Johann Rafelski (editor)
(editors proof version)
Preprint pagination different from to be published version

Melting Hadrons, Boiling Quarks

From Hagedorn temperature to ultra-relativistic heavy-ion collisions at CERN

with a tribute to Rolf Hagedorn

Springer Open

Berlin Heidelberg New York

Barcelona Budapest Hong Kong

London Milan Paris

Santa Clara Singapore Tokyo

ISBN 978-3-319-17544-7
ISBN 978-3-319-17545-4 (eBook)
DOI 10.1007/978-3-319-17545-4
to appear September 2015

Johann Rafelski *Editor*

Melting Hadrons, Boiling Quarks

From Hagedorn Temperature to Ultra-Relativistic Heavy-Ion Collisions at CERN

With a Tribute to Rolf Hagedorn

This book shows how the study of multi-hadron production phenomena in the years after the founding of CERN culminated in Hagedorn's pioneering idea of limiting temperature, leading on to the discovery of the quark-gluon plasma – announced, in February 2000 at CERN.

Following the foreword by Herwig Schopper – the Director General (1981-1988) of CERN at the key historical juncture – the first part is a tribute to Rolf Hagedorn (1919-2003) and includes contributions by contemporary friends and colleagues, and those who were most touched by Hagedorn: Tamás Biró, Igor Dremin, Torleif Ericson, Marek Gázdzicki, Mark Gorenstein, Hans Gutbrod, Maurice Jacob, István Montvay, Berndt Müller, Grazyna Odyniec, Emanuele Quercigh, Krzysztof Redlich, Helmut Satz, Luigi Sertorio, Ludwik Turko, and Gabriele Veneziano.

The second and third parts retrace 20 years of developments that after discovery of the Hagedorn temperature in 1964 led to its recognition as the melting point of hadrons into boiling quarks, and to the rise of the experimental relativistic heavy ion collision program. These parts contain previously unpublished material authored by Hagedorn and Rafelski: conference retrospectives, research notes, workshop reports, in some instances abbreviated to avoid duplication of material, and rounded off with the editor's explanatory notes.

In celebration of 50 Years of Hagedorn Temperature

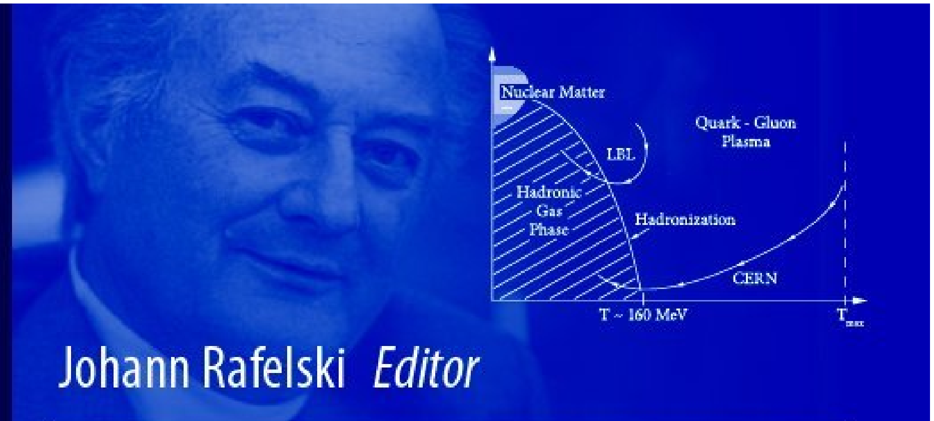
Physics

ISBN 978-3-319-17544-7

▸ springer.com

Rafelski *Ed.*

Melting Hadrons, Boiling Quarks – From Hagedorn Temperature to Ultra-Relativistic Heavy-Ion Collisions at CERN



Johann Rafelski *Editor*

Melting Hadrons, Boiling Quarks

From Hagedorn Temperature to Ultra-Relativistic Heavy-Ion Collisions at CERN

With a Tribute to Rolf Hagedorn

Springer Open

Foreword

This book fulfills two purposes which have been neglected for a long time. It delivers the proper credit to a physicist, Rolf Hagedorn, for his important role at the birth of a new research field, and it describes how a development which he started just 50 years ago is closely connected to the most recent surprises in the new experimental domain of relativistic heavy ion physics.

These developments, focused on the first 20 years 1964-1983, are faithfully and competently described in this book, prepared by Johann Rafelski, a close collaborator and co-author of Hagedorn. Its contents include much of the material they developed in close collaboration, including little known and even secret manuscripts.

I got to know Rolf Hagedorn in the 1960s when I did my first experiment at CERN. In contrast to many other theorists working at that time often on abstract and fundamental problems, Hagedorn was accessible to an experimental physicist. He explained to me his main ideas concerning the heating up of strongly interacting matter in high energy collisions in a way easily understandable for an experimentalist. The concept that the energy content of strongly interacting matter could increase without surpassing a certain temperature was matured in the head of Hagedorn over several years. It was refined and finally found its definite formulation in the form of the Statistical Bootstrap Model (SBM).

Of course, along this path he recognized that the energy content can only be increased without increasing the temperature if new degrees of freedom become available. As to their nature, at first Hagedorn could only speculate. Quarks and gluons were not yet known and the theory of strong interactions QCD which could justify the new phase of matter, a quark-gluon plasma, did not exist. But as these new concepts arose they were incorporated into Hagedorn's description of hot and dense nuclear matter.

On the experimental side, in the 1970s and 1980s, the study of heavy ion reactions grew out of the nuclear physics and eventually became an interdisciplinary field of its own that is presently achieving new peaks. Hagedorn can rightly be considered as one of the founding fathers of this field, in which the 'Hagedorn Temperature' still plays a vital role.

The rapid progress was due not only to such new theoretical ideas, but to experiments at increasing energies at laboratories like Brookhaven National Laboratory in the USA, Dubna in Russia, and CERN in Europe. At CERN difficulties arose in the 1980s, because in order to build LEP at a constant and even reduced budget, it became necessary to stop even unique facilities like the ISR collider at CERN. Some physicists considered this an act of vandalism.

In that general spirit of CERN physics program concentration and focus on LEP it was also proposed to stop the heavy ion work at CERN, and at the least, not to approve the new proposals for using the SPS for this kind of physics. I listened to all the arguments of colleagues for and against heavy ions in the SPS. I also remembered the conversations I had had with Hagedorn 15 years earlier. In the end, T. D. Lee gave me the decisive arguments that this new direction in physics should be part of the CERN program. He persuaded me because his physics argument sounded convincing and the advice was given by somebody without a direct interest.

I decided that the SPS should be converted so that it could function as a heavy ion accelerator, which unavoidably implied using some resources of CERN. But the LEP construction and related financial constraints made it impossible to provide direct funds for the experiments from the CERN budget. Heavy ion physicists would have to find the necessary resources from their home bases and to exploit existing equipment at CERN.

This decision was one of the most difficult to take since contrary to the practice at CERN, it was not supported by the competent bodies. However, the reaction of the interested physicists was marvelous and a new age of heavy ion physics started at CERN. After a series of very successful experiments at the SPS, it is reaching a new zenith in the ALICE experiment at the LHC, which is mainly devoted to heavy ion collisions. Other LHC experiments (ATLAS and CMS) are also contributing remarkable results.

Since the first steps of Hagedorn and his collaborators, a long path of new insights had to be paved with hard work. The quark-gluon plasma, a new state of matter, was identified at last in the year 2000. This new state of matter continues to surprise us: for example, at the newly built RHIC collider at BNL, it was determined that at the extreme conditions produced in high energy collisions, nuclear quark-gluon matter behaves like an ideal liquid.

I remember Hagedorn as a lively colleague fully dedicated to physics but also fond of nature and animals, especially horses. He was original, and able to explain his novel ideas and in doing this he was laying the foundations that had led to the development of the study of nuclear matter at extreme conditions at CERN.

At first, Hagedorns research interests were somewhat outside the mainstream and he could not find many colleagues to join his efforts. However, with remarkable persistence he followed up his ideas and it is very sad that he could not see the main fruits of his concepts during his lifetime. How happy would Rolf Hagedorn have been if he could have learned what wonderful new world of nuclear matter at extremely high temperatures came out of his relatively simple and original ideas he formulated 50 years ago!

CERN, Geneva and University Hamburg, Germany *Herwig Schopper*

Preface

Half a century ago, Rolf Hagedorn pioneered the field of research that this book describes: the interpretation of particle production in hadronic interaction in terms of statistical and thermal methods. While several before him, including E. Fermi and L. Landau, provided seminal contributions, Hagedorn was the first to devote his career to the subject, and to recognize the pivotal importance of the hadronic mass spectrum which led him to propose the Hagedorn temperature.

The appearance of the Hagedorn Temperature governing elementary hadronic interactions and particle production has been and remains a surprise. It could be that a full understanding of the Hagedorn temperature hides within the vacuum structure and the related quark-confinement mechanism, or, that it is still beyond our current paradigm of the laws of nature.

When our understanding was evolving, ideas were developing so quickly that there was no time to enter the cumbersome process of assembling ongoing work into refereed papers. The conference reports were often the only place where novel work was published, building progress on earlier presentations. Therefore many of the steps taken in creating this knowledge may have not been seen by the following scientific generation. Some of the evolving insights supersede earlier work which today's generation uses in their research, an example being the precise form of the Hagedorn mass spectrum. The republication here of these pivotal reports is therefore of scientific as well as historical interest.

In the time line of the subject there were two pivotal milestones. The first milestone occurred in 1964/65, when Hagedorn, working to resolve discrepancies of the statistical particle production model with the experimental *pp* reaction data, produced his "distinguishable particles" paper. Due to a twist of history, this work is published here for the first time; that is, 50 years later. Hagedorn then went on to interpret the observation he made. Within a time span of a few months he created a model of how the large diversity of strongly interacting particles could arise, based on their clustering properties, and in the process invented the Statistical Bootstrap Model.

The second milestone followed a decade later when we spearheaded the development of an experimental program to study 'melted' hadrons, and the boiling quark-gluon plasma phase of matter. The diverse roots of this program go back to the mid 1970s, but the intense theoretical and experimental work on the thermal properties of strongly interacting matter, and the confirmation of a new quark-gluon plasma paradigm started in 1978 when the SBM mutated to become a model for melting nuclear matter. This development motivated the experimental exploration in the collisions of heavy nuclei at relativistic energies of the phases of matter in conditions close to those last seen in the early Universe.

This volume has three parts. In the first part through personal recollections and historical documents, the developments culminating in the discovery of quark-gluon plasma are described, focused often on the role of Rolf Hagedorn in making this happen. It would be, however, inappropriate to present in this part only the scientific side. I have included testimonials about Hagedorn, a man of remarkable character.

The second part contains the original pivotal documents that describe the emergence of the Hagedorn temperature concept, and the Statistical Bootstrap Model as a new scientific field, paving the way for the understanding of the dissolution of hadrons into quark-gluon matter. The third part is devoted to the heavy ion collision path which led to the new paradigm of locally deconfined, hot quark-gluon plasma phase of matter, and strangeness as its observable. Quark-gluon plasma is the primordial stuff filling the Universe before matter as we know it was created.

This volume then provides the reader both a scientific and a historical perspective on melting nuclei and boiling quarks; on Rolf Hagedorn; and of how CERN, despite its initial disinterest, became the site where this new physics happened. Looking back, I can say that events in Fall 1964 – Spring 1965 marked the beginning of the path to quark-gluon plasma discovery, which CERN announced as a "New State of Matter" in February 2000.

Rolf Hagedorn was the person with whom I interacted most intensly in these formative years of the field. I thank other senior, contemporary and junior theorists directly or indirectly involved in our effort: Peter Carruthers, John W. Clark, Michael Danos, Walter Greiner, Joseph Kapusta, Peter Koch, Jean Letessier, István Montvay, Berndt Müller, Krzysztof Redlich, Helmut Satz, and Ludwik Turko. Their role is acknoweledged in individual chapters. This being a book about, and with Rolf Hagedorn, he is the main focus.

Acknowledgments

I thank Victoria Grossack for constant, competent editorial support; Dr. Stephen Lyle for essential input into latex composition of the volume; CERN for sponsoring this work as a full open access publication, released under a creative commons license; Springer and in particular Dr. Christian Caron for his interest in, and guidance given to this project; and the US Department of Energy, Office of Science, Office of Nuclear Physics under award number DE-FG02-04ER41318 for support of my research activities.

Tucson and Geneva, Winter 2014/15 *Johann Rafelski*

Contents

Acronyms

An effort is made in this volume to avoid excessive use of acronyms. However, when appropriate we follow the use in original articles of the following universally recognized abreviations which have aquired proper name character.

