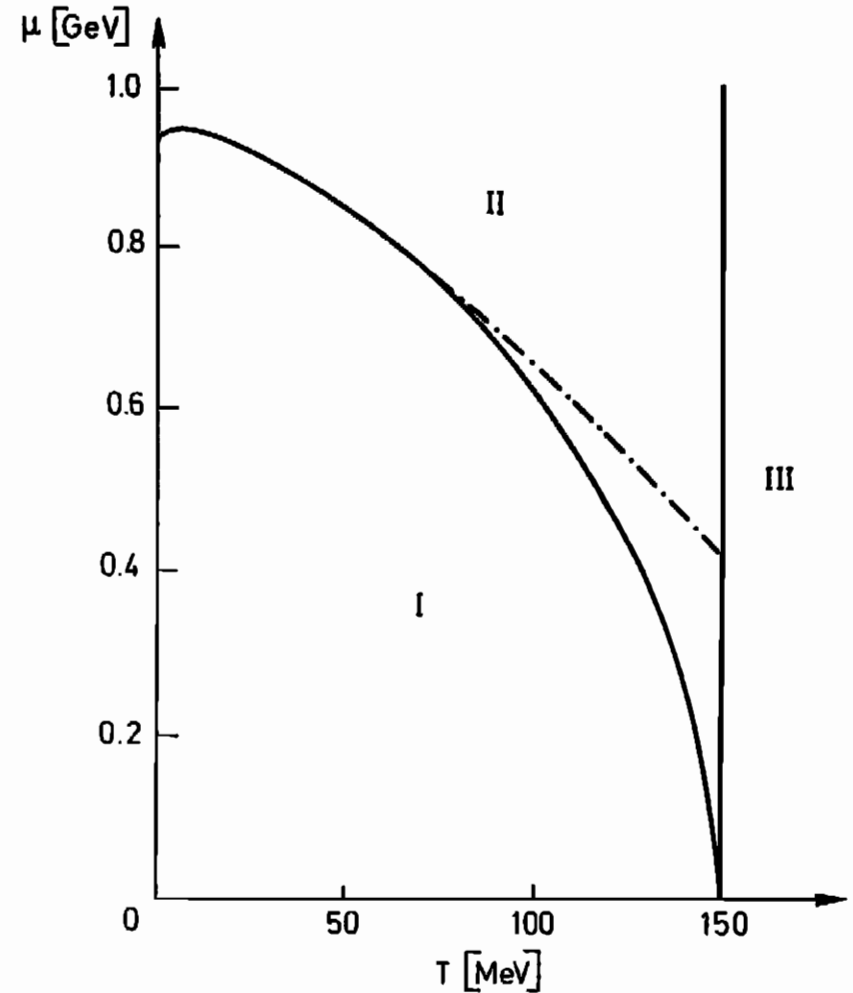


Fig. 5.1 The critical curve $\mu_c = \mu_c(T_c)$ in the μT plane separating the gaseous phase (I) from the "liquid" phase (II). The dot-dashed line would be the critical curve if pions were excluded. Region (III) is inaccessible ($T > T_0$): infinite energy density. For $T = 0$ the critical chemical potential equals the nucleon mass; note that this is not its maximum value.

and consider the function $\mu_c = f(T_c)$, where $T_c = \beta_c^{-1}$ that follows from Eq. (5.41). As shown in Fig. 5.1 this line divides the (μ, T) plane into two parts. For $\mu < \mu_c(T_c)$ we have $\varphi < \varphi_0$ and we know that the grand canonical description is valid there. At $\varphi = \varphi_0$ we are on the critical curve corresponding to a singularity of $\ln Z$. We record the interesting behaviour of $\mu_c(T_c)$ for small T_c (large β_c). This can be found analytically employing the asymptotic expansion for the Hankel functions

$$K_m(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[1 + \frac{4m-1}{8z} + \dots \right] \quad (5.43)$$

We find, using Eqs. (5.41)-(5.43), the relation for large β_c :

$$\frac{2\pi^2(\ln 4 - 1)}{A m_N^4 \sqrt{\pi/2}} = \frac{\exp(\beta_c(\mu_c - m_N))}{(\beta_c m_N)^{3/2}} + \left(\frac{m_N}{m_\pi}\right)^4 \frac{\exp(-\beta_c m_\pi)}{(\beta_c m_\pi)^{3/2}} \quad (5.44)$$

We see that as long as $\mu_c - m_N \ll m_\pi$, the last term in Eq. (5.44) is negligible compared to the first term. This is just the case for large β_c (small T_c), since when neglecting it we obtain solving Eq. (5.44):

$$\mu_c = m_N - \frac{3}{2} T_c \ln(T_c/T_c') \quad (5.45)$$

$$T_c' := 2\pi m_N \left(\frac{\ln 4 - 1}{A m_N^4} \right)^{3/2}$$

Depending on the choice of A as discussed in Section 2 we find $T_c' \approx 15-40$ MeV. From Eq. (5.45) we see that μ_c increases as a function of T_c initially, until $T_c^{\max} = eT_c'$. Beyond this point it drops continuously until $\mu_c = 0$ at $T_c = T_0 \approx 149$ MeV.

We note that the behaviour of the chemical potential for $T \neq T_0$ is similar even when the pion term is switched off entirely (dashed line in Fig. 5.1). This is visible in Eq. (5.44), where the last term is almost everywhere small, except when $\beta_c \sim 1/m_\pi$.

The limiting temperature T_0 is now a solution of the equation (5.41) with $\lambda_c = 1$. However, since the nuclear term (m_N) is exponentially small at $\beta_c \approx 1/m_\pi \approx 1/T_0$ we expect that the limiting temperature is little changed from that of pionic bootstrap. The change of T_0 induced by the possible baryon production is obtained by expanding Eq. (5.41) around β_0 . We find

$$\Delta\beta = -\left(\frac{m_N}{m_\pi}\right)^4 \frac{K_1(\beta_0 m_N)}{\beta_0 m_N} \left/ \left[\frac{1}{\beta_0} \frac{K_1(\beta_0 m_\pi)}{\beta_0 m_\pi} + \left(\frac{m_N}{m_\pi}\right)^4 \frac{K_1(\beta_0 m_N)}{\beta_0 m_N} \right] \right. \quad (5.46)$$

As $\Delta\beta$ is positive, $\approx 4.2 \times 10^{-4} \text{ MeV}^{-1}$, the change of T_0 is negative: the limiting temperature is slightly lowered (by about 11 MeV).

There are three domains shown in Fig. 5.1. In domain I enclosed by the function $\mu_c(T_c)$ the grand canonical description is valid; in domain II, above the critical curve, we have $\varphi > \ln 4 - 1$, but $T < T_0$. Thus this is a new bootstrap region opened up by the introduction of nuclear matter -- the description of the physical quantities should be canonical, since the grand canonical partition function does not exist for $\varphi > \varphi_0$. It is possible, however, to consider the analytical continuation of the grand canonical function into this domain -- inverse L transform can then be used to find the canonical quantities. Henceforth, we will call region I the gaseous phase (because it contains the

region of small density) and region II "liquid phase" [because it is approached if at fixed temperature the baryon density (that is λ or μ) increases]*). Region III characterized by $T > T_0$ is a domain that cannot be reached from the physical phases in those bootstrap models that give divergent energy density at $T = T_0$. As we have demonstrated in Section 3 an alternative treatment of the bootstrap model can allow a transition even to this region -- we will, however, not discuss further this possibility here.

We cannot exclude that in models with more general input functions (see Eqs. (4.10) and (4.18)) a further phase develops for large baryon densities. However, this is not so within our simple model of pions and nucleons where we neglect most of the details of nuclear structure. In particular, for $T \rightarrow 0$ and for μ corresponding to $\nu \sim 1$ we might need more detailed input than we have considered in the present simplified study.

5.7 Numerical Methods

We need to compute the different derivatives with respect to β and λ of Eq. (5.40). Since $\phi(\beta, \lambda) = G(\varphi(\beta, \lambda))$, we only need the function $G(\varphi)$ and its derivatives with sufficient precision in order to calculate the quantities of physical interest. This is done by considering the expansion (2.48) of $G(\varphi)$ at φ_0 :

$$G(\varphi) = G_0 - (\varphi_0 - \varphi)^{1/2} - \frac{1}{6}(\varphi_0 - \varphi)^{3/2} - \frac{1}{36}(\varphi_0 - \varphi)^{5/2} + \Delta G(\varphi) \quad (5.47)$$

*) In Section 6 we shall see that in region II in fact two phases co-exist: liquid and vapour, in equilibrium. We write "liquid" for short, but put it within quotation marks to emphasize that it is not the pure liquid phase.