Laboratories

BNL	Brookhaven National Laboratory, Long Island, New York
CERN	Derived from French language, *Conseil Europén pour la Recherche Nucléaire*, and maintained as the proper name for the *International Particle Physics Laboratory* located across French-Swiss Border near to Geneva
Dubna	International laboratory in Russia named after the location, providing beams of near relativistic heavy ions
GSI	German acronym for "Gesellschaft für Schwerionenforschung", translates as Center for Heavy Ion Research, at Darmstadt suburb Wixhausen close to Frankfurt
LBNL	Lawrence Berkeley National Laboratory; earlier name LBL
LPI	(Moscow) Lebedev Physical Institute

Accelerators, Experiments

AFS	Axial Field Spectrometer, an ISR experimental area 1977 - 1982
AGS	Alternate Gradient Synchrotron, used today as injector for RHIC at BNL, formerly a fixed target relativistic heavy ion source
ALICE	LHC experiment dedicated to study of QGP
Bevalac	Two accelerators at LBL connected with transfer line, delivering a beam of near relativistic heavy ions at LBL
ISR	Intersecting Storage Ring, the first hadron collider ever built, located at CERN
LEP	Large Electron-Positron collider, was housed in the same tunnel as the LHC today
LHC	Large Hadron Collider
NAxy	NA refers to the experimental 'North Area' located in France, formerly the CERN-II campus, while 'xy' is a sequential number like 35, 49, 61, etc.

PS Proton Synchroton, the first high energy particle accelerator at CERN, served as injector to ISR, remains the injector of SPS and thus LHC
PHENIX One of two ‘large’ experiments at RHIC, see also STAR
RHIC Relativistic Heavy Ion Collider
SPS Super Proton Synchroton, an accelerator ring used today mainly as injector to LHC, but still providing heavy ion beams for fixed target experiments
STAR One of two ‘large’ experiments at RHIC, see also PHENIX
WAxy WA refers to the main CERN campus experimental ‘West Area’ while xy is sequential number like 85, 94, 97, etc

Scientific Abreviations

AA Nucleus-nucleus, used as in ‘heavy ion collision’ between nuclei of nucleon number A
BE Bootstrap Equation
BES Beam energy scan: RHIC experimental program where RHI collisions in a wide energy range are explored, reaching to lowest accessible energy
BeV Old for ‘GeV’ when a ‘billion’ was used in sense of ‘giga’
CM Center of mass or, in relativistic context, center of momentum
fm 10^{-15} meter named after Enrico Fermi, nearly the radius of the proton
GeV Giga (10^9) electron Volt, a particle physics unit of energy about 1.07 times energy equivalent of the proton mass
HG Hadron gas: same as HRG, often used in this simplified name form
HRG Hadron (also, equivalently, Hagedorn) resonance gas
LQCD Lattice-QCD as in numerical solution of QCD repreented on a lattice space-time
MeV Mega (10^6) electron Volt, there are a 1000 MeV in a GeV, see above
pA Proton-nucleus, used as in ‘collision’ with a nucleus of nucleon number A
pp Proton-proton, used as in ‘collision between’
RHI Relativistic heavy ion – typically ‘collisions’, distinct from RHIC, the collider
QCD Quantum chromo-dynamics
SBM Statistical Bootstrap Model
QGP Quark-gluon plasma
SHM Statistical Hadronization Model
T_H Hagedorn temperature, T_0 in Hagedorn’s and other contemporary work

Other Abreviations

DG The CERN Director General is often referred to as ‘DG’
SPIRES ‘Stanford Physics Information Retrieval System’; bibliographic data base about literature in the field of HEP (High Energy Physics) and related areas, originating at SLAC (Stanford Linear Accelerator Center)

Part I

Reminiscences: **Rolf Hagedorn** *and* **Relativistic Heavy Ion Research**

edited by Johann Rafelski

Contributions by:

Tamás Biró, Igor Dremin, Torleif Ericson,
Marek Gaździcki, Mark Gorenstein,
Hans Gutbrod, Maurice Jacob,
István Montvay, Berndt Müller,
Grażyna Odyniec, Emanuele Quercigh,
Johann Rafelski, Krzysztof Redlich,
Helmut Satz, Luigi Sertorio, Ludwik Turko,
Gabriele Veneziano

The year 1964/65 saw the rise of several new ideas which have shaped fundamental physics for the past 50 years. Quarks and the Higgs particle were invented, and the limiting Hagedorn temperature T_{H}, the melting point of hadrons, was recognized. Of course back in Fall 1964 – Spring 1965, if someone were asked how these new ideas could turn into the standard model of particle physics; or lead to the discovery of a new phase of matter: quark-gluon plasma – the response would have been stupendous silence.

The simple question – why cannot quarks be put on open display? – demonstrates that there is more to understanding the laws of physics than the classification of the standard model particle zoo and the measurement of its many parameters. The manifestation of all laws of physics and especially of strong interactions require incorporation of the response of the vacuum state, the modern day quantum and relativity compatible 'æther'. This is a shift in the paradigm, and thus we have to work much harder at explaining the advance in our understanding that is being made.

Rolf Hagedorn was the scientist whose dedicated, determined personal commitment formed the deep roots of this novel area of physics. I can say with certainty that in Fall 1964 he had no clue what would happen to his T_{H} in the next 20 years. This book, especially Part I, shows how from a humble beginning a path to the new paradigm of strong interactions emerged, as well as how this research program found its way onto the menu of major laboratories, in particular CERN, where the quark-gluon plasma first became experimental reality.

Rolf Hagedorn's work in the field of hot hadronic matter dominated this research field during the first 15 years: his talks and publications often gave the decisive turn to events. In order to fully appreciate the physics of *Melting Hadrons, Boiling Quarks*, one must explore the thinking of this man. It is appropriate to ask his former collaborators and those involved in the research program today to make contributions describing past events and/or their present status. In doing this one is naturally led to invite each contributor to write about Hagedorn, both as a scientist and an extraordinary human being.

Fifteen essays by 17 authors offer reflections on Rolf Hagedorn, his science, and the growth of Hagedorn's ideas to the current quark-gluon plasma experimental program. These contributions show the only place where Hagedorn worked, CERN, from its creation to the present day, as seen through eyes of Hagedorn and his contemporaries – it so happens that Hagedorn was one of the first CERN employees. Some contributions are drawn from material presented on Hagedorn's 75th birthday – updated and refreshed by the authors, with the exception of the essay by Maurice Jacob which is printed posthumously; hence I adapted it to the current format.

I believe that these first 125 pages give an accurate picture of how Hagedorn's journey in science brought CERN to the opportunity to pursue the quark-gluon plasma discovery. Each contribution is the work of its author: I did not act as a referee but as a friend and colleague, guiding when possible the author to what I did not yet see in the contents. But how it is said is entirely the doing of each contributor – in that way I believe a sincere, personal and complemental account has emerged. I thank all for their kind and understanding cooperation.

Chapter 1
Spotlight on Rolf Hagedorn

Johann Rafelski

Abstract I describe several events that characterize my work, and my personal relationship, with Rolf Hagedorn closing with biographical remarks.

1.1 Working with Hagedorn

Meeting Hagedorn

I had the privilege of interacting closely with Rolf Hagedorn during the last 25 years of his life. The pivotal role that Hagedorn played in my development was as my teacher of relativistic statistical and thermal physics, and of particle production. The timing of our collaboration was singular due to the coincidence with the scientific rise of quark-gluon plasma research. Though we published only about half-a-dozen papers together, we worked together on many of publications that were later published by us independently, an approach consistent with the unique personality of Hagedorn that will emerge from these pages. In my work, I could build on the personal strengths and scientific achievement of Hagedorn in helping to develop a new research area, the formation and observation of quark-gluon plasma.

I first met Rolf Hagedorn (Fig. 1.1) in the winter 1975/76 when I attended his Colloquium on the Statistical Bootstrap Model presented at the University Frankfurt. Hagedorn offered a fascinating description of thermal multiparticle physics, and after his talk he found a way to answer all questions. At that time I knew little about subjects such as the Statistical Bootstrap Madel, or relativistic statistical mechanics, or about the experimental data in which Hagedorn was so deeply interested. In fact I even lacked a thorough understanding of thermal physics, not unusual in the particle or the nuclear context in the early 1970s.

Department of Physics, The University of Arizona, Tucson, AZ 85721, USA

Fig. 1.1 Rolf Hagedorn (center), with Johann Rafelski June 30, 1994 *CREDIT: CERN Photo 1994-06-064-024.*

After the talk I privately asked Hagedorn a few naive questions. Hagedorn took everything seriously, and gave clear explanations to the questions which could be answered. At that time I was working on the quark structure of hadrons and it seemed to me that the work of Hagedorn should connect with this topic, as well as with another topic shaping my scientific background at the time, the field of heavy ion collisions. It became clear to me that I could learn what I needed from Hagedorn. I asked if I could visit him at CERN and he suggested I consider a short-term position application. I arrived as a CERN-Fellow on September 1, 1977.

From our first meeting, my personal impression was that Hagedorn was a modest, determined person. Important for the success of our collaboration was that he was remarkably structured in organizing his work and in presenting the outcome of his research: Hagedorn did not need to make a draft in order to create an immaculate write-up of a manuscript. All his work, personal or professional, was from the first to the last word clear and presentable. His letters rarely had corrections, and if so, he made these visible and readable – to show that he changed his mind. In seminars his questions were precise and thus could be answered. All this went along with the perfect arrangement of his desk and the office in general; everything had a place, as can be seen in Fig. 1.2.

Our collaboration was in the first years that of a teacher and a student: Rolf Hagedorn presented his ideas and theoretical work slowly, repeating details until, in his eyes, I understood everything. Sometimes we sat in his office for hours, from the morning till evening. I occasionally worried that I was wasting too much of his time, and tried out other collaborations. But I always returned, attracted to both the person and the subject. I can say that Hagedorn taught me in a year what took him nearly 20 years to discover. This has been a gigantic advantage that still marks my abilities to this day.

I think our different career paths, different fields of expertise, and different approaches to physics, meshed in a special way: for example when Hagedorn began

Fig. 1.2 Rolf Hagedorn at his office desk, 1978 *Photo: Johann Rafelski*.

his formal physics education at Göttingen his age was the same as mine upon my arrival at CERN. We were curious about each other's research, which was complementary. Hagedorn was a natural teacher looking for a student, and I wanted to learn what Hagedorn knew. Hagedorn liked a structured classroom – as we shall see, he even attended a class on retirement; I like jumps into deep water – leaving me no other choice but to swim. As I 'swam' along with Hagedorn, he strongly influenced my development as a physicist. Meeting him helped me in my choice of research field, and I worked for many years in the field that he pioneered.

A Short Story about Hagedorn Temperature

On 3 February 1978, Rolf Hagedorn handed me a copy of his unpublished manuscript, "Thermodynamics of distinguishable particles". This original had a big red dot-mark, showing it was the original, not to be lost, with the number "0" meaning less than "1" (see below). Hagedorn kept just one red-marked copy and mentioned that another was in the CERN archives. He told me that I was to keep a copy to myself – a promise I can now break having found the document on the CERN Document Server (CDS). This was the initial unpublished paper proposing an exponential hadron mass-spectrum and the limiting (Hagedorn) temperature.

Discussing with me this first paper, see Chapter 19, on limiting temperature – CERN preprint TH-483 dated 12 October 1964 Hagedorn recollected: "After Léon van Hove (see Fig. 1.3) read the manuscript, he asked me to compute requirements for the hadron mass-spectrum. This led me to recognize that not every, even exponential, mass-spectrum produces limiting temperature." Hagedorn made it clear that did not like this ad-hoc fine-tuning. By October 27, 1964 Hagedorn concluded that

Fig. 1.3 Rolf Hagedorn (on left) in discussion with Léon van Hove (on right), December 1968 *Image credit: CERN Image 68-12-143*.

his result was too model-dependent to publish and placed the justification for his decision in the CERN archives, see Chapter 17.5.

The CERN-TH 520 preprint dated 24 January 1965, "Statistical Thermodynamics of Strong Interactions at High Energies" – marked with a big "1" in the Hagedorn collection is today the renowned "Hagedorn paper". It is relevant to recollect what dates on CERN-TH preprints meant: In those days, a hand-written manuscript was handed to Tania Fabergé, the Theory Division (TH) secretary, Fig. 1.4; it received a sequential TH-preprint number and the day's date, as recorded in the TH log-book.

During my days at CERN after 1977 a normal length paper sat in the typing queue in the TH office until it reappeared in my mailbox or I got it back from Marie-Noëlle Fontaine, see Fig. 1.4, with date and number clearly visible on the front page. Somewhere along the line a senior member of TH would look at the work. This was a mild internal refereeing that also helped a young fellow like me to meet senior division members. This is how I made new friends in the Theory Division including John Bell, Maurice Jacob, and Jacques Prentki. I do not mention here Rolf Hagedorn or Léon van Hove, both of whom I met before my arrival at CERN. It is quite possible that the interaction between van Hove and Hagedorn that caused the withdrawal of the '0' paper was just such an internal refereeing exercise.

Another point in this story is that between the CERN-TH date and the actual mailing out of the paper to publishers, and the distribution as a CERN preprint, perhaps eight weeks had to pass. Hagedorn's article was received by Nuovo Cimento Supplemento on 12 March 1965, and the issue no.2 of vol. III, 1st series (1965), was printed on 28 January 1966. This was an average delay for the journal[1]. Hagedorn's monumental work received, as I believe, its first citation in an experimental *Physical Review Letters* submitted in March 1967 and printed in July 1967.

The contents of the paper '1' was widely available by means of CERN preprint distribution to most particle physics libraries in Spring 1965. Thus more than two years had passed between the report of the birth of Hagedorn limiting temperature,

[1] I thank Tullio Basaglia of CERN library for careful log of the time line of publications published in NC Supplemento.

Fig. 1.4 Top: Tania Fabergé, bottom Marie-Noëlle Fontaine talking with Gilbert Lévrier at their CERN-TH desks, Fall 1978; *Photos: Johann Rafelski*.

and someone distant noticing this new idea and the citation itself being visible. By our contemporary measure, absence of a citation in the first two years means that Hagedorn's monumental invention of limiting temperature had 'impact' zero. Even so, within a decade, Hagedorn (limiting) temperature had become a household term in the physics community and the SBM paper was cited several hundred times.

Hot Nuclear Matter in the Statistical Bootstrap Model

After I settled at CERN in Fall 1977, we immediately turned to our joint project: Hagedorn remembered our Frankfurt discussions and resumed my education about particle production and the statistical bootstrap as if we had never been interrupted. I brought into the collaboration my know-how about heavy ion collisions, confinement, and quarks. Within a few weeks we saw the scope of the work that would emerge from our discussions in the coming years.

Our collaboration had clear objectives: to develop an understanding of both, the hot and dense baryon-rich hadronic matter, see Chapter 23, and to determine particle spectra emanating from the hot fireball of hadronic matter created in relativistic heavy ion collisions, see Chapters 26 and 27. Considered after the fact, perhaps we should have established our priority by publishing on dissolving hadrons into quark matter in collisions of heavy ions a few weeks after we started. However, what we knew in Fall 1977 was not good enough for Hagedorn, who desired a fully consistent model, see Hagedorn's retrospective on our work in Section 25.4 on page 293 ff.

Hagedorn wanted a solid theoretical model of hot nuclear matter, fully consistent with all he knew about particle production, something the world could trust for years to come. Building such a 'good' model is an iterative, time-consuming process. We had to explore alternatives and needed to identify potential inconsistencies. Slowly we progressed towards a fully comprehensive SBM based model of hot nuclear matter. Looking back at those long sessions in the Winter of 1977/78 I see a blackboard full of clean, exactly formed equations – and a sign instructing that no one should clean the board; Hagedorn expected we would resume next morning. One day I took a few pictures of Hagedorn in his office as is shown in Figure 1.5.

As I learned from Hagedorn to recite by heart all the results of relativistic thermodynamics, days of work became weeks, and weeks became months, and the word about our effort spread ever wider. Our daily discussions helped the iterative discovery process. It was the arrival of István Montvay, see Chapter 5, that sped up the paper writing; he joined in our discussions, contributed many important insights, and was helpful in making us appreciate how much we knew and how important it was to share our insights with our colleagues. So by early Summer 1978 we started as a team of three the writing up of the SBM model of hot nuclear matter.

Our results were ready for presentation in late Summer 1978, and we made two extensive conference presentations of our effort. In mid-October 1978 Hagedorn presented at the Erice workshop, an event organized also to remember his 60th birth-

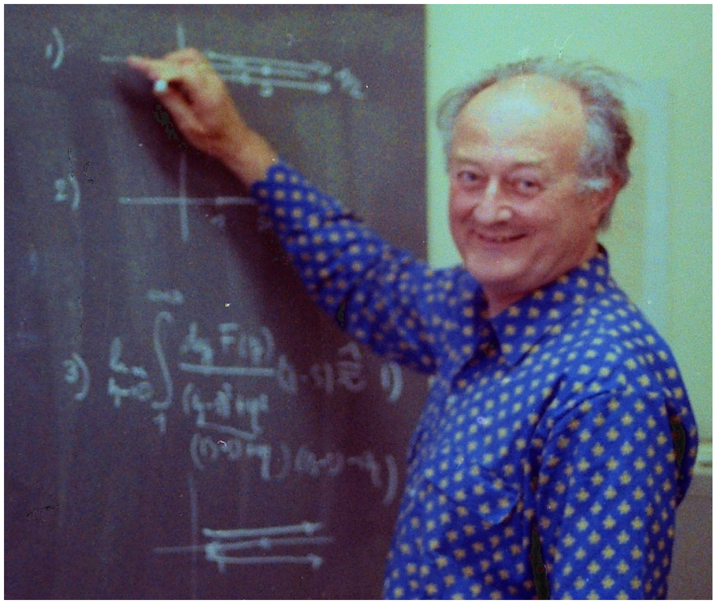


Fig. 1.5 Rolf Hagedorn at the blackboard Fall 1978 *Photo: Johann Rafelski*.

day by Helmut Satz and Luigi Sertorio[2], while I presented in January 1979 at the Bormio series of Winter meetings, see Chapter 23. Given that many people tracked the widely distributed CERN preprints, and Hagedorn was a name that many followed, our results were soon well known. We made no effort to prepare a formal publication. 'Everybody' knew of our work; the meeting proceedings (often printed by an University Press e.g. the Bormio 1979 volume) and CERN preprints were just what the Internet provides today, a free flow of scientific information.

Higher Level Computer Language

When we started to convert our ideas into results that would lead to publications, one could imagine that this was the moment when a younger collaborator would carry the load of the work. In fact before my arrival at CERN I had plenty of practice working with teletype terminals (who still remembers these?), programming in Fortran, and drawing results by hand. It turned out that for Hagedorn this was ancient technology. Rolf Hagedorn enjoyed tremendously the moment, and was resolved to prove to me how much simpler and faster it was possible to make the

[2] Proceedings *Hadron Matter at Extreme Energy Density* were edited by N. Cabibbo and L. Sertorio as Volume 2 in a new *Ettore Majorana International Science Series*, published by Plenum Press (New York 1980).

Fig. 1.6 Rolf Hagedorn, seen here working at CERN at a computer console in 1968, when the GAMMA (Graphically Aided Mathematical MAchine), a precursor to SIGMA computer language, he helped develop was launched; *Credit: CERN photo 68-12-141*.

required computations with SIGMA (System for Interactive Graphical and Mathematical Applications).

This was the computer language he had helped develop for use in direct interaction with the computer, working at its console, see Fig. 1.6, or by the time we worked together, at a remote terminal. Indeed, Hagedorn was able to complete the required calculations rapidly, and to obtain the graphic representation of our results on screen and to print these out practically ready for publication.

Development of a user-friendly computer interface, and of an easy-to-use higher level language was another pioneering idea that Hagedorn spearheaded at CERN. Arguably, this development at CERN by Hagedorn and a few collaborators (Carlo Vandoni and Juris Reinfelds in particular) of direct user-computer interactive approach, and user-oriented language, spearheaded in the CERN Computer Department the traditions which seeded the birth of the Internet at CERN twenty years later.

Relativistic Heavy Ion Collisions

In the 1970's the emerging European center for the new field of relativistic heavy ion collisions was GSI, the German Heavy Ion laboratory located between Frankfurt and Darmstadt in the village of Wixhausen, today part of suburban Darmstadt. In 1977 the experimental work was carried out in the US at the Lawrence Berkeley

Laboratory's Bevalac, see Chapters 12 and 13. GSI was the site where the experiments were prepared and data was processed, and many researchers called it home.

On the way to CERN in Spring and Summer 1977 I was able to spend some of my time at GSI. During this short period I made friends who were later important in developing the relativistic heavy ion program at CERN. Particularly relevant was meeting Rudolph Bock, who was the pre-eminent force for relativistic heavy ion physics in Europe. His experimental group worked at the Bevalac, many of these individual researchers would later shape the CERN research program, see Chapter 13.

Due to the prior meeting with Hagedorn that left such a deep mark, I recognized a scientific opportunity to merge my interest in quarks and heavy ion collisions with Hagedorn's statistical particle production model. The visit to GSI reinforced this viewpoint. By pursuing our collaborative goals Hagedorn and I 'discovered' theoretically the phase boundary to quark-gluon plasma (QGP) right before the relativistic heavy ion collision experimental program, see Chapter 25.

When Hagedorn and I started our collaboration, it was based on the intersection of our common capabilities and interests, without direct concern for a possible CERN heavy ion experimental program which seemed to be set to happen at GSI or/and LBL. Seminars and corridor discussions spread the word of our effort. In particular, the cafeteria in the main CERN building has been and is a place where people come and go, and where stories are told. Our effort to capture the physics of hot nuclear matter in the perspective of particle production and the detailed understanding of its boundary with hot quark-gluon matter formation was soon known to many, as was our view of the relevance of our work to relativistic heavy ion collision physics.

We made an effort to let the world know what we were doing. Hagedorn, in charge of several lecture series at CERN, asked me to bring in a few people who could tell him and everyone interested about the potential for experiments with relativistic heavy ion collisions. This created the opportunity to speak in private to those directly interested in our work. We could see how these visiting experts react. I suggested inviting several experimentalists from GSI and LBL; among them was Hans Gutbrod, who soon found a lasting home at CERN, see Chapter 13.

In this line of thought, Maurice Jacob connected us with the CERN experimental groups working at the Intersecting Storage Rings (ISR), see Chapter 28. Bill Willis and many others were convinced that nuclear collisions within the ISR experimental program could provide access to new physics. This interest awakened CERN management to the new 'heavy ion' scientific opportunity, although at first the general community's reception was pretty frosty. A different way to achieve the same scientific goal was found, made possible by Herwig Schopper, by using SPS as a heavy ion machine, see Chapter 29. This approach was compatible with CERN pushing ahead into the LEP era, and preparing the technology and building the large underground tunnel that today houses the LHC.

In late 1979, while our heavy ion effort was going on, I moved from CERN to Frankfurt. Following tradition I was invited to present an inaugural lecture, which was scheduled for June 18, 1980. The translation of the German abstract I submitted reads: "**Quark Matter–Nuclear Matter:** The fusion of constituents of protons and neutrons –quarks– into quark matter is expected to form a new phase of nuclear matter. Based on our our recent theoretical work this is expected to occur at temperature and density accessible to experimental study."

This lecture was a preliminary version of the presentation I would give in a few months at the Bielefeld workshop in August, and a few weeks later at the GSI-Laboratory, see Chapter 27. At this meeting for the first time a discussion of experimental signatures was a keynote topic and thus it has been since designated to be the first of the "Quark Matter" meeting series. My proposed strange particle signature of QGP was a major component of this presentation. In the following months and years I developed the strange particle signatures of the new QGP phase, see Chapters 31 and 32, while Hagedorn focused his own work soon on models of this transition, see Chapter 24. Even though we published a lot of our work in separate publications, we exchanged our manuscripts and heeded mutual advice.

Strangeness and the Discovery of Quark-Gluon Plasma

Upon arrival at CERN on September 1, 1977 Tania assigned me to a three-person office, and one of my office mates was CERN-Fellow from the UK, Brian Combridge. Brian enters the annals of physics by being the first to correctly evaluate the production of charm quarks in *pp* collisions. This process turned out to be dominated by the two-gluon fusion reactions.

My time with Brian as an office mate was pivotal in two ways: I learned much about the working of perturbative QCD, and about the importance of glue in the production of heavy quark flavor. When Rudolph Bock asked me to present how QGP can be discovered at the GSI workshop in October 1980, I placed emphasis on strangeness flavor as a possible signature. Looking back, I believe I turned to strangeness because Brian Combridge primed me with the story of charm production. This was a natural step given that the temperature of QGP formation Hagedorn and I computed was close to strange quark mass estimates.

The first strangeness signature of QGP arguments seen at the end of Chapter 27 rely on the assumption of a strangeness abundance equilibrium. One of participants at the GSI workshop, József Zimányi, went home to work out if this hypothesis could be true. Within a few months I learned that the outcome of this investigation, involving Tamás Biró (see Chapter 5), challenged my strangeness chemical equilibration hypothesis in QGP.