Equation (5.47) defines the remainder ΔG which can be taken to have the polynomial form

$$\Delta G(\varphi) = \sum_{n=0}^N a_n \varphi^n \quad (5.48)$$

Since we know the inverse function $\varphi = \varphi(G)$, we can easily fit the coefficients a_n . We find that even for quite small N ($= 3$) already a very satisfactory result is obtained. This is partially due to the fact that Eq. (5.47) with ΔG neglected is, in itself, a very good approximation of G since the maximum error occurs at $\varphi = 0$ and is

$$\Delta G(0) = G_0 - \varphi_0^{1/2} - \frac{1}{6}\varphi_0 - \frac{1}{36}\varphi_0^{3/2} = 5.7 \cdot 10^{-4}$$

Also, at $\varphi = \varphi_0$ the proper analytic behaviour is obtained from Eq. (5.47) for $G(\varphi)$ and its first and second derivatives. Thus to one per mille accuracy the expansion (5.47) is already quite adequate, however we have included in numerical calculations the polynomial terms, Eq. (5.48), in order to achieve relative accuracy of 10^{-6} . Another merit of the expansion (5.47) (5.48) is its analytical integrability in Eq. (5.40). Thus we have succeeded in obtaining $\ln Z$ in terms of known functions. The computation of the different physical quantities, though tedious, is an elementary exercise now. The results were obtained and graphically processed by the CERN Interactive Computing System SIGMA²⁵. An independent check of our calculations has been done with the Yellin expansion which follows easily from Eqs. (2.49) and (2.57). These two completely different methods yield the same numerical results for the critical curves separating gas and "liquid".

5.8 Baryon Density in the Gaseous Phase

We begin the discussion of our results by considering the baryon number density ν , Eq. (5.27) along the gas-"liquid phase" limit. As a unit of V we will choose the "elementary" volume of one baryon, $V_N = m_N A$, as introduced in Eqs. (4.4), (2.26). In the nuclear bootstrap we fix the value of A by requiring V_N to be the volume available to a nucleon in cold nuclear matter at the saturation point. We find

$$\begin{aligned} V_N &= \frac{4\pi}{3} (1.2 \text{ fm})^3 = 9 \cdot 10^2 [\text{GeV}^{-3}] \\ A &= .96 \cdot 10^3 [\text{GeV}^{-4}] \end{aligned} \quad (5.49)$$

which is in the range of A we have estimated in Section 2 (in Eq. (2.28) it gives $\mu \approx 11.3 m_\pi$ *)). The baryon number contained in the elementary volume V_N follows now from Eqs. (5.27) and (5.40)

$$V_N \nu = -\frac{1}{m_N} \frac{\partial}{\partial \beta} \int_0^\lambda \frac{d\lambda'}{\lambda'} [G(\varphi_\pi + \lambda' \varphi_N) - G(\varphi_\pi)] \quad (5.50)$$

We find upon differentiation

$$\begin{aligned} V_N \nu &= -\frac{1}{m_N} \frac{\partial \varphi_\pi}{\partial \beta} \frac{\partial}{\partial \varphi_\pi} \int_0^\lambda \frac{d\lambda'}{\lambda'} [G(\varphi_\pi + \lambda' \varphi_N) - G(\varphi_\pi)] - \\ &\quad - \frac{1}{m_N} \frac{\partial \varphi_N}{\partial \beta} \frac{1}{\varphi_N} [G(\varphi_\pi + \lambda \varphi_N) - G(\varphi_\pi)] \end{aligned} \quad (5.51)$$

At the critical line we just have $\varphi_\pi + \lambda_c \varphi_N = \varphi_0$, so

*) All figures have been computed using the A value of Eq. (2.27). The value given in Eq. (5.49) would yield T_0 around 200 MeV.

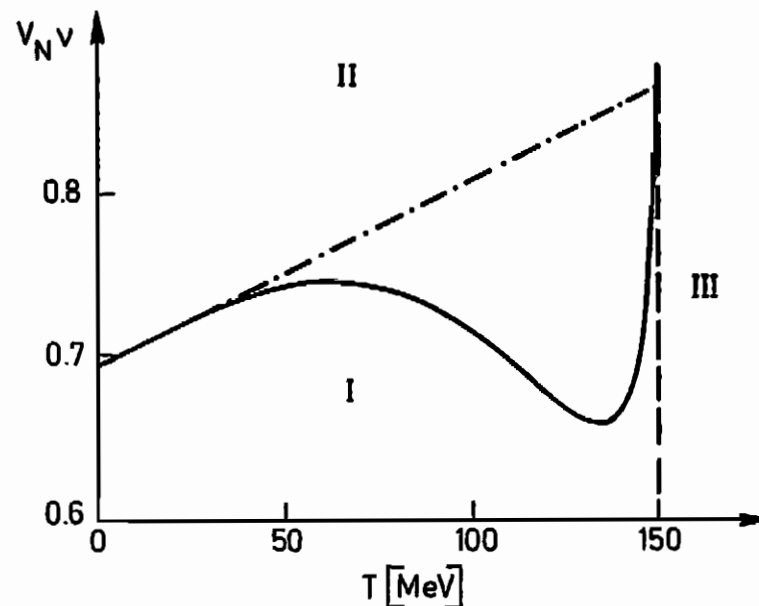


Fig. 5.2 Critical baryon number per nucleon volume $V_N \nu$ as a function of the temperature. The dot-dashed line results if pions are excluded. The unexpected shape of the critical curve is seen to be due to the co-existence of pions. Region (I) is gaseous, (II) fluid. For $T > T_0$ the baryon density diverges. Note that ordinary nuclei lie in the liquid phase but that at $T \rightarrow T_0$ the gaseous phase is reached at any density.

$$\begin{aligned}
 V_{NV} \Big|_{\text{crit.}} &= -(\ln 2 - G(\varphi_N)) \frac{1}{m_N} \frac{\partial}{\partial \beta} \ln \varphi_N \Big|_{\text{crit.}} - \\
 & - \frac{1}{m_N} \frac{\partial \varphi_N}{\partial \beta} \frac{\partial}{\partial \varphi_N} \int_0^\lambda \frac{d\lambda'}{\lambda'} [G(\varphi_N + \lambda' \varphi_N) - G(\varphi_N)] \Big|_{\text{crit.}}
 \end{aligned}
 \tag{5.52}$$

The first term is the only one remaining in the absence of pions and is shown as a dashed-dotted curve in Fig. 5.2. Since for $T \leq T_0$ we have $m_N/T \gg 1$, the asymptotic form (5.43) for the Bessel function in φ_N can be used to determine v . Therefore we find

$$\begin{aligned}
 V_{NV} \Big|_{\text{crit. no pions}} &= \left(1 + \frac{3}{2} T / m_N\right) \cdot \ln 2
 \end{aligned}
 \tag{5.53}$$

Even including pions this expression is correct for low temperatures since as before $\varphi_{\pi, \text{crit}} \ll \lambda_{\text{crit}} \varphi_{N, \text{crit}}$. The complete expression (5.52) is shown in Fig. 5.2. We see that the onset of the pion component lowers the phase transition density, but at high temperatures the density increases sharply again. We remark that we have computed the results of Fig. 5.2 once as a limit in the gaseous phase and separately as a limit in the "liquid phase". The exact agreement of both computations serves as a check of the numerical procedures involved.

We notice that for $T < T_0 - \delta$ (with δ a few MeV) the transition from gaseous to "liquid phases" occurs always below one (one baryon per unit volume is by definition the normal nuclear density). This justifies a posteriori our choice for the names of the different phases. The sudden rise of the transition density as T approaches T_0 , if confirmed in more elaborate models, could help to obtain information about the magnitude of T_0 from nuclear physics experiments.