Unfortunately, I missed the Summer 1981 seminar József Zimányi gave in Frankfurt; I saw his work only after it was written up and circulated as a preprint in November 1981. I was interested in technical details of Biró and Zimányi work since rumors were spreading that Zimányi had shown in his lecture that the Rafelski-Hagedorn work was wrong. Indeed, the results of Hagedorn-Rafelski showing the dominance of particle production process in relativistic colliding nuclear matter were in plain contradiction to the thrust of the relativistic heavy ion collision work by some of my colleagues in Frankfurt. They assumed that the collision energy was flowing into hydrodynamic compression of nuclear matter. Later experiments established that Hagedorn-Rafelski results showing particle production were correct.

The Frankfurt nuclear matter compression hypothesis also derailed, as noted at the end of this paragraph, much of the work of Biró-Zimányi. However, the real issue with this work was elsewhere. From a first view of their preprint it was clear to me that the input from the all-important Brian Combridge's work on QCD flavor production was not present: the Feynman diagram figure showed that the kinetic model for production of strangeness flavor included only light quark annihilation on antiquarks. Thermal antiquarks are themselves quite rare in the baryon dense QGP under consideration in Frankfurt, and this antiquark based process was thus very, very slow. What was missing was the 2-gluon fusion process.

The QCD-glue strangeness chemical equilibration paper was prepared in collaboration with Berndt Müller before the end of 1981 and published soon after. These critical results are described in Chapters 31 and also in 32. For the following several years I worked out many details of strangeness and strange antibaryon signature working with Berndt, and with a student, Peter Koch. These results stimulated the experimental work described in Chapter 15 by Emanuele Quercigh, with some results shown in Chapter 33. The large 20-fold enhancement found was, in my eyes, the cornerstone of the CERN February 2000 QGP discovery story.

Among my favorite of Hagedorn's letters is the one dated 19 September 1995. It was written upon reception from publishers of a copy of the proceedings volume of a meeting[3]. I had placed Hagedorn's name on the distribution list, hoping he would enjoy the contents, but not expecting that he would read it front to back, which he did.

Hagedorn writes: "I just received here *Strangeness in Hadronic Matter* thank you sincerely. So much has happened since you told me for the first time about your ideas and considerations of strangeness in QGP[4]. Your idea has proved itself to be fruitful, exciting and – hopefully! – at the end decisive. Shall I live to see the unambiguous evidence and prove of the existence of quark-gluon plasma? Maybe this does not matter, I am anyway fully convinced, where else can the phase transformation (which surely is present) otherwise lead?"

This event was followed by continued discussion between us about the rapidly emerging results from CERN strangeness experiments. Hagedorn especially appreciated – in the historical perspective of his own work – the universality of strange particle and antiparticle transverse energy spectra showing that both particles and antiparticles had a common thermal source. In the following few years we agreed that the observed patterns of strange antibaryon enhancement, and the universal nature of evaporation spectra of particles confirm QGP discovery, a point more thoroughly described by Emanuele Quercigh in Chapter 15.

---

[3] J. Rafelski, *Strangeness in Hadronic Matter* AIP Conference Proceedings **340**, American Institute of Physics (New York 1995)

[4] Hagedorn proposes the date "1983(?)" but it must have been sometime 1979/80

Retirement

Another impression that I wish to place under the spotlight, as it also has affected many others at CERN since, concerns Rolf Hagedorn planning his retirement. I worked closely with him while he was 58-64 years old, and at CERN the retirement age was then, and still is today, 65. One day Hagedorn told me that he took a course on ‘How to Retire’. He became convinced that he must follow one piece of his classroom advice: he ought to reduce his work load gradually even before reaching the age of 65, so that when he reached 70 he would approach near zero level of scientific activity; the time lost to CERN before his retirement age could be more than made up by his work after retirement.

Hagedorn believed that the worst scenario for him would be to work full steam till the age of 65, and then to completely drop the pen. He thought this was unhealthy and for most scientists anyway impossible, as it is difficult for aged men to work full steam but also for a scientist to stop thinking. Hagedorn would never proceed without proper agreement with CERN; in other words, he saw the need to create an emeritus status which allows continuation in the research program for retired personnel. Maurice Jacob says in Chapter 3 a few things about the ensuing negotiations that have led to the recognition of the Emeritus Scientist status at CERN. According to Maurice, Hagedorn was the first to receive this recognition upon his retirement.

Rolf Hagedorn retired on 31 July, 1984, after more than 30 years of service to CERN. On 30 July, 1984, the Director General (Herwig Schopper) addressed Hagedorn in his letter: “Following a proposal made by your Division Leader (Maurice Jacob), I am happy to grant you the possibility of continuing your research activities at CERN on an unremunerated basis. . . . I should like to take this opportunity to congratulate you on your recent completion of 30 years of service and thank you for the contribution you have made to the success of this Laboratory. Wishing you good health and a happy retirement I remain, *Mit allen guten Wünschen* Yours sincerely, *Herwig Schopper*.”

I should add that while the initial plan Hagedorn had was to phase out by the age of 70, he changed the formula: on one hand our collaboration progressed and the topic of heavy ion collisions turned hot in 1979 just as he planned to begin to work less, while at the same time, CERN was still working towards developing the emeritus status. In essence the onset of the plan was postponed to the date of his formal retirement in 1984. Thus in 1994 Hagedorn still remained hard at work – “The long way to the Statistical Bootstrap Model” is his 1994 account of the development of SBM, see Chapter 17.

From personal correspondence I know that Hagedorn followed scientific developments with great interest until a few months before his death. To be specific his last long typed letter (with many attachments) which we worked through at CERN in person, dated 25 August, 2002, he apologizes about typing as handwriting was becoming difficult. He comments on several recent contributions I made with his usual precision “. . . a few typos are marked in red” in the attached preprint he read and annotated. In another comment on a recently published book[5] he says "I read the section on H(adronic)-Gas, I could not write it better". I believe he meant every word.

Hagedorn turns then to the main topic of his letter and our meeting: I passed on to him the entire draft volume (800 pages!) of the material selected, and commentaries written for the annotated reprint volume *Quark-Gluon Plasma: Theoretical Foundations* which I was working on with Joseph Kapusta and Berndt Müller. Hagedorn made many comments which were incorporated in the final version of the volume. However, our volume was not finished until after Hagedorn's death, and thus we dedicated this work to him[6].

As just noted, the health of Hagedorn was failing. At the end of the August 2002 two-day meeting he came out to say that he did not expect to meet me again; he already had been told about an aggressive cancer. Indeed I could not visit him before he passed, a little more than six months after this meeting.

1.2 The Righteous Man

There was a clear sense of absolute morality around Hagedorn. He would not leave a stone unturned to correct an error or an injustice. When reviewing the work of others he did not hide in anonymity; instead he sought permission of the editors to contact the authors to explain to them in person any required corrections, or to sign the positive reviews. But there were other expressions of his strong convictions:

Helping Those In Need

Rolf Hagedorn was always ready to help those in need. When a colleague and collaborator arrived from beyond the iron curtain, and told him that he had decided not to return home, becoming a refugee, Hagedorn spent weeks looking for a place that would take him. When the state of war in Poland made life there difficult in the early eighties, and one of our friends and collaborators was imprisoned, Rolf worked incessantly to ease the burden on his friend in Poland. Hagedorn, a former prisoner of war, knew well what internment meant to a scientist.

Similarly, the fate of the prominent Soviet dissidents Andrei Sakharov and Youri Orlov preoccupied him much of the time, and he left nothin undone to further their cause. Certainly the Soviet empire was not brought down by Rolf Hagedorn, but he was definitely an important force that helped our colleagues in the East fight for their freedom: knowing that people like Rolf were there to stand behind them in bad times was a great support for their cause.

[5] J. Letessier & J. Rafelski, *Hadron and Quark-Gluon Plasma*, (Cambridge, UK, 2002)

[6] J. Kapusta, B. Müller & J.Rafelski, *Quark-Gluon Plasma: Theoretical Foundations; An annotated reprint collection*, Elsevier, (Amsterdam 2003).

When the cards turned and the iron curtain fell, a different type of injustice attracted Hagedorn's attention. Now Hagedorn stood up for the rights of those that the revolution in the East suddenly left in limbo: before curtain's fall some scientists were among the 'privileged'; overnight they became jobless, and were scorned as collaborators of the communist regime. Hagedorn knew better, and had the moral privilege by having stood in for the freedom and the truth at earlier time. His cause was again the right one, and he prevailed in his battles, for example for the rights of his close collaborator Johannes Ranft.

Le Chambon – a short story outside the physics context

Nobody asked who was Jewish and who was not. Nobody asked where you were from. Nobody asked who your father was or if you could pay. They just accepted each of us, taking us in with warmth, sheltering children, often without their parents-children who cried in the night from nightmares; by Elizabeth Koenig-Kaufman, a former child refugee in Le Chambon[7].

As years passed I have become more aware of the relation that Hagedorn must have had with Le Chambon. What I learned about begins in Fall 1963, when, as Maurice Jacob describes it in Chapter 3 they met in India; at that time, Hagedorn had enrolled his young daughter at the Boarding School Le Collége-Lycé Cv́enol International in Le Chambon. A school at the time filled with children associated with the holocaust survivors, a school located in a small village 4.5h long hours by car from CERN, just about 'nowhere' in the midst of France. If you were not aware what happened in Le Chambon, this school is hard to find and for this reason the school is about to close today. However, this is a place Hagedorn and his wife liked to ride horses in the hills of the French Central Plateau, a place he turned to within a relatively short time after his arrival at CERN, at a time at which the story of this remarkable place was not yet told. Is this just a coincidence?

Let me tell the rest of the story. Aside of music which Hagedorn loved as is also described by Maurice Jacob in Chapter 3, Hagedorn liked "…a little History of Arts and related topics" which is a quote from his 1954 personal short biography as printed below. This in turn led to his interest in photography, a topic also raised by Maurice. Once he had time, in retirement, Hagedorn learned to create his own color prints. He mentioned this to me, later I realized, as a warning. In September 1989 – about 10-11 years after taking me, along with his wife and mine, on a long weekend trip to Le Chambon-Sur-Lignon – he had sent a set of A4 large prints from this event, which he was keen to tell me, he made himself. As Maurice Jacob ably describes Hagedorn liked photography, so there are many more pictures he could print besides the Le Chambon event, but he did not.

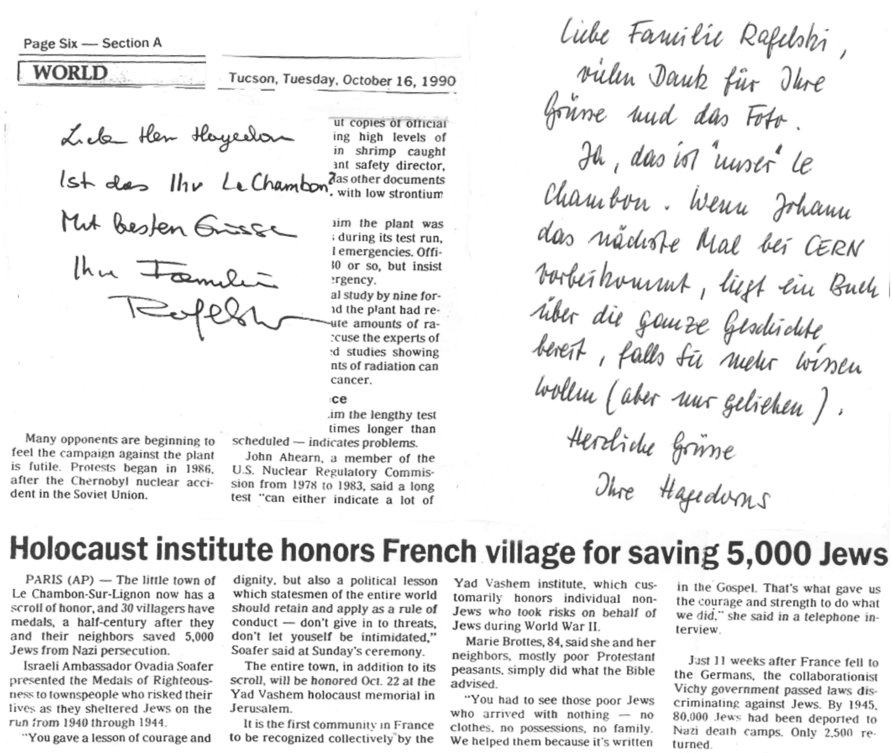
I kept these prints on my desk while developing a plan to thank Hagedorn appropriately. In the interim the photographs reminded me of our visit to Le Chambon-

[7] Opening of the entry "Le Chambon-sur-Lignon" at web-based "Holocaust Encyclopedia"

Sur-Lignon a decade earlier, and the name of this village, long forgotten, was in consequence in my mind. I did recall that these photographs were taken at local spots important to Hagedorn who took us to these places and who set up the photo shots in detail.