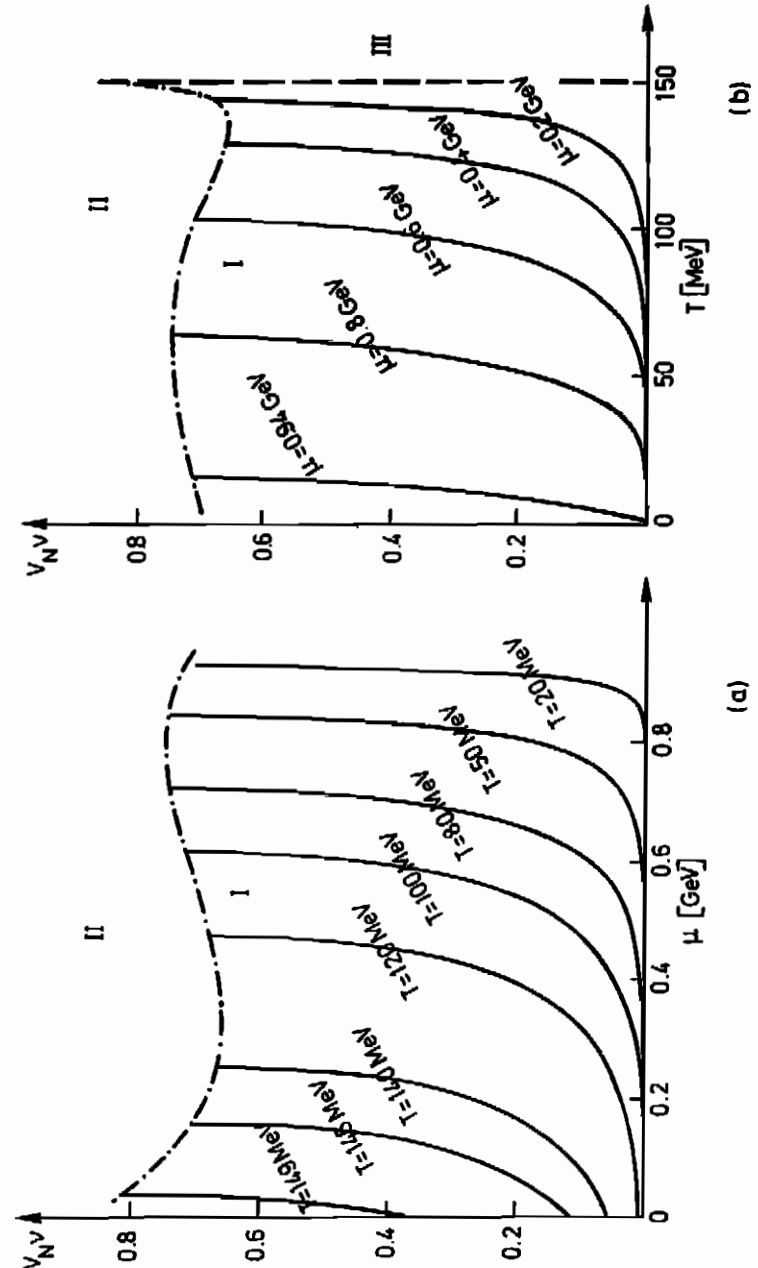


Fig. 5.3 (figure caption on following page).

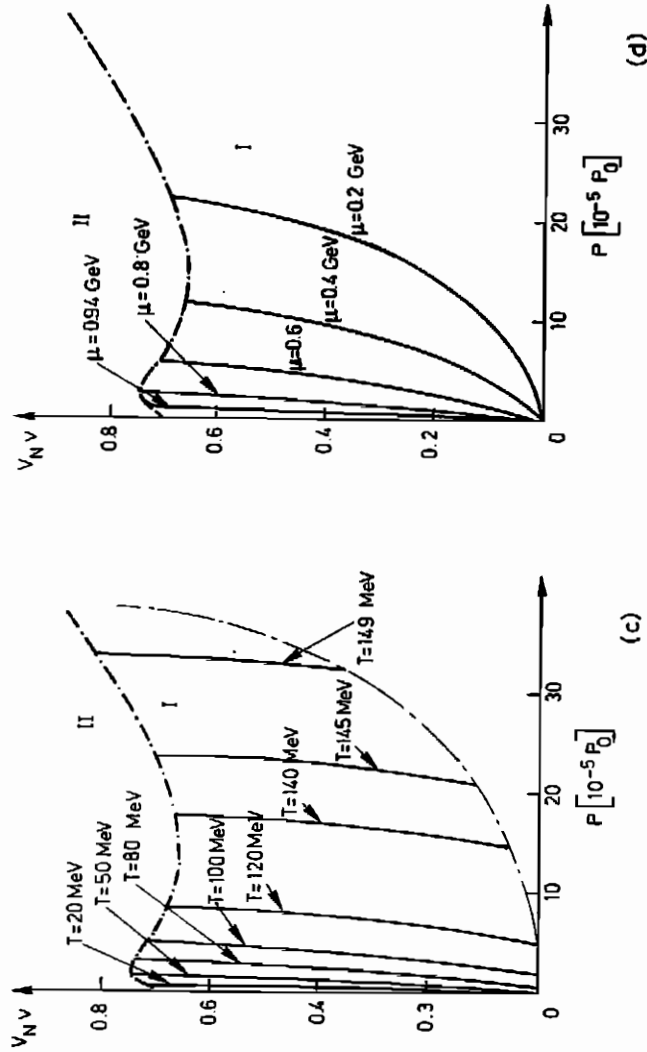


Fig. 5.3 Baryon number per nucleon volume $V_N v$ in various representations up to the critical curve. (a) against chemical potential with isotherms; (b) against the temperature with μ as parameter; (c) against the pressure with isotherms; (d) against the pressure with μ as parameter. $P_0 = A^{-1} \approx$ proton rest energy density ("internal proton pressure"). The dash-dotted line is the critical curve, region (II) the liquid phase. The white lower right corner in (c) is due to the impossibility of having no baryons at high temperature (unsymmetry of our input term).

In Fig. 5.3 we show the baryon density in the gaseous phase. In (a) as a function of chemical potential with temperature being the parameter (isotherms), in (b) as a function of temperature, with the chemical potential as a parameter. In (c) and (d) we have eliminated the chemical potential from (a) and replaced it by the pressure, Eq. (5.26), in units of $P_0 = m_N/v_N = A^{-1}$, Eq. (5.49). Since in Fig. 5.3a $\mu = 0$ implies a finite baryon density in particular noticeable for $T \geq 120$ MeV we cannot find for $T > 120$ MeV the corresponding pressure in Fig. 5.3c below a certain density. This is a consequence of taking a single baryon (without antibaryon) in the input term (5.38). By this we have allowed free baryon production without compensating antibaryon production. The simple input (5.38) was chosen to reduce numerical computation: we did not anticipate such a visible effect, otherwise we would have included antibaryons. We stress that this minor inconsistency does not change the main results of the model.

5.9 Baryon Energy in the Gaseous Phase

The energy contained in the unit volume V_N can be obtained easily from Eqs (5.25) and (5.40):

$$V_N \mathcal{E}(\beta, \lambda) = \frac{m_N}{m_\pi^2} \frac{\partial^2}{\partial \beta^2} G(\varphi_\pi) + \frac{1}{m_N} \frac{\partial^2}{\partial \beta^2} \int_0^\lambda \frac{d\lambda'}{\lambda'} \int_0^{\lambda'} \frac{d\lambda''}{\lambda''} [G(\varphi_\pi + \lambda'' \varphi_N) - G(\varphi_\pi)] \quad (5.54)$$

Both (5.54) and (5.50) are functions of μ and T and we can eliminate numerically either one of these physical parameters in Eq. (5.54) and replace it by v , Eq. (5.50). Since T has a better intuitive meaning, we eliminate the chemical potential from (5.54) and consider

$$\varepsilon(\beta, \lambda) = \varepsilon(\beta, \lambda(\beta, \nu)) = \varepsilon_\nu(\beta, \nu) \quad (5.55)$$

dropping henceforth the lower index ν . The results are shown in Fig. 5.4. Here the isotherms $T = \text{constant}$ are shown for $V_N \varepsilon$ as a function of $V_N \nu$. We record the nearly linear behaviour (in the gas phase) of the energy density: $\varepsilon \sim C_1 + C_2 \nu$ with temperature dependent constants C_1, C_2 . We recall that for very small $\nu(T)$ our neglect of antibaryons is not justified. But above $V_N \nu = 0.1$ and $T \leq 120$ MeV our results should be independent of this approximation.

Even better insight can be obtained inspecting the energy per baryon, excluding the rest mass:

$$E_{\text{kin}}^{\text{nr}} = \left[\varepsilon(\beta, \nu) / \nu \right] - m_N \quad (5.56)$$

shown in Fig. 5.5. For small temperatures ($T < 30$ MeV) and densities this should be just the usual $3/2 T$, which we actually find for $T = 20$ MeV. For higher temperatures, as we can see in Fig. 5.5b, this is the lower limit of the thermal and interaction energy $\varepsilon_{\text{b}}^{\text{nr}}$. For $T = 50$ MeV and higher, we have a large pion component; thus the energy per baryon (total energy divided by total baryon number), which also includes the energy of the pions, stays high above the lower limit $3/2 T$. We note that our interaction energy is, by definition, positive -- our nuclear mass m_N for the input nucleon should, in principle, include all the binding effects at saturation, thus be really $m_N - E_B$. Therefore, at densities lower than the saturation density in the gaseous phase, the thermal energy $3/2 T$ is the lower limit on the energy per baryon. Furthermore, we note that within our model the thermal energy dominates the picture between ~ 20 and ~ 60 MeV, at which point the onset of pion and resonances excitation becomes important.

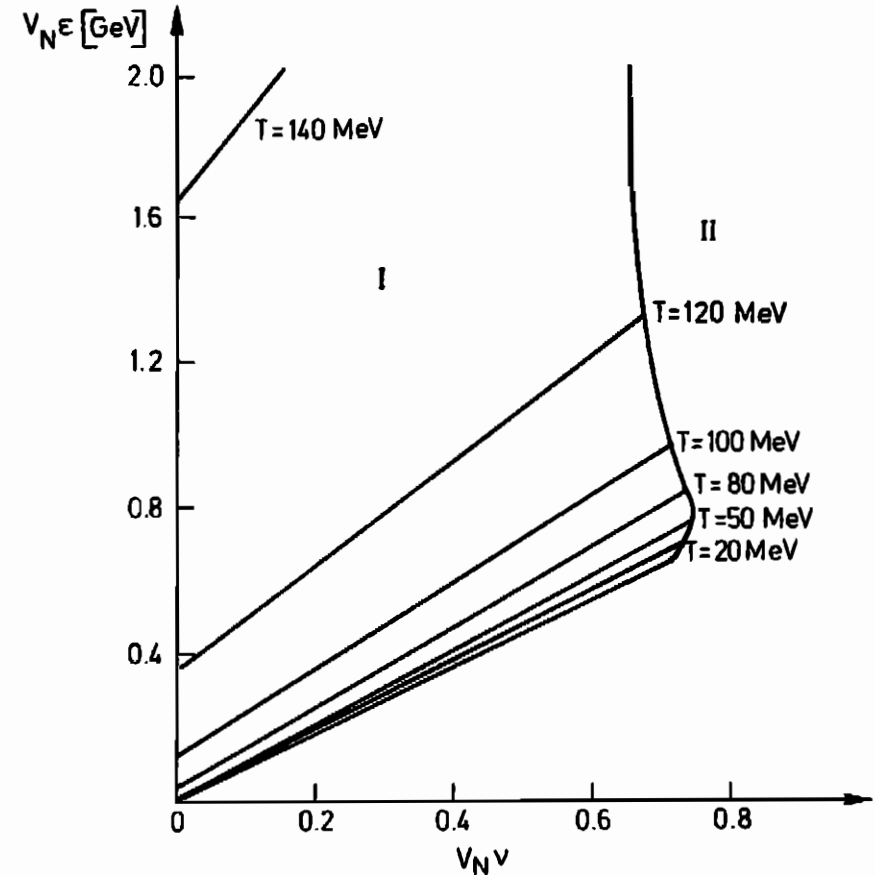


Fig. 5.4 Energy per nucleon volume $V_N \varepsilon$ as a function of baryon number per nucleon volume. Isotherms up to the critical curve separating gas (I) from liquid (II). As the rest mass is included in the energy per nucleon volume, the lower part of the diagram remains empty.

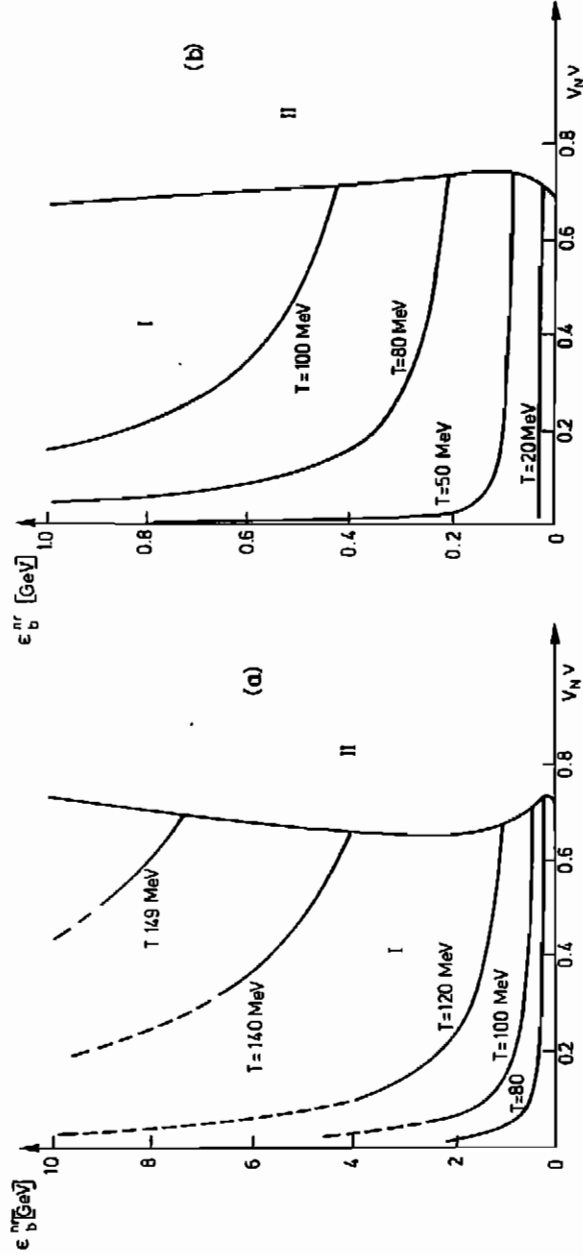


Fig. 5.5 Energy per baryon (minus rest mass) as a function of baryon number per nucleon volume. (a) from 0 to 10 GeV - (b) from 0 to 1 GeV with isotherms up to the critical curve separating gas (I) from fluid (II). Note that energy per baryon is not the energy per carried by a baryon but : total energy of the system divided by total baryon number minus rest mass : $E/b - m_N$. At very low temperatures ($T = 20$ MeV) it is $\approx 3/2 T$.

It is straightforward to isolate the thermic term from Eq. (5.54). In fact recalling the rules of chain differentiation we obtain from Eqs. (5.25), (5.27) and (5.40)

$$\epsilon = - \frac{1}{V} \frac{\partial \varphi_\pi}{\partial \beta} \frac{\partial}{\partial \varphi_\pi} \ln Z - v \frac{\partial}{\partial \beta} \ln \varphi_N \quad (5.57)$$

The first term expresses the pion-nucleon interacting component and, as discussed in a previous subsection, is small at temperatures below 60 MeV. The second term is just the "free" nucleon term at density v , which in the non-relativistic limit gives us the usual $3/2 T v$.

5.10 Baryon Pressure

We turn now to a brief discussion of the pressure of the nuclear matter in our model. In Fig. 5.6 we show the same results as in Fig. 5.3c, but now P as a function of the inverse baryon density for fixed temperatures $T = 20, 50, 80, 100, 120, 140, 145$ MeV. For the interpretation of this figure we note that from Eqs. (5.18), (5.26) and (5.27) it follows that $P = f(\lambda, T)$ and $V/(V_N \langle b \rangle) = g(\lambda, T)$ so that in this figure we plot $f(\lambda, T)$ against $g(\lambda, T)$. We may view this as a P - V diagram with fixed $\{\langle b \rangle, T\}$ or as a P - $\langle b \rangle^{-1}$ diagram with fixed $\{V, T\}$. Adopting the first interpretation we see that P falls with increasing V (as expected), but not like $1/V$ (as for the ideal gas), since that would require that $g(\lambda, T) \cdot f(\lambda, T)$ be a function of T alone in contradiction to Eq. (5.18). The absolute magnitude of pressure rises very rapidly as we approach the critical temperature, in accordance with the conclusions that can be drawn from Table 3.1.

On first sight, more surprising are the results presented in Fig. 5.7, where the energy per baryon (excluding the rest mass) is plotted as a function of the pressure at constant temperature.