Reading my Tucson daily paper on October 16, 1990, nearly a year after receiving the photo prints, I noticed that this rural French Village Le Chambon-Sur-Lignon, where we had spent a memorable weekend with the Hagedorns, was subject of a long report: "...entire town will be honored on October 22, 1990 at the Yad Vashem holocaust memorial in Jerusalem." What, an entire village? That was a circumstance out of the ordinary. I shipped this article shown in Fig. 1.7 to Hagedorn. Since I was sending the original to Hagedorn, I copied the article, fortunately along with a short pertinent note to keep the details in memory.

My note said (in German) *Dear Hagedorn, Is this your Le Chambon-Sur-Lignon? With best regards, your family Rafelski*. On the right in Fig. 1.7, in return mail *Dear Family Rafelski, many thanks for your greetings and the photo. Yes, this is "our" Le Chambon-Sur-Lignon. Should you want to know more, a book addressing the entire*


Page Six — Section A

WORLD

Tucson, Tuesday, October 16, 1990

Liebe Herr Hagedorn
Ist das Ihr Le Chambon?
Mit besten Grüsse
Ihre Familie
Rafelski

ut copies of official ing high levels of in shrimp caught ant safety director, ?as other documents . with low strontium

aim the plant was ; during its test run, l emergencies. Offi- 30 or so, but insist ergency.

al study by nine for- id the plant had re- ute amounts of ra- :cuse the experts of :d studies showing nts of radiation can cancer.

ce

im the lengthy test times longer than scheduled — indicates problems.

Many opponents are beginning to feel the campaign against the plant is futile. Protests began in 1986, after the Chernobyl nuclear accident in the Soviet Union.

John Ahearn, a member of the U.S. Nuclear Regulatory Commission from 1978 to 1983, said a long test "can either indicate a lot of

Liebe Familie Rafelski,
vielen Dank für Ihre
Grüsse und das Foto.
Ja, das ist "unser" Le
Chambon. Wenn Johann
das nächste Mal bei CERN
vorbeikommt, liegt ein Buch
über die ganze Geschichte
bereit, falls Sie mehr wissen
wollen (aber nur geliehen).
Herzliche Grüsse
Ihre Hagedorns

Holocaust institute honors French village for saving 5,000 Jews

PARIS (AP) — The little town of Le Chambon-Sur-Lignon now has a scroll of honor, and 30 villagers have medals, a half-century after they and their neighbors saved 5,000 Jews from Nazi persecution.

Israeli Ambassador Ovadia Soafer presented the Medals of Righteousness to townspeople who risked their lives as they sheltered Jews on the run from 1940 through 1944.

"You gave a lesson of courage and dignity, but also a political lesson which statesmen of the entire world should retain and apply as a rule of conduct — don't give in to threats, don't let youself be intimidated," Soafer said at Sunday's ceremony.

The entire town, in addition to its scroll, will be honored Oct. 22 at the Yad Vashem holocaust memorial in Jerusalem.

It is the first community in France to be recognized collectively by the Yad Vashem institute, which customarily honors individual non-Jews who took risks on behalf of Jews during World War II.

Marie Brottes, 84, said she and her neighbors, mostly poor Protestant peasants, simply did what the Bible advised.

"You had to see those poor Jews who arrived with nothing — no clothes, no possessions, no family. We helped them because it's written in the Gospel. That's what gave us the courage and strength to do what we did," she said in a telephone interview.

Just 11 weeks after France fell to the Germans, the collaborationist Vichy government passed laws discriminating against Jews. By 1945, 80,000 Jews had been deported to Nazi death camps. Only 2,500 returned.

Fig. 1.7 Included with my annual Holiday Greetings 1990/1 was a photo I prepared for Hagedorn, Fig. 6.4 on page 48, and the article from our daily paper I noted by accident in Tucson local paper, The Arizona Daily Star of October 16, 1990.

story will be waiting for Johann at the time of his next visit to CERN (but only on loan). Yours very sincerely Hagedorns. I was not to come to CERN for some time and when I returned I did not raise the matter of Le Chambon, and Hagedorn did not either, as I am sure today, he waited for me to step forward. The book he mentions must have been: *Lest Innocent Blood Be Shed (Story of the Village of Le Chambon)* by Philip Hallie – first printed in 1978.

As I write this down, I further recollect how we were having breakfast one morning in Le Chambon, I realize today that these middle-aged people of Le Chambon who in a collective act of valor saved maybe 3,000 to 5,000 Jewish refugees from certain death, were serving us coffee. At the time I noted that the Hagedorns had a strong personal connection in the hotel we were staying. When we visited Le Chambon-Sur-Lignon, Hagedorn was among people with the same mentality as his own. People who did the right thing, even in face of punishment by death. I believe today that the people of Le Chambon-Sur-Lignon, and Hagedorns, were more deeply connected.

It is known that the activities of the population of Le Chambon benefited from a mole among Nazis sending out timely warnings, and preventing adverse encounters which would have tragic consequences for the refugees, and the entire population of Le Chambon. I can only speculate that Hagedorn knew about Le Chambon before he arrived at CERN; perhaps someone told him during his years in Africa, or while he was in the war prison camp in US. But judging by the way the Hagedorns took us around the hills of Le Chambon, in depth of my heart I think that a more direct involvement of Hagedorns in the war story of Le Chambon is possible.

1.3 Rolf Hagedorn: Biographical Information

Rolf Hagedorn Curriculum Vitae[8] *1954*

I was born in 1919 at Wuppertal-Barmen. My parents, Max and Linda Hagedorn (born Reinecke), are both alive today. At the age of six I entered the elementary school and four years later the "Städtisches Realgymnasium Wuppertal-Barmen" where I passed the final examination (Abiturium) Easter 1937. Thereafter I joined the Reichsarbeitsdienst[9] and in November 1937 the Armed Forces since I intended to complete my military service before beginning the university course in order not to interrupt it. But at the end of 1939, when I was to be released, the war began and I had to stay with the Luftwaffe. During the war, I spent a long time in North Africa where I was captured May 1943 and was brought to the USA. There in the prisoner-of-war Camp Crossville (Tennessee) I began studying Mathematics, Physics and also a little History of Arts and related topics. Further I had to teach a high school

[8] CERN job application form dated January 1954

[9] Translated: Reich Labor Service. From June 1935 onwards, men aged between 18 and 24 had to serve six months before entering their military draft service.

class in Physics on a basic level. After having returned in January 1946, I continued my studies in Göttingen, where I decided to become a theoretician. Consequently I was a pupil of Prof. R. Becker. In 1950 I passed the diplom-examination with a work on the theory of Lamb-shift in nonrelativistic Quantum electrodynamics (not published). This was followed by a paper in the theory of Barium-titanate as a thesis in spring 1952. Since June 1952 I have been working at the Max-Planck-Institute für Physik, Göttingen, on nuclear physics, especially on the evaporation stars in nuclear emulsions.

CERN Appointment

Hagedorn began his CERN appointment on 1 April, 1954 in the "Proton Synchrotron (PS) Division". He was offered a permanent appointment on 28 September 1960 effective 1 January, 1961 working in the "Theoretical Studies (TH) Division". His job description: "The main activity of Dr. Hagedorn should concern theoretical investigations and computations of direct importance to the planning and interpretation of experiments in the CERN high energy physics program, in particular the investigations based on the use of statistical models for particle production; or other similar studies as may become of interest in the evolution of this subject. In addition he should devote a reasonable fraction of his time to the study of other parts of theoretical physics so as to be prepared to adjust the orientation of his work to the unpredictable needs of the future development of high energy physics."

CERN Obituary[10]: Rolf Hagedorn 1919–2003

This official CERN document was drafted by Torleif Ericson and Johann Rafelski and published by CERN Courier, and in abbreviated format in the official CERN bulletin, this shorter version follows (with added precise birth and death data):

Rolf Hagedorn, the theorist who introduced the concept that hadronic matter has a melting point, died on 9 March, 2003 in Geneva. He was born 20 July, 1919 in Wuppertal, Germany.

After studies in Göttingen he came to CERN in Geneva in 1954 as an accelerator theorist. He joined the CERN Theory Group after its transfer in 1957 from Copenhagen to Geneva (Fig. 1.8) and he was a senior physicist in the Division when he retired in 1984.

He continued his research after retirement, and up to very recently he made pertinent contributions in developments in the field of relativistic heavy ion collisions.

As an accelerator physicist he developed the theoretical predictions for the particle spectra initially observed when the CERN PS first began operation, which was

[10] Copyright CERN 2014 - CERN Publications, DG-CO; Bulletin Issue 14/2003, http://cds.cern.ch/record/46337

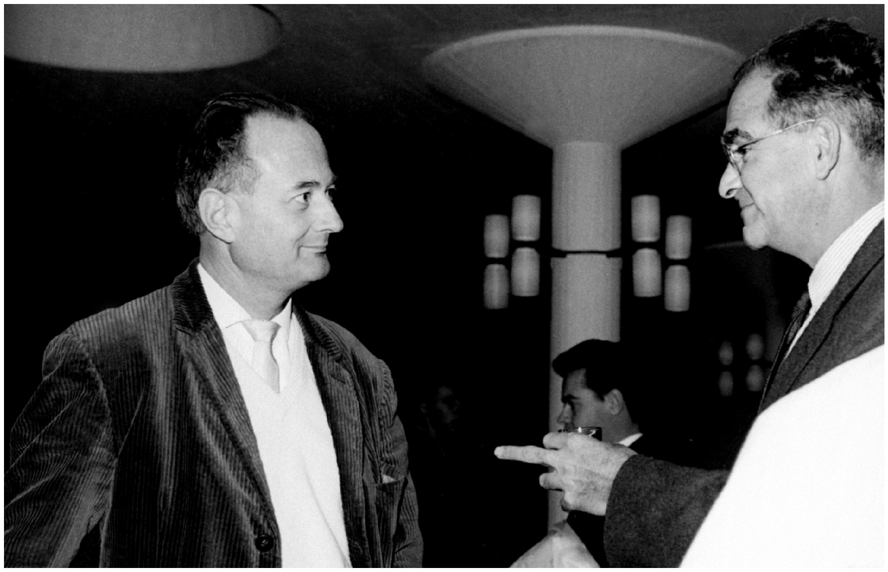


Fig. 1.8 Rolf Hagedorn 1964, on left, in conversation with Victor Weisskopf, Director General of CERN; *Credit: CERN photo 64-11-103*.

important for the optimization of secondary beams. He then developed the statistical theory of meson production in considerable detail up to very high energies. It was a consequence of these studies that he found that one should expect a limiting temperature in hadronic collisions, the Hagedorn temperature. This picture has had a major impact on theoretical thinking and on our understanding of the properties of hot hadronic matter, which is important now in the heavy ion program. Since the picture is applicable to any exponentially rising particle mass spectrum it is also influencing the development of string theories.

Among contributions to CERN, Hagedorn developed one of the earliest user-friendly interactive computing programs for algebraic manipulations, the SIGMA.

Rolf Hagedorn was a person of the highest scientific integrity and standards of reasoning. He was always willing to help colleagues and his comments were usually penetrating and deep.

He will be much missed by friends and colleagues.

Chapter 2
Rolf Hagedorn: The years leading to T_H

Torleif Ericson

Abstract On the occasion of Rolf Hagedorn's 75$^{\text{th}}$ birthday it gives me a special honor and pleasure to have this opportunity to add these remarks about the seminal influence Hagedorn has had on the scientific community.

2.1 CERN Theory Division in 1960s

Having been friends and office neighbors at CERN for more than 35 years (Fig. 2.1) I want here to recall the atmosphere at CERN when I first met Hagedorn in 1960 and some episodes that I remember from our time together at CERN. I will also say a few words about the miraculous events that brought Rolf Hagedorn into physics in the first place, and soon after to CERN.

Rolf Hagedorn brings to my mind a particular word: fortuitous. The word 'fortuitous' has several meanings. First of all fortuitous means something that happens by chance, it's statistical, and the word statistical of course brings Rolf Hagedorn to mind immediately. But fortuitous has also the overtone of something of good fortune, good luck, and that again is very much what I associate with Rolf's impact on all of us. It also applies very much to his path into science and choice of research area, which has also been associated with a couple of steps of chance and good fortune.

I first met Hagedorn, when I arrived at CERN as a postdoc in 1960. CERN was a quite different place at the time, not at all like CERN today. It was considerably smaller, although Rolf probably would say it had already become a very large organization before and during the time he was there. But in 1960, when I came to

Adapted from: *Hot Hadronic Matter: Theory and Experiment*, NATO ASI Series B: Physics Vol **346**, Jean Letessier, Hans. H. Gutbrod and Johann Rafelski, Plenum Press (New York 1995)

CERN-TH, 1211 Geneva 23, Switzerland

Fig. 2.1 Torleif Ericson (left) and Rolf Hagedorn listen to an after dinner lecture. Across the table Tatjana Fabergé is onlooking.*CREDIT: CERN Photo 1994-06-067-031*.

CERN, the Theory Division had only some 30 people or so and now we are 170. So everybody knew everybody and we discussed our research with everybody else.

I had the good fortune to be assigned an office next door to Rolf's, and luckily, fortuitously, even in later years we have always had offices side by side, so we have been in very close contact. I had been working myself on statistical reactions in nuclear physics, and in particular on compound nuclei and related topics, and it was of course interesting to see what this could mean in particle physics. I knew before I arrived about Hagedorn, at least vaguely.

Once at CERN I quickly wanted to know what Rolf Hagedorn was up to. I immediately found out that he was a true strong interaction man — I recall that when I first went to his office, I did not look at his door. I just knocked and stepped in — ignoring a huge sign on the door saying 'Ne pas déranger' —'Do not disturb'! I did not know that Rolf had the habit of taking a nap for an hour or so at noon, and I just barged in there. As you can imagine, there was a pretty strong interaction! I was a newly arrived, fresh postdoc, while he was a staff member fast asleep. But ever since I only had very pleasant interactions with Rolf.

What was the Theory Division at this time? Let us look at the people who were around. Many of these friends came to celebrate Hagedorn's 75th birthday. There was one important fixture of the Theory Division through all the years, our beloved Tatiana Fabergé, see Fig. 1.4. She was already then running the TH-division, and has done it ever since, independent of whoever happened to be the division leader.

The late Léon Van Hove came to take up the leadership of the Theory Division on nearly the same day I arrived and he was already a major physics figure. He had been on and off at CERN before, but that was about the time he started to take care of us. He is unfortunately no longer with us, but his wife Jenny is here. I am very happy about that, since Léon was a very old and dear friend of Rolf's.

Jacques Prentki was member of the division from the early days. John and Mary Bell had arrived at CERN about a year before — Mary was not part of the Theory Division, but was in close contact with it. Since Hagedorn also was interested in accelerators, the Bells were also in close contact with him. André Martin had shown up the year before and was like me a fellow in the Theory Division as was Daniele Amati. Both became staff members a couple of years later. Roland Omnès had left just before I arrived, as did Frans Cerulus who had interacted very strongly with Rolf. André Petermann was busy with calculating QED corrections.

Another staff member was the late Vladimir Glaser, who was a very brilliant abstract theoretician, who had been earlier a postdoc along with Rolf in Germany. He was a staff member for many years and a great specialist in finding ingenious counter-examples to theoretical conjectures. That covers the staff members, and most of the remainder were fellows, with a few visitors from the USA. It was really quite a small group, very different from the 170 postdocs and senior people who are in the group today.

2.2 Hagedorn's Path to and at CERN

The war years

Let me now tell a few words about the scientific career of Rolf Hagedorn (Fig. 2.2), and about how he got into physics in the first place. This is a very interesting story, which I have heard from Rolf Hagedorn in various forms at various times throughout the years. Until rather late in his life there was little that indicated that Rolf would make an exceptional career as a scientist.

Hagedorn's life as a young man was deeply marked by the upheavals of the war years in Europe and the greater evils that took place at that time. He graduated from high school not long before the war, was immediately drafted into the army, and soon after the WWII started shipped off into North Africa as an officer in the Rommel Afrika Korps. He has told me how much he enjoyed the vast spaces and the quiet, pitch dark nights with brilliant stars in the desert.

As you know the Afrika Korps was captured following the Allied invasion of North Africa in 1943 and Rolf spent the rest of the war in an officer prison camp in the United States. They were all very young people in the camp and there was not much to do, so they set up their own 'university'. There were of course no senior professors, so these young men taught each other what they happened to know. Presumably this was similar to Viki Weisskopf's saying: 'Physicists are persons who explain to each other what they do not know'. So maybe that was when Rolf Hagedorn became a physicist. At first he got training not so much in physics, but

Fig. 2.2 In midst of the experimental advances in heavy ion physics, Summer 1994, a workshop on “Hot Hadronic Matter: Theory and Experiment” was held at Divonne, France. The proceedings were published as: *Hot Hadronic Matter: Theory and Experiment*, NATO ASI Series B: Physics Vol **346**, Jean Letessier, Hans. H. Gutbrod and Johann Rafelski, Plenum Press (New York 1995). The picture shows Rolf Hagedorn, on June 30, 1994 thanking those attanding. *CREDIT: CERN Photo 1994-06-065-004*.

in other subjects such as mathematics, since he met an assistant of Hilbert’s, who knew a lot of mathematics.

At Göttingen

When Hagedorn returned to Germany in January 1946, German universities had been destroyed. In fact the students usually had to start at the universities literally carrying bricks to build up the institutes. Rolf in his mid-late-twenties following nearly nine years of service, war and prison camp was not a very young student.

Because of his training in the prison camp he succeeded, after some non-trivial effort, to be accepted as a fourth semester student at the university of Göttingen — one of the few left standing and at the same time a university with a great tradition in physics and mathematics. After having completed his studies with the usual diploma and doctorate with a thesis under Richard Becker on solid-state physics, he was accepted at the Max Planck Institute for Physics as a postdoc.

This was still located at Göttingen at the time, and not yet in Munich, and the director was Werner Heisenberg. Hagedorn was a member of an exceptional group and I think some names might interest you: Bruno Zumino, Harry Lehmann, Wolfhart Zimmermann, Kurt Symanzik, Gerhard Lüders, Reinhard Oehme, Vladimir Glaser, Carl Friedrich von Weizäcker and a couple more. I suppose at the time Hagedorn thought that this was a pretty normal group, but there was something very special at Göttingen at that time, for all of them have made important marks on physics in the following years.

In 1954 Hagedorn got the opportunity to go for some months to CERN, this very new place coming up in Geneva, which was not yet well known, and in fact not even formally and totally approved. He arrived at CERN to help with accelerator design problems, to calculate non-linear oscillations in particle orbits. The pioneering work on linear orbit theory had just been completed by Gerhard Lüders, who wished to go back to Göttingen.

Lüders asked Heisenberg to send somebody to replace him, and Heisenberg asked Rolf if he was interested in going for a couple of months. Being paid only 300 DM per month at the Max Planck Institute (not much for a family of three) Hagedorn jumped at the occasion. Important events in life happen like that: you come for a couple of months but you end up staying for the rest of your life. That's exactly what happened to Rolf.

At CERN

During this early period Hagedorn calculated non-linear orbital oscillations of the CERN–PS with some clever novel approximation techniques extended from one dimension to two dimensions. When the CERN theory group came to Geneva from Copenhagen, where it had been located at first, Rolf joined it, and it was how we met in 1960. In the small CERN environment of the time, it was easy to move between very varied activities and he had of course been a theoretical physicist all the time.

I want to emphasize that it is this unusual background that has marked Rolf deeply in his scientific evolution; a Ph.D. in theoretical solid state physics, followed by work on cosmic ray interaction in Heisenberg's institute, on to orbit theory at CERN and finally, to the CERN theoretical physics division, he has seen most of the physics of the day. Without such a varied experience he may have had a very different and probably less original impact on physics.

In these early years I had many discussions with Hagedorn about statistical hadronic physics and its basis, but I never quite got into the field. He explained to me repeatedly the background of the Fermi statistical model and its assumptions. I always thought I understood it, when he explained it, but afterwards I never completely understood it. I do not know today if I understood it or did not; it was on and off.

While I did not quite personally get involved into Hagedorn's research line, I followed it closely. I speculated if the kind of statistical fluctuations effects I worked on in low energy nuclear physics could be applied to these high energy situations. Steve Frautschi developed some of these ideas later on. I instead turned to intermediate energy physics, which took off just then.

One day, which I remember vividly, some time in late 1964, I encountered Hagedorn who was just bubbling over to a degree I have never seen before or since. His eyes were quite bright describing to me all these fireballs: fireballs going into fireballs living on fireballs forever and all in a logically consistent way. This must have been only a few days after the invention of the statistical bootstrap picture. I had the impression of a man who had just found the famous philosopher's stone, and that must have been how Hagedorn felt about it. Clearly, he recognized the importance of his novel idea.

2.3 Appreciation

We all know that Rolf Hagedorn was not a physicist who shot from the hip in face of a new problem. He took his time; he worked it over; he wanted to absorb it. Maybe this reflects the thoughts of long nights in the desert and the prison camp, in which time of reflection is a great virtue. I am sure that this pattern was a mark of this background, being brought up in an environment with plenty of time for informal discussions, with few formal classes pounding down your neck and professors pushing you to produce something very quickly. As result you learn to absorb a problem slowly, and, in case of Hagedorn, you do it profoundly.

Maybe that is why, in a strange way, Hagedorn differed from a large number of other prominent physicists I have met during my life. Most of the physicists are recognized rapidly after their contribution. After that quick recognition, their impact as time goes on gets disseminated and integrated, and people often notice it less and less. With Hagedorn, the opposite happened. It is like the best of wines. It is not so palatable in the early years, but as time goes on, it becomes more and more remarkable.

Statistical Bootstrap was looked upon with considerable skepticism in the beginning, at least within the CERN Theory Division. But as time has gone on, it has taken on bigger and bigger dimensions and has become more and more important. This is the sign of truly original work, of something that had real influence on our thinking.

Chapter 3
Music, and Science: Tribute to Rolf Hagedorn*

Maurice Jacob

Abstract I present here Rolf Hagedorn (Fig. 3.1) as a man, and present his achievements as a physicist. He has made several very important contributions: to particle and nuclear fields of research: The Hagedorn Temperature and the Statistical Bootstrap Model are concepts that are here to stay, and which have stimulated much further research. But Rolf Hagedorn is also a wonderful person and, saying that, does not require a specialist.

Fig. 3.1 Maurice Jacob (in right of center back part of picture) flanked by Ms Jenny (Léon) Van Hove on his left, and by Ms Mary (John) Bell on his right, observes Hagedorn's reaction to his 75th birthday gift, photo taken shortly before this address; *CREDIT: CERN Photo 1994-06-067-018*.

---

* Presented in 1994 by Maurice Jacob, *deceased 2 May 2007*
CERN-TH, 1211 Geneve 23, Switzerland
corresponding editor
J. Rafelski, Department of Physics, The University of Arizona, Tucson, AZ 85721, USA

Adapted from: *Hot Hadronic Matter: Theory and Experiment*, NATO ASI Series B: Physics Vol **346**, Jean Letessier, Hans. H. Gutbrod and Johann Rafelski, Plenum Press (New York 1995)

3.1 Personal Remarks

Visit to India

I first met Rolf Hagedorn in Madras in the fall of 1963. I had only seen him briefly earlier, and knew of him as a member of the CERN theory group. Madras, where we spent three full months together, is the place where I really got to know him. We had independently responded to the call of Alladi Ramakrishnan and were each giving a one-term lecture course at Matscience. Hagedorn was teaching relativistic mechanics based on his CERN lectures and a book which he had recently written on the subject.

We spent plenty of time together, sharing this fantastic Indian experience, and I greatly appreciated the friendship which he extended to me despite our difference in age. I strongly felt the sensitivity with which he was reacting to everything. I was impressed by his thoughtful kindness and his benevolent understanding in front of many features of Indian life which were bringing a complicated mixture of great admiration and some of the times feelings of revulsion to our western eyes. His long war years had shaped up his compassion for humankind.

Rolf was a magnificent traveling companion and the great pleasure which he had in discovering these gorgeous temples of southern India was communicative. I remember him summarizing his visits saying, “Now I have seen something beautiful. I am happy”. He was a zealous photographer and he collected a magnificent series of photographs which I admired later. He had clever ways to take close-up pictures of people without embarrassing them. I admired his skills.

I never saw Rolf lose his temper. His only strong reaction which I can recall was during a night which we spent in a guest house in an Indian wildlife reserve at Bandipur. This was during a wonderful four-day trip which Lise and I shared with him through the magnificent Mysore area. During the middle of the night the guest house, where we had been alone in the evening, was invaded by a very noisy group of visitors who stormed it past midnight as if it was their own. I still remember Rolf shouting, “Since it is a guest house you should have realized that there could be guests inside”.

Art and music

I recall his appreciation for the magnificent Indian music which we heard in Madras. We listened together to “monuments” such as M.S. Subulakshmi and Ravi Shankar. The carnatic music of Subulakshmi is very different from our classical western music but Rolf would say, “Music can take many forms but one can always recognize and fully appreciate great music”.