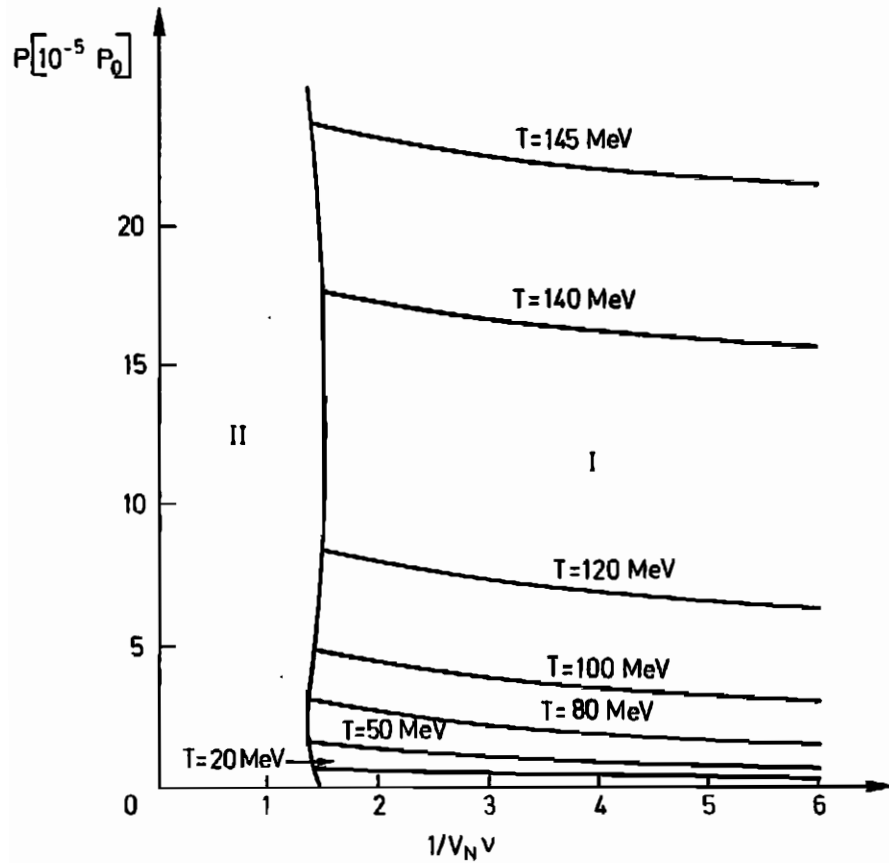


Fig. 5.6 Pressure against inverse of baryon number per nucleon volume (i.e., $V/(\langle b \rangle \cdot V_N)$: volume per baryon in units of the nucleon volume). Isotherms up to the critical curve separating gas (I) from fluid (II). $P_0 \cong$ rest energy density of proton ("internal proton pressure") = $m_N/V_N = 1/A$.

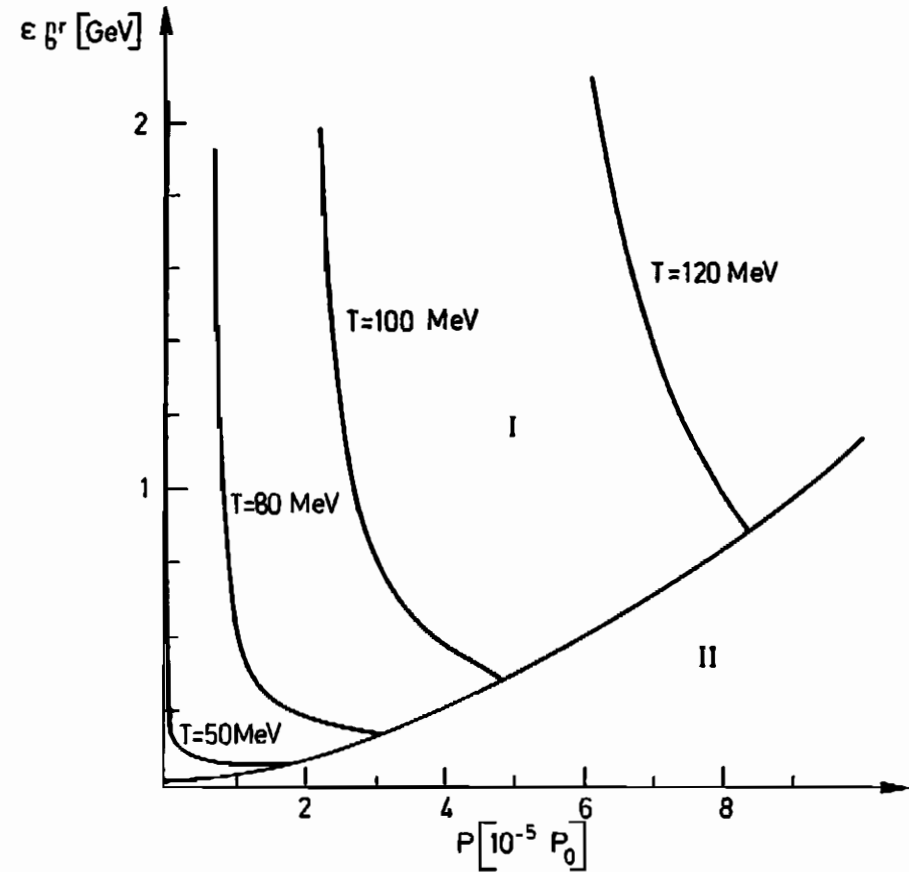


Fig. 5.7 Energy per baryon (minus rest mass) as a function of pressure. Isotherms up to the critical curve separating gas (I) from fluid (II). The energy per baryon plotted here is not the energy carried by the baryon (cf. caption to Fig. 5.5).

These functions decrease as the pressure increases, because we include in $\epsilon_b^{\pi N}$ the energy of the pions, resonances, etc., and express it per baryon. Consider the isotherm $T = 100$ MeV: along it, the pion component remains the same, no matter how many baryons we force into the system. If we increase the baryon number (by increasing μ), more baryons must share the same amount of pionic energy; hence the energy per baryon decreases. At the same time the pressure increases, since there are now more particles. Therefore the "energy per baryon" is a misleading concept at temperatures where pions contribute significantly to the energy density; it should be used only to characterize the initial state of nucleus-nucleus collisions.

6. THE "LIQUID PHASE"

6.1 The Canonical Partition Function

The grand canonical partition function Z of nuclear matter in the case of the input term (pion and nucleon) defined by Eq. (5.38) can be written

$$\begin{aligned} Z(\beta, V, \lambda) &= Z_{\pi}(\beta, V) Z_{\pi N}(\beta, V, \lambda) \\ \ln Z_{\pi}(\beta, V) &:= V m_{\pi}^3 \frac{K_2(\beta m_{\pi})}{2\pi^2 (\beta m_{\pi})} G_1^{(1)}[\varphi_{\pi}(\beta)] \\ \ln Z_{\pi N}(\beta, V, \lambda) &:= \frac{V}{V_N} \frac{K_2(\beta m_N)}{K_1(\beta m_N)} G_1^{(0)}[\lambda \varphi_N(\beta), \varphi_{\pi}(\beta)] + \\ &+ \frac{V m_{\pi}}{V_N m_N} \frac{K_2(\beta m_{\pi})}{K_1(\beta m_{\pi})} \varphi_{\pi}(\beta) G_2^{(1)}[\lambda \varphi_N(\beta), \varphi_{\pi}(\beta)]. \end{aligned} \quad (6.1)$$

Here the functions $\varphi_{\alpha}(\beta)$ are defined as in Eq. (5.41)

$$\varphi_{\alpha}(\beta) = \frac{V_{\alpha} m_{\alpha}^3}{2\pi^2} \frac{K_1(\beta m_{\alpha})}{\beta m_{\alpha}} \quad (6.2)$$

while the modified functions $G_K^{(k)}(\varphi_1, \varphi_2)$ are defined as

$$G_K^{(k)}(\varphi) := \frac{d^k G(\varphi)}{d\varphi^k} \quad (6.3)$$

and

$$\begin{aligned} G_0^{(k)}(\varphi_1, \varphi_2) &:= G_1^{(k)}(\varphi_1 + \varphi_2) - G_1^{(k)}(\varphi_2) \\ G_{K+1}^{(k)}(\varphi_1, \varphi_2) &:= \int_0^{\varphi_1} \frac{d\alpha}{\alpha} G_K^{(k)}(\alpha, \varphi_2); \quad K \geq 0 \end{aligned} \quad (6.4)$$

From the definition in Eq. (6.4) it follows immediately that

$$\begin{aligned} \frac{\partial G_K^{(k)}(\varphi_1, \varphi_2)}{\partial \varphi_1} &= \frac{1}{\varphi_1} G_{K-1}^{(k)}(\varphi_1, \varphi_2) \\ \frac{\partial G_K^{(k)}(\varphi_1, \varphi_2)}{\partial \varphi_2} &= G_K^{(k+1)}(\varphi_1, \varphi_2) \end{aligned} \quad (6.5)$$

The pion bootstrap partition function $Z_{\pi}(\beta, V)$ in Eq. (6.1) does not affect any of the following considerations. The corresponding contributions for the thermodynamical quantities can be simply added at the end to the ones obtained from $Z_{\pi N}$. Therefore, in what follows we shall consider only $Z_{\pi N}$ containing the nucleons and the πN interaction.

In order to study the thermodynamical properties of the "liquid phase" we have to abandon the grand canonical description because of the singularity corresponding to the gas-"liquid phase" transition. We can go either to the grand microcanonical or to the canonical ensemble with fixed baryon number. (The procedure is rather similar

the one in Ref. 26), where the Bose-Einstein condensation was considered in the grand-microcanonical ensemble.) We choose the canonical description which is simpler here (but the grand-microcanonical description is also possible and equivalent).