If I may at this point venture a hypothesis, I would say that music may have even played a role in Rolf joining the CERN Theory group. Rolf Hagedorn had come in 1954 to CERN to be at first an accelerator physicist, working with Schorr on accelerator theory. But his office was next to that of Jacques Prentki and Bernard D'Espagnat.

Jacques and Rolf could share their love for music during many discussions and also talk physics. Jacques told me that he remembers Rolf talking in a seminar about "an ensemble of accelerators". We notice here the influence of thermodynamical concepts! But, back to music: I was told that they were surprised to discover a common appreciation for Heinrich Schütz.

A student of Monteverdi, Schütz is considered by the experts as the initiator of Baroque music. He may not compare to many to the giants who followed him and in particular to Bach but Rolf would say, "Bach is like the Alps when Schütz is rather like the Jura, and . . . I like the Jura". This we know he does, living in the countryside where the slope of the Jura starts to rise sharply, and sharing with his wife this pleasant and quite country life with cats, horses and lots of music.

3.2 Contribution to Research

Torleif Ericson, for many years his neighbor in the TH Division, has told us about Rolf Hagedorn's place at CERN. I will turn now to the difficult task of trying to summarize in a few phrases Hagedorn's contributions to our field of research.

First with generalities, one may say that it is in the line with the brilliant Germanic tradition in statistical thermodynamics and Rolf may find well-deserved pride in having his name associated with a temperature. It is also in line with the phenomenological approach whereby one tries to understand and predict according to models. It is finally in line with the desire of any theorist to achieve a powerful synthetic view, providing a rationale for the observed phenomena. The Statistical Bootstrap Model, which is Hagedorn's brain child, fits perfectly the latter point.

Thermal particle production

Particle production processes are now so well known that they are taken for granted. Nevertheless the fact that very high energy collisions generally result in the production of many secondaries first came as a surprise to many. Having admitted that this is the case, the idea to try to apply the wide body of knowledge of statistical thermodynamics to such production processes may naturally come to mind.

However, difficulties quickly speak for themselves. Great minds like Fermi and Landau indeed made clever attempts but with unsatisfactory results. Particle physics and statistical physics were long separated. This is no longer the case! In particular, we now know of the great successes that were later met at the interface of statistical physics and field theory.

The contribution of Rolf Hagedorn concerns the application of statistical physics to the phenomenology of hadronic interactions, a field of research in which at CERN

Léon Van Hove was also much interested, and which he described as follows: “A meeting ground between particle and statistical physics, a dialog between theory and experiment”.

Rolf Hagedorn’s work started long before I came to CERN. My understanding is that Bruno Ferretti, who was head of the Theory Division when Hagedorn joined it, asked him to try to predict particle yields in the accelerator high energy collisions of the time. This he started with Frans Cerulus. There were few clues to begin with, but they made the best of the fireball concept which was then supported by cosmic ray studies and used it to make predictions about particle yields and therefore the secondary beams to be expected from the machine beam directed at a target.

Many key ingredients brought soon afterward by experiment helped refine the approach. Among them one should quote the limited transverse momentum with which the overwhelming majority of the secondary particles happen to be produced. They show an exponential drop with respect to the transverse mass. One should also quote the exponential drop of elastic scattering at wide angle as a function of incident energy. Such exponential behaviors strongly suggest a thermal distribution for whatever eventually comes out of the reaction and it is to Rolf’s great credit to have clung to this thermal interpretation and to have used it to build production models which turned out to be remarkably accurate at predicting yields for the many different types of secondaries which originate from high energy collisions.

Limiting temperature

What could actually be “thermalized” in the collisions? Many objections were raised at the time. Applying straightforward statistical mechanics to the produced pions was indeed giving the wrong results. But, even if there was a thermalized system at all, why was the temperature apparently constant? Shouldn’t one have expected it to rise with incident energy or with the mass of the excited fireball?

It was Rolf’s great merit to interpret the apparently limiting temperature which could be associated with the transverse mass distribution of the secondaries as resulting from an exponential mass spectrum for the many resonant states in which hadrons can be excited into before these resonance would fragment into less massive ones to eventually give, at the end of the line, the observed secondary particles.

The rise of the temperature is associated with the population of higher and higher energy levels by the elements of a system. If there is an exponentially increasing number of level offering themselves to be filled, the temperature saturates. It is the entropy which eventually increases with the collision energy but the temperature gets then stuck to a limiting value. This is the Hagedorn temperature. It is of the order of 150 MeV, close to the pion mass.

The impressive number of states which have now to be considered at the same time leads to a new writing of equations based on statistical physics. The factorial n factor, which was plaguing statistical calculation focusing on pions only and which was introduced to rightfully avoid multiple counting in phase space integrals, now had to be dropped, since each one of the many states was unlikely to have a population exceeding one. Agreement between experiment and statistical calculations prevailed at long last.

In his popular book, “The Quark and the Jaguar”, Murray Gell Mann explains how progress in physics often results from the dropping of a condition which was long considered as mandatory and which had not been properly challenged. This applies very well to this $1/n!$ factor which as Rolf concluded for the hadronic system effectively had not to be there after all.

Statistical Bootstrap Model

Despite the great success of the Hagedorn approach at predicting particle yields, we may still have reservations at speaking about a temperature in collisions among elementary particles but, as we shall see later, this applies to heavy ion collisions which are attracting an increasing interest and attention. But now comes Rolf’s great achievement in pioneering the development of the Statistical Bootstrap Model. Rolf has beautifully described its genesis in *The Long Way to the Statistical Bootstrap Model* see Chapter 17 in this volume.

To put it in a nutshell, one may say that each of the many resonant states in which hadrons can be excited through a collision is itself a constituent of a still heavier one while being also composed of lighter ones. What Hagedorn showed is that when one puts logic and hard work into the idea one cannot escape an exponential spectrum of resonant states. The temperature of such a system is then limited from above.

This limit is the Hagedorn temperature. If one takes a more global view, talking about “fireballs” (in the old language) or “clusters” (in the more modern vernacular) rather than of resonances, the conclusion is that the temperature of such objects is independent of their mass. One can then also understand why the limiting temperature is of the same order as the mass of the smallest mass state, the pion.

The concept of an exponential spectrum is now part of our understanding of hadron phenomena. It has been reached through different approaches such as that offered by dual models. It fits beautifully the hadronic level counting which can now be followed up to over 4000 cataloged resonances Rolf was first at pinning it down through his Statistical Bootstrap Model. The Statistical Bootstrap Model has been at the origin of many further works which have refined it. Rolf was thus at the origin of an important and very fruitful line of research.

Can one go beyond the limiting temperature set by Hagedorn? The answer is yes, but one has to consider a phase transition whereby one leaves the hadronic phase to reach a new phase where the hadron constituents, the quarks and the gluons, are no longer confined. The limiting temperature becomes a phase transition temperature which can be calculated by means of lattice gauge theory method.

The ongoing experimental and theoretical work bears witness to all the fascinating activities which now go on applying statistical and thermodynamical concepts to heavy ion collisions. Rolf Hagedorn can take fully justified pride in having pioneered and followed this line of approach for a long time and this despite the many stumbling blocks which he had to overcome. We can rejoice with him that many of his views have been vindicated by recent and promising developments.

3.3 Active Retirement

We can also rejoice with him that his "pro forma" retirement in 1984, which has kept him close to CERN and we enjoy his frequent visits. We are happy to see him following closely and participating in present research. His talk at this conference bears magnificent witness to that.

In connection with Rolf "pro forma" retirement, I should conclude by saying that I was thrilled to have Rolf as a test case when pushing through the Management Board, together with Adolf Minten and Allan Wetherell, a special status for "retired scientists willing to continue research".

Rolf Hagedorn was the first person at CERN to be granted the new status. Great was my joy when I could leave with him the letter from the CERN director general Erwin Schopper written to that effect which had followed by a mere few days the approval of the new scheme.

Let me try to summarize Hagedorn's research through a modest limerick (Fig. 3.2):

There are many hadrons with strong interaction
Which behave thermally in their curious motion
Rolf Hagedorn showed long ago
How S-B-M can make it go
Exponential spectrum is the explanation.

Fig. 3.2 Maurice Jacob (center) follows Rolf Hagedorn accompanied by Ms Mary (John) Bell *CREDIT: CERN Photo 1994-06-065-026*.

Chapter 4
On Hagedorn

Luigi Sertorio

Abstract Theoretical physics is not based on dogmatic truths. It is the search for ever improving formal constructions that can agree with the new experimental discoveries. At each step new unknowns appear, new domains are opened. This is the immense fascination of fundamental research.

4.1 In Times Past

When CERN was created 60 years ago I was a student of physics. The technology applied in the creation of the CERN particle accelerator and the super powerful computers was the frontier of excellence in the years after the second world war. The nucleon-nucleon collision in the region of the GeV per particle is the creation in the laboratory of processes not existing on planets and on stars. In the Earth we have molecular interactions in the range of fractions of eV, from extremely cold to fire. In the Sun we have nuclear interactions in the range of MeV, inside the core. With collisions in the GeV region we explore interactions that are beyond the dynamics of the known island of permanence, stars and planets, therefore we explore something that belongs to cosmology. The transition from nuclear to subnuclear physics is not only the search for high energy scale of interactions, or small space scale of structures, but is a new challenge to the dualism microscopic and macroscopic, quantum theory and gravitation.

I met Hagedorn at CERN in the years 1970-1973. He was a great man who loved the truth, not the authority. His mind was rigorous, calm, curious, and wide. He did not sit on his success but continuously extended his horizon of research. His name was connected to the concept of fireball and limiting temperature. Fermi was the first to apply the language of statistical physics for the subnuclear interactions. Hagedorn extended Fermi's approach, taking into account the multitude of hadron resonances. He obtained a very satisfactory description of the experimental particle production

---

INFN, Torino, Italy

data produced at CERN. The discovery of Hagedorn's limiting temperature in 1964 happened just in exactly the same year which brought cosmology and the Big Bang to the experimental realm by the observation of the cosmic background thermal photons.

At that time three theoretical approaches coexisted, the analytic S matrix, the Hamiltonian model, and thermodynamics, the work of Hagedorn. The S matrix approach needed a simplicity in the domain of resonances that was not compatible with the experimental data. The Hamiltonian theory developed into the construction of an Hamiltonian with an ever increasing number of discrete symmetries. Cosmology is facing a triple dilemma: gravitation for the understanding of the large scale morphology, quantum formalism for the microscopic world, and thermodynamics that acts in between. Thermodynamics contains the arrow of time, while gravitation and Hamiltonian quantum mechanics contain time reversal symmetry. This is the problem of future physics. Finally notice that the terrestrial biosphere also contains the arrow of time through the immensely complex link of the interaction between the living organisms and the inorganic molecular thermodynamics. I turned my interest to the study of the biosphere. Not easy, not immediate. Other physicists of my generation followed this course. Among the senior scientists that encouraged such turn I can mention Bethe and Hagedorn.

I had the fortune of interacting with Hagedorn in the three years at CERN and also later on. But at this point let me open a parenthesis.

Every important step in the advancement of physics has two articulations: the broader theoretical horizon that is opened, and the technology that is implied in the research itself. Technology has in its turn military and civilian consequences. This is true since the time of Archimedes and his offensive concave mirrors, solar weapons. Each scientific advance leads to a new weapon: the birth of metallurgy moved from swords to guns; modern molecular thermodynamics and exothermic reactions among molecules nourished the development of all kinds of explosives. Then arrives quantum mechanics with the discovery of nuclear reactions, very rapidly exploited in the creation of the fission bomb, and a little later the fusion H-bomb. On the civilian use side, the thermodynamic control of the nuclear exothermic reactions was achieved in the domain of fission, not yet for fusion.

What does this have to do with Hagedorn? In the second half of the twentieth century both the USA and the Soviet Union developed high level laboratories devoted to nuclear weapon research and production. The dream was the subnuclear bomb, a thousand times more powerful than the H-bomb. The eyes were focused on particle accelerators where protons are smashed. The message coming from Hagedorn was very clear: no exothermic reactions, but instead particle fragmentation. Stop to the bombardiers. The impact of these events that I summarize, and that are unknown to the general public, has been very important for the large scale international financing of research. We understand that several young scientists turned their energies to the understanding of complex systems. This phenomenon was particularly relevant in USA and Soviet Union. In particular the research on the biosphere takes strength in those years.

4.2 Wide Field of Interests

I had several conversations with Hagedorn on general subjects which are my preeminent field of interest today: these are the life in the Universe and the understanding of time and evolution. What is life? In some sacred ancient books is a gift of God; later we find the concept of evolution, further on with the birth of quantum photochemistry the concept of biosphere and the deep complexity of genetics can be formulated. This immense domain of problems fascinated Hagedorn and was the attraction of my interaction with him.