The connection between the grand canonical and canonical partition function is (L transform and its inverse)

$$Z_{\pi N}(\beta, V, \lambda) = \sum_{b=0}^{\infty} \lambda^b Z_{\pi N}(\beta, V, b) \quad (6.6)$$

$$Z_{\pi N}(\beta, V, b) = \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda^{b+1}} Z_{\pi N}(\beta, V, \lambda)$$

We introduce as integration variable $\lambda \varphi_N$ instead of λ and write the second relation in the form

$$Z_{\pi N}(\beta, V, b) = \varphi_N(\beta)^b \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda^{b+1}} \Gamma\left\{\varphi_{\pi}, \frac{V}{V_N}, \alpha_N, \alpha_{\pi}; \lambda\right\} \quad (6.7)$$

where

$$\Gamma\left(\varphi_{\pi}, \frac{V}{V_N}, \alpha_N, \alpha_{\pi}; \lambda\right) := \exp\left\{\frac{V}{V_N} \left[\alpha_N G_1^{(b)}(\lambda, \varphi_{\pi}) + \alpha_{\pi} \varphi_{\pi} G_2^{(b)}(\lambda, \varphi_{\pi})\right]\right\} \quad (6.8)$$

$$\alpha_N := \frac{K_2(\beta \mu_N)}{K_1(\beta \mu_N)}; \quad \alpha_{\pi} := \frac{m_{\pi} K_2(\beta \mu_{\pi})}{m_N K_1(\beta \mu_{\pi})}$$

The integration in the λ plane goes, in principle, along an arbitrary curve encircling the origin. In practice it can be easily evaluated if, e.g., there is a saddle point of the integrand on the real axis. In the gas phase the saddle point exists and we can proceed according to the well-known Darwin-Fowler method which can be found in textbooks (see, e.g., Ref. 27). Therefore, we only state the final result without going into details. In the thermodynamical limit $V \rightarrow \infty$, $b/V = \nu$ fixed we have

$$\begin{aligned} \frac{1}{V} \ln Z_{\pi N}(\beta, V, b) &= \\ &= \nu \ln\left(\frac{\varphi_N(\beta)}{\lambda_s}\right) + \frac{1}{V} \ln Z_{\pi N}\left(\beta, V, \lambda = \frac{\lambda_s}{\varphi_N(\beta)}\right) \end{aligned} \quad (6.9)$$

where $\lambda = \lambda_s$ is the saddle point determined by the equation

$$\nu V_N = \frac{K_2(\beta \mu_N)}{K_1(\beta \mu_N)} G_0^{(b)}(\lambda_s, \varphi_{\pi}(\beta)) + V_{\pi} \frac{m_{\pi}^4}{m_N} \frac{K_2(\beta \mu_{\pi})}{2\pi^2 \beta \mu_{\pi}} G_1^{(b)}(\lambda_s, \varphi_{\pi}(\beta)) \quad (6.10)$$

This equation has a solution only if the baryon number density ν is smaller than the critical value $\nu_*(\beta)$. Otherwise (i.e., in the "liquid phase") there is no saddle point and the relation in Eq. (6.9) is no longer valid.

The thermodynamical quantities can be obtained from the canonical partition function as

$$\begin{aligned} \varepsilon &= -\frac{1}{V} \frac{\partial}{\partial \beta} \ln Z(\beta, V, b) \\ P\beta &= \frac{\partial}{\partial V} \ln Z(\beta, V, b) \end{aligned} \quad (6.11)$$

Using Eq. (6.9) in the gas phase and adding, of course, the pion gas contribution coming from Z_{π} , the same equations of state can be derived for the gas phase as in the grand canonical ensemble (cf. Subsections 5.5 to 5.10).

6.2 The Canonical Partition Function $Z_{\pi N}$ in the "liquid phase"

Let us turn to the "liquid phase". As can be seen from Eq. (6.10), if the baryon number density ν approaches the phase transition value $\nu_*(\beta)$, the saddle point at $\lambda = \lambda_s$ goes to the singularity in the λ plane at

$$\lambda_0(\beta) = \ln 4 - 1 - \varphi_{\pi}(\beta) \quad (6.12)$$

For $\nu > \nu_*(\beta)$ (i.e., in the "liquid phase") the saddle point disappears. As Eq. (2.48) shows, $G(\varphi)$ has a square root cut in the complex λ -plane extending from φ_0 to ∞ . This cut generates a cut in the λ -plane extending from $\lambda_0(\beta)$ to ∞ .

In this case we can integrate along the curve C shown in Fig. 6.1. The integral splits into the contribution from the cut and from the circle.

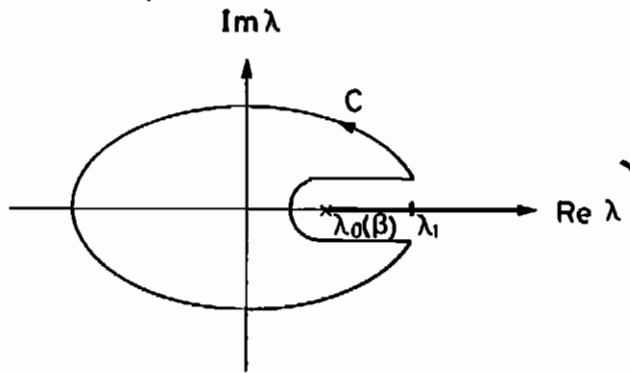


Fig. 6.1 Path of integration for Eq. (6.7) when the saddle point has disappeared at $\lambda_0(\beta)$.

Along the cut, the relation $G(\varphi^*) = G(\varphi)^*$, which follows from Eq. (2.40), yields the discontinuity relation

$$\lim_{\varepsilon \rightarrow 0} G_K^{(k)}(\lambda \pm i\varepsilon) =: H_K^{(k)}(\lambda, \varphi_2) \pm i h_K^{(k)}(\lambda, \varphi_2) \quad (6.13)$$

The functions $H_K^{(k)}(\lambda, \varphi_2)$ and $h_K^{(k)}(\lambda, \varphi_2)$ defined by Eq. (6.1) can be expanded in a power series in the variables $[\lambda - \lambda_0(\varphi_2)]$ and $[\lambda - \lambda_0(\varphi_2)]^{1/2}$, respectively. Using Eqs. (2.48), (6.4), (6.5) one finds

$$H_1^{(0)}(\lambda, \varphi_2) = G_1^{(0)}(\lambda_0(\varphi_2), \varphi_2) + (\lambda - \lambda_0(\varphi_2)) \frac{H_0 - G(\varphi_2)}{\lambda_0(\varphi_2)} + \dots \quad (6.14)$$

$$h_1^{(0)}(\lambda, \varphi_2) = \frac{2}{3\lambda_0(\varphi_2)} (\lambda - \lambda_0(\varphi_2))^{3/2} + \dots$$

$$H_2^{(0)}(\lambda, \varphi_2) = G_2^{(0)}(\lambda_0(\varphi_2), \varphi_2) + (\lambda - \lambda_0(\varphi_2)) \frac{G_1^{(0)}(\lambda_0(\varphi_2), \varphi_2)}{\lambda_0(\varphi_2)} + \dots \quad (6.15)$$

$$h_2^{(0)}(\lambda, \varphi_2) = \frac{2}{3\lambda_0(\varphi_2)^2} (\lambda - \lambda_0(\varphi_2))^{3/2} + \dots$$

We shall see below that this is all we need in the thermodynamic limit ($V \rightarrow \infty$, $b \rightarrow \infty$, $V/b = \nu$).

In terms of the functions $H_K^{(k)}(\lambda, \varphi_2)$ and $h_K^{(k)}(\lambda, \varphi_2)$ we now obtain from Eqs. (6.7) and (6.8) by integrating along the curve C shown in Fig. 6.1 the following expression:

$$Z_{\pi N}(\beta, \nu, b) = \varphi_N(\beta)^b \frac{1}{\pi} \int_{\lambda_0(\beta)}^{\lambda_1} \frac{d\lambda}{\lambda^{b+1}} \times \quad (6.16)$$

$$\times \exp \left\{ \frac{\nu}{V_N} \left[\alpha_N H_1^{(0)}(\lambda, \varphi_{\pi}) + \alpha_{\pi} \varphi_{\pi} H_2^{(0)}(\lambda, \varphi_{\pi}) \right] \right\} \times$$

$$\times \sin \left\{ \frac{\nu}{V_N} \left[\alpha_N h_1^{(0)}(\lambda, \varphi_{\pi}) + \alpha_{\pi} \varphi_{\pi} h_2^{(0)}(\lambda, \varphi_{\pi}) \right] \right\} + \int_{\text{circle}}$$

Below, we shall show that the contribution of the circle can be neglected in the thermodynamic limit ($V \rightarrow \infty$, $v = b/V$ fixed), therefore we consider only the cut where we write

$$f(\lambda, vV_N) := \frac{1}{\lambda} \exp \left\{ \frac{1}{vV_N} \left[\alpha_N h_1^{(0)}(\lambda, \varphi_N) + \alpha_\pi \varphi_\pi h_2^{(0)}(\lambda, \varphi_N) \right] \right\} \quad (6.17)$$

therefore

$$Z_{\pi N}(\beta, V, b) = \varphi_N(\beta) \frac{1}{\pi} \int_{\lambda_0(\beta)}^{\lambda_1} \frac{d\lambda}{\lambda} e^{b \ln f(\lambda, vV_N)} \times \quad (6.18)$$

$$\times \sin \left\{ \frac{b}{vV_N} \left[\alpha_N h_1^{(0)}(\lambda, \varphi_N) + \alpha_\pi \varphi_\pi h_2^{(0)}(\lambda, \varphi_N) \right] \right\}$$

From the study of the behaviour of the functions $H_K^{(k)}$ and $h_K^{(k)}$ near the tip $\lambda = \lambda_0(\beta)$ of the cut we can infer that $f(\lambda, vV_N)$ is a decreasing function of λ near $\lambda_0(\beta)$; hence if $b \rightarrow \infty$, only the tip of the cut is important (if the circle is put not very far from the tip). As a consequence, we can use the approximations in Eqs. (6.14) and (6.15) to evaluate the integral. After a somewhat long but straightforward calculation we get the simple result

$$\frac{1}{b} \ln Z_{\pi N}(\beta, V, b) \Rightarrow v \ln \left(\frac{\varphi_N(\beta)}{\lambda_0(\beta)} \right) + \frac{1}{v} \ln Z_{\pi N}(\beta, V, \lambda = \frac{\lambda_0(\beta)}{\varphi_N(\beta)}) \quad (6.19)$$

(Note that in the thermodynamic limit finite powers of b can be and, in fact, were neglected in $Z_{\pi N}$.) Comparing Eqs. (6.19) and (6.9) we can see that the canonical partition function in the "liquid phase" is the same as in the gas phase; only λ_s has to be replaced by the v independent value $\lambda_0(\beta)$.

6.2.1 The Contribution from the Circle

In order to really establish Eq. (6.19) we have to prove yet that the circle contribution in Eq. (6.16) can indeed be neglected. For convenience let us rescale the variable λ in the function Γ in Eq. (6.8) and introduce

$$\bar{\Gamma}(\varphi_N, \frac{V}{V_N}, \alpha_N, \alpha_\pi; \lambda) := \Gamma(\varphi_N, \frac{V}{V_N}, \alpha_N, \alpha_\pi, \frac{\lambda}{\sigma(\beta)}) \quad (6.20)$$

with $\sigma(\beta) > 0$ arbitrary for the moment. We have, of course,

$$Z_{\pi N}(\beta, V, b) = [\varphi_N(\beta) \sigma(\beta)]^b \frac{1}{2\pi i} \oint_{\lambda^{\frac{b}{b+1}}} \bar{\Gamma}(\varphi_N, \frac{V}{V_N}, \alpha_N, \alpha_\pi; \lambda) \quad (6.21)$$

The singularity of the function $\bar{\Gamma}$ in the λ plane is at

$$\lambda = \sigma(\beta) \lambda_0(\beta) \quad (6.22)$$

We shall show that in the "liquid phase" the circle can be omitted in the thermodynamic limit assuming that the saddle point contribution is dominating the corresponding integral in the gas phase. (This is sufficient as the Darwin-Fowler method is well established in the gas phase.) Let us choose a baryon number density v_δ in the gas phase such that

$$v_\delta = v_*(\beta) - \delta(\varepsilon) \quad (6.23)$$

where $\delta(\varepsilon)$ is such that for v_δ the saddle point is at

$$\lambda_s = \lambda_0(\beta) - \varepsilon \quad (6.24)$$

Later on we let ϵ go to zero [hence $\delta(\epsilon) \rightarrow 0$]. For the density v_δ we choose the integration path $C_{v_\delta} + S_{v_\delta}$ going across the saddle point, whereas for the density $v + v_*(\beta)$ we use $C_v + S_v$ (see Fig. 6.2).

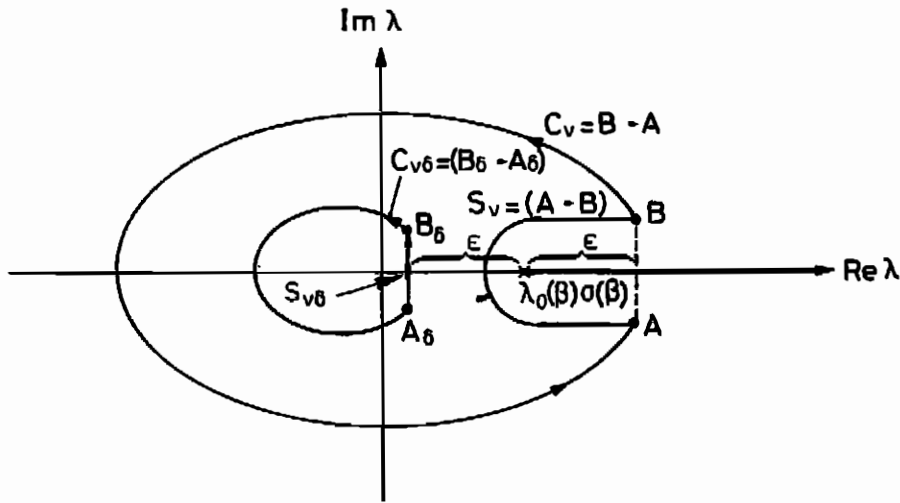


Fig. 6.2 Paths of integration used in the proof that the contribution of the circle C_v can be neglected in the thermodynamic limit.

We know from the dominance of the saddle point in the gas phase that in the thermodynamic limit

$$\int_{C_{v_\delta}} (\dots) / \int_{S_{v_\delta}} (\dots) \rightarrow 0 \tag{6.25}$$

Here and below (...) stands for the integrand in Eq. (6.21) belonging to the appropriate density (v_δ or v). On the other hand, the above explicit calculations show (see Eqs. (6.9) and (6.19) that in the case $\sigma(\beta) = \lambda_0(\beta)^{-1}$ we have

$$\int_{S_{v_\delta}} (\dots) / \int_{S_v} (\dots) \rightarrow 1 \tag{6.26}$$

At the same time

$$\int_{C_v} (\dots) \cong \int_{C_{v_\delta}} (\dots) e^{-i\alpha V(v - v_\delta)} \tag{6.27}$$

The last strongly oscillating factor cannot be cancelled as $V \rightarrow \infty$, therefore Eqs. (6.25)-(6.27) imply

$$\int_{C_v} (\dots) / \int_{S_v} (\dots) \rightarrow 0 \tag{6.28}$$

This proves the dominance of the tip of the cut in the "liquid phase".

Equation (2.40) implies $G(\lambda + 4\pi i) = G(\lambda) + 2\pi i$. Hence in addition to the cut shown in Fig. 6.1 there are further cuts from $\lambda_0 + k \cdot 4\pi i$ to $\infty + k \cdot 4\pi i$; $k = \pm 1, \pm 2, \dots$. For $b \rightarrow \infty$ these cuts cannot contribute, since they are far away from the paths shown in Figs. 6.1 and 6.2 (note: $4\pi/\lambda_0 \approx 30$).

6.3 The Equations of State for the "liquid phase"

Using the expression obtained in Eq. (6.19) for the canonical partition function of the "liquid phase" it is easy to derive the equations of state from Eq. (6.11). Let us introduce the notation $v_*(\beta)$ for the critical baryon number density, $\epsilon_*(\beta)$ for the critical (relativistic) energy density and $P_*(\beta)$ for the critical pressure. In terms of the functions $G_K^{(k)}$ we have:

$$v_*(\beta)V_N = \frac{K_2(\beta m_N)}{K_1(\beta m_N)} G_0^{(0)}(\lambda_0(\beta), \varphi_\pi(\beta)) + \frac{V_\pi m_\pi^4}{m_N} \frac{K_2(\beta m_\pi)}{2\pi^2 \beta m_\pi} G_1^{(1)}(\lambda_0(\beta), \varphi_\pi(\beta))$$

$$\begin{aligned} \varepsilon_*(\beta) &= v_*(\beta) m_N \frac{K_2(\beta m_N)}{K_1(\beta m_N)} + V_\pi m_\pi^7 \left[\frac{K_2(\beta m_\pi)}{2\pi^2 \beta m_\pi} \right]^2 G_1^{(2)}[\varphi_\pi(\beta)] + \\ &+ m_\pi^4 \frac{K_2(\beta m_\pi)}{2\pi^2 \beta m_\pi} R(\beta m_\pi) G_1^{(1)}[\varphi_\pi(\beta)] + \end{aligned}$$

$$+ G_1^{(0)}(\lambda_0(\beta), \varphi_\pi(\beta)) \frac{m_N}{V_N} \left[\frac{K_2(\beta m_N)}{K_1(\beta m_N)} R(\beta m_N) - \left(\frac{K_2(\beta m_N)}{K_1(\beta m_N)} \right)^2 \right] +$$

$$+ G_1^{(1)}(\lambda_0(\beta), \varphi_\pi(\beta)) \frac{V_\pi m_\pi^4}{V_N} \frac{K_2(\beta m_\pi)}{2\pi^2 \beta m_\pi} \frac{K_2(\beta m_N)}{K_1(\beta m_N)} +$$

$$+ G_2^{(1)}(\lambda_0(\beta), \varphi_\pi(\beta)) \frac{V_\pi m_\pi^5}{V_N m_N} \frac{K_2(\beta m_\pi)}{2\pi^2 \beta m_\pi} R(\beta m_\pi) +$$

$$+ G_2^{(2)}(\lambda_0(\beta), \varphi_\pi(\beta)) \frac{V_\pi^2 m_\pi^3}{V_N m_N} \left(\frac{K_2(\beta m_N)}{2\pi^2 \beta m_\pi} \right)^2 ;$$

$$\begin{aligned} P_*(\beta)\beta V_N &= \frac{K_2(\beta m_N)}{K_1(\beta m_N)} G_1^{(0)}(\lambda_0(\beta), \varphi_\pi(\beta)) + \\ &+ \frac{V_\pi m_\pi^4}{m_N} \frac{K_2(\beta m_\pi)}{2\pi^2 \beta m_\pi} G_1^{(1)}(\lambda_0(\beta), \varphi_\pi(\beta)) + m_\pi^3 V_N \frac{K_2(\beta m_\pi)}{2\pi^2 \beta m_\pi} G_1^{(1)}[\varphi_\pi(\beta)] \end{aligned}$$

(6.29)

Here the function

$$R(x) := \frac{3}{x} + \frac{K_1(x)}{K_2(x)} \quad (6.30)$$

was also used which actually gives the average energy per particle in a relativistic ideal (Boltzmann) gas by $mR(m\beta)$.

From Eqs (6.19) and (6.11) it follows that in the "liquid phase"

$$\varepsilon(\beta) = \varepsilon_*(\beta) + [v - v_*(\beta)] \left[m_N \frac{K_2(\beta m_N)}{K_1(\beta m_N)} + \frac{V_\pi m_\pi^4}{2\pi^2 \lambda_0(\beta)} \frac{K_2(\beta m_\pi)}{\beta m_\pi} \right] \quad (6.31)$$

$$P(\beta) = P_*(\beta)$$

For the derivation of these expressions the differentiation rules in Eq. (6.5) have to be used. From Eq. (6.31) we see that in the "liquid phase" the critical gas characterized by $v_*(\beta)$, $\varepsilon_*(\beta)$ and $P_*(\beta)$ co-exists with arbitrarily large nucleon clusters responsible for the phase transition. The average energy per baryon number in the large cluster's component is

$$\bar{\omega}_{liq} := \varepsilon(\beta) - \varepsilon_*(\beta) = m_N \frac{K_2(\beta m_N)}{K_1(\beta m_N)} + \frac{V_\pi m_\pi^4}{\lambda_0(\beta) 2\pi^2 \beta m_\pi} \quad (6.32)$$

The large cluster's phase does not contribute to the pressure, hence P is constant along the isotherms (and equal to the pressure of the critical gas). Hence, $P_*(\beta)$ is the vapour pressure when the nuclear gas and its condensate co-exist. What we have called "liquid" turns out to be really a co-existence of two phases. We understand fully the pure gaseous phase and the state of two co-existing phases, but not yet the pure "liquid". For this the volume of the clusters must be introduced also on the thermodynamical level. The property $P = P_*$ and the coincidence of the isotherms with the $\lambda = \text{constant}$ curves in the "liquid" phase ($\lambda = \lambda_0(\beta)$) is also true for the Bose-Einstein condensation²⁶⁾. This shows a structural

similarity between our "nuclear liquid" and the Bose-Einstein condensate. The difference lies in the internal excitability of the large clusters manifested, e.g., in Eq. (6.32), while the large Bose-Einstein clusters are in the ground state ($\omega_{liq} = m =$ the mass of the boson).

7. SUMMARY AND OUTLOOK

We have proposed a bootstrap model for the description of nuclear matter at finite temperature. We bootstrapped the level density (related to the mass spectrum) of nuclear matter. We consider the latter to consist of arbitrary clusters of: nucleon clusters and their excitations, of pions, of mesonic and baryonic resonances, and of the corresponding anticlusters. In a simplified, approximate version of our model which contains only Boltzmann pions and nucleons as elementary building blocks, we compute numerically the physical properties of nuclear matter. In this model, there is only one free parameter A taken to be equal to the inverse rest energy density of the nucleon.

In studying this model we obtain a non-perturbative, exact solution (describing actual qualitative physical properties of nuclear matter) despite the high and ambitious aims outlined in the Introduction. We consider our present calculations nevertheless more as an exploratory study than as a final result. However, some of the general features we find are rather model-independent and should survive further elaborations:

- 1) Considering the grand canonical partition function, depending on the chemical potential and temperature, we find three different situations:
 - a) a gaseous state (containing for $\mu \rightarrow 0, T = 0$, the empty vacuum), and characterized by the presence of easily moveable but strongly interacting nuclei and pions, all in arbitrary states of excitation;

- b) a "liquid" phase at larger baryon densities, and
 - c) a supercritical (unphysical) region above $T = T_0 = 150$ MeV, where the energy density becomes infinite.
- 2) The transition to the "liquid" phase occurs at about 0.65-0.75 of the normal nuclear number density and at finite energy density, except when T approaches T_0 , where the pure gaseous phase persists through high density and where the energy density becomes very large. We stress once more that what we have called throughout this paper the "liquid phase" is really the co-existence of two phases, vapour and liquid, in equilibrium. This is obvious from Eq. (6.31) which says that the pressure depends only on the temperature but not on the density, just as it is usual for the vapour pressure. We are presently unable to describe the pure liquid.
 - 3) In our actual description we find a limiting temperature $T_0 \approx 150$ MeV. At this temperature the energy density diverges. We have noted, however, that this is a subtle point which touches on the limits of the validity of our present interpretation of the mass spectrum. In this respect we recall that the volume of fireballs grows now with the fireball mass -- thus the average density should be finite for $T \rightarrow T_0$ (see end of sub-section 3.3). In further elaborating the alternative description mentioned in Section 3, we would have found a model with a finite energy density at T_0 so that the presently forbidden region beyond T_0 would have become accessible, though only in a microcanonical, non-thermodynamical description.

- 4) Above $T = 20$ MeV but below 60 MeV we find that the energy per baryon obeys roughly the simple relation $\sim 3/2 kT$. Below 20 MeV our model includes too little nuclear structure to have enough predictive power; above 60 MeV pion degrees of freedom absorb an increasing amount of the total energy so that the "energy per baryon" (= total energy/number of baryons) exceeds more and more the energy which the baryons carry themselves.

Perhaps the most important aspect of our work at the present level is the better understanding of several conceptual aspects new to the bootstrap model, namely:

- a) the definition of the mass spectrum, as the level density $\sigma(p, b, V)$ with the co-moving volume $V||p$;
- b) the connection between the bootstrap function ϕ (obeying the bootstrap equation) and $\ln Z$ (describing arbitrary macroscopic systems);
- c) the now possible existence of a physical world for finite energy densities beyond $\epsilon(T_0)$.

Looking ahead, we hope to enlarge our model by making the input more elaborate, by maintaining the particle-antiparticle symmetry and by considering the particular position of alpha clusters. It seems that a more profound study of the "liquid phase" will be rewarding since much of the structure of the liquid (maybe even the existence of a new "solid" phase) depends on the amount of nucleon structure we include in the input terms. An obvious first step in this direction is the possible introduction of effective masses ($<$ free masses) of the bound nucleons, a feature very likely relevant to the understanding of the saturation of nuclear matter in the bootstrap description. We must also incorporate Fermi and Bose statistics and investigate the version leading to a finite energy density at T_0 .

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