Consider the process of star formation and the new stars of second generation which come with planets. The planets are, in the universe, the tiny domains of permanence for the existence of inorganic molecules, and possibly the existence of the fragile complex organic molecules. We are born within the terrestrial biosphere; no wonder if along the millennia we have developed a geocentric vision of life. But modern cosmology forces us to rethink our place in the Universe. Hagedorn's mind was fascinated by these questions.

In 1994 I gave Hagedorn (Fig. 4.1) a copy of my book *Thermodynamics of Complex Systems*. Months later we had a meeting at CERN and he returned the book to me covered with corrections, recalculations, plus handwritten pages and pages of remarks and proposals. The size of the book was doubled. He said: "you should rewrite it, there is much more to say, I will help with my criticism". Several, also personal, difficulties stepped in, and I was unable to continue in the way he was suggesting. However, the light of his encouragement was extremely important for the continuation of my work since.

This is an example of my personal experience with Hagedorn. This was not the exception but the rule how Hagedorn interacted with his friends, and with his colleagues, even those he had not yet met but who sent him their work asking for his opinion.

My friendship with Hagedorn was deep, and it was with very great personal satisfaction that I organized The Erice meeting in October 1978, where we celebrated his work and his 60th birthday. I could also be present and support in many essential ways the 1994 Divonne meeting where we celebrated Hagedorn's 75th birthday, a splendid fest of science supported by NATO Scientific Affairs Division.

Hagedorn was always available to discuss physics, in every situation. Once, I don't remember the year, we met in his office at CERN and at the end of the afternoon we moved to the parking lot. He was going to drive to his home nearby in France; I was on the way to take my return trip to Torino driving my loved Alfa Romeo. The parking area goodbye developed into a conversation on the definition of boundary for the complex system biosphere, that lasted over one hour. The problem is still open today. And that was the last time I talked with Hagedorn.

Fig. 4.1 Standing: Rolf Hagedorn – Across of Maurice Jacob and Luigi Sertorio; Between Luigi and Maurice: Ms Mary Bell, next to Maurice Jacob: Mrs Van Hove; On Hagedorn Side Torleif Ericson (left edge) and Chris Lewellyn Smith (bottom edge) Mrs Helga Rafelski on left of Hagedorn and Mrs. Zinoviev on right. *Image credit: CERN Image 199406-066-018.*

4.3 Retrospective

When Hagedorn made his path breaking discovery, CERN was young, about ten years of age, a concentration of talent, creative interactions. Today CERN is 50 years older, has grown to be a revered international organization from which we expect a leadership of excellence. The present work on relativistic heavy ions collision is inspired by the revolutionary ideas of Hagedorn. A flow of ideas and new problems were originated in those beautiful years, and are a challenge for our future.

It is a great honor for me to be able to write these notes in memory of the 50th birthday of the creation of the paradigm of limiting temperature of hadrons, and in this way to contribute to the lasting memory of Hagedorn's path breaking contribution made at CERN. As I have tried to explain the work of Hagedorn had a broad impact, not only limited to the CERN community. I can certainly say that the time of Hagedorn was a great time.

Chapter 5
Hungarian Perspective

István Montvay and Tamás Biró

Abstract Rolf Hagedorn is introduced from the personal perspective of two Hungarian physics generations. A colleague (IM) and a student (TB) recount memories and events from the early-70-s to mid-80-s, and evaluate Hagedorn's impact on present particle and nuclear Hungarian physics community.

5.1 Influence Spreads to Hungary

The Statistical Bootstrap Model was presented by Rolf Hagedorn in the proceedings of the 1974 Balatonfüred Symposium on High Energy Hadron Interactions coedited by one of us (IM). However, to the best of our memory, Hagedorn never was able to actually visit Hungary. Yet when we look around today, a disproportionately large fraction of Hungarian Physics is engaged in the fields he pioneered. Both the emigrees, as well as those who made their lives in Hungary behind the iron curtain have espoused the soft hadron production and related fields. This mystery has many origins and the best way to address this is by taking under the microscope a few momentary events reported by key eyewitnesses. Maybe our two complementary contributions do not completly answer this question but we make a first step.

Hungary joined CERN in 1992, but Hungarian groups have participated in numerous experiments at CERN almost since its foundation. These collaborations were coordinated by the KFKI Research Institute for Particle and Nuclear Physics (RMKI) and Institute of Nuclear Research (ATOMKI) of the Hungarian Academy of Sciences, of the Departments of Atomic and Theoretical Physics of Loránd Eötvös University in Budapest and the Institute of Experimental Physics of the University of Debrecen. Hungarian research groups have contributed to many experiments at CERN. And, indeed, some theorists including István Montvay could also visit CERN for extended periods of time.

IM: DESY, Hamburg, Germany; and TB: Wigner RCP, Budapest, Hungary

5.2 Memories by István Montvay

I met Rolf Hagedorn in 1972 when I was the first visiting scientist from Hungary invited for one year at CERN. This visit was for me an outstanding possibility to work at one of the leading research institutes of the world. First, I wrote papers on quark and parton models but I was soon fascinated by the theories of multiparticle production in high energy hadron collisions, in particular, by the statistical bootstrap model of fireballs and hadron thermodynamics of Hagedorn which predicted a highest temperature of hadronic matter due to the exponentially increasing density of states.

This "Hagedorn temperature" showed up in the experimentally observed universal transverse momentum distribution of the produced hadrons (mainly pions) at high collision energies. Under the guidance of Hagedorn, I discovered the possibility to explicitly solve the integral equations for fireball decay. (I. Montvay, Phys. Lett. B **42** (1972) 466; Nucl. Phys. B **53** (1973) 521.) With time my collaboration with Hagedorn became more intensive. At the end of my visit at CERN we wrote a paper together on a model study in hadron statistical bootstrap. (R. Hagedorn and I. Montvay, Nucl. Phys. B **59** (1973) 45.).

During the time at CERN and in Geneva I learned a lot, especially on topics related to statistics and thermodynamics. Hagedorn's help was invaluable - not only in physics but also in everyday life in the "western world" to a large extent unknown until then by me and by my wife. At the time the situation in Europe was very differnt. For example our visas were valid only for Switzerland but not for France where much of CERN is located. Hagedorn had a solution to the problem: I remember how he smuggled us in his car through the French border at Saint-Genis. This was very exciting for us both because we could enjoy Hagedorn's hospitality in Sergy-Haut, and several times in the excellent French restaurants in the surroundings. We realized that though these deeds were somewhat illegal, should we be caught we would not serve time in a gulag. That was a very different world compared to home.

The second time I had the opportunity to work with Hagedorn was in 1978 when stopping at CERN on my way out of Hungary going on "vacation", and I soon settled down together with my family at first in Bielefeld and then in Hamburg at DESY.

I rejoined Hagedorn's team at CERN. We were, three of us together with Johann Rafelski, a very intense and enjoyable collaboration addressing different questions of the thermodynamics of nuclear matter within the framework of the statistical bootstrap model. We worked out a detailed picture of high temperature hadronic phases: a gaseous phase at low nucleon density and a liquid phase at high density. In our work Hagedorn's concept of "highest temperature" was evolving from an absolute highest temperature towards the "highest temperature of hadronic form of matter". In a study of the properties of Hagedorn singularity, Cabibbo and Parisi considered the possibility of its interpretation as a phase transition. This feature, however, was absent in the original Hagedorn's Statistical Bootstrap Model (SBM) they considered.

Our effort extended and modified physical ideas of the SBM allowing us the study of phase boundary. A detailed description of the outcome of our lively discussions at the blackboard in Hagedorn's office at CERN was written down in a nascent format (R. Hagedorn, I. Montvay and J. Rafelski, CERN-TH-2605, Dec 1978, 99pp) for the Proceedings of the Erice Workshop held in October 1978 on "Hadronic Matter at Extreme Energy Density" (edited by Nicola Cabibbo and Luigi Sertorio, Plenum 1980). This work developed soon into a complete and consistent model description of the phase transition from hadronic to quark-gluonic phase.

While I enjoyed the phenomenological approach taken by Hagedorn and Rafelski, my interest grew in more exact, numerical lattice characterization of the properties of strong intereactions. This work took me via Bielefeld to DESY in late 1978. In my work I continued the Hagedorn legacy, but in a very different, much more abstract setting, much different from the intuitive Hagedorn work.

5.3 Tamás Biró Grows up with Hagedorn

I came to strong interaction physics and the nascent field of QGP just when István was moving out of Hagedorn circle to pursue a more exact description of fundamental interaction on lattice. I joined the field and was noticed right away because of my interest in understanding strangeness production, an observable of the quark-gluon plasma phase that appeared in the proposals for relativistic heavy ion experiments made by Hagedorn and Rafelski in 1980. Under supervision of my advisor J.Zimányi I received my diploma in 1980, and my Ph.D. in 1982. To be able to pursue the hot hadron and quark matter with strangeness I had to read the relevant papers and thus I had gained the 1970's view of thermal strong interaction physics.

Although I never met Rolf Hagedorn personally, I was growing up in physics with his ideas spreading wide over particle and nuclear physics. A statistical, horribile dictu thermodynamical approach to high energy elementary systems and the very notion of a quark-gluon plasma became widespread in the 1980's and has only grown and matured since then.

The young Hungarian particle physicists, István Montvay, Peter Hasenfratz, Julius Kuti and Zoltan Kunszt - and a few others, mainly former students of George Marx, led the theory research in Budapest in the 1970's, but after their departure from Hungary their influence dispersed onto the international stage. Nuclear physicists took over introducing the new fields, the heavy ion experiment and quark-gluon phase production, the quest for this Holy Grail of matter exceeding the Hagedorn temperature. The noteworthy participants were József Zimányi, István Lovas and Judit Németh in Budapest and soon also in Debrecen. As relations of Hungary with western countries gradually eased, more and more physicists visited western institutes and our research ideas developed much in parallel with the world's leading institutes.

Being a student of József Zimányi I remember a paper from 1979, which he co-authored with his former colleauge in Budapest, István Montvay, discussing nuclear reactions in terms of hydrodynamics of hot nuclear matter (I.Montvay and

J.Zimányi, NPA 316 (1979) 490). My same-generation-colleague Anna Hasenfratz next door to my office was working on the connection of lattice QCD renormalization with more conventional schemes together with her brother (A.Hasenfratz and P.Hasenfratz, PLB 93 (1980) 165). As early as 1981 Julius Kuti headed the first lattice SU(2) calculation on an East-German computer in Hungary (J.Kuti, J. Polonyi and K. Szlachanyi, PLB 98 (1981) 199).

These examples show that the thermal interpretation of strongly interacting bulk hadronic, later quark, matter and the thinking in terms of a phase transition became ubiquitous as I entered the field of physics in the 1980's. The initiation and renaissence of experimental relativistic heavy-ion programs for decades to come followed this development. It triggered anew theoretical work in terms of thermodynamics and hydrodynamics. The phrases 'hadrochemistry' and 'quarkochemistry' were coined for the study of the time evolution of hot nuclear matter, and lattice gauge theory, followed by lattice QCD conquered the world of computing with theoretical physics purposes.

To date the physical picture has been refined: it turned out that QCD describes a crossover type transition between the hadronic and quark-gluon plasma stages of elementary matter; even signs of the finite heat bath have been rediscovered in the curving of particle spectra and number distribution. Yet, the transition still occurs near to the temperature obtained by Hagedorn in his model in 1964: around $T_{\mathrm{H}} \approx$ 160 MeV.

In my own work I recently was able to pick up the thread of Hagedorn limiting temperature, within the newly rising context of non-extensive statistical physics and quark coalescence (T.S. Biró and A.Peshier, PLB 632 (2006) 247). This work was motivated by the earlier studies, including one by Hagedorn (R. Hagedorn, La Revista del Nuovo Cimento, 6 (1983) 1), that a cut power-law distribution is a much better fit to particle spectra than the exponential – used in the original limiting temperature considerations. We studied production of hadrons arising from power-law tailed distribution of massless partons formed in coalescence. We found that this generates an exponential growth of the multiplicity of hadrons with mass m. The cut power-law distribution leads to a nonlinear equipartition formula, $E/N \sim T/(1-T/T_0) + T/(1-2T/3T_0) + T/(1-T/3T_0)$, showing that $T < T_0$ even at infinite energy.

5.4 Hagedorn Remembered

Rolf Hagedorn's impact on physics is farther reaching in the indirect than in the direct way. At first he was pretty much alone, but persisted in the application of thermodynamics in the field of particle production and strong interaction physics. However, within a decade, in the late seventies and eighties of the 20th century, the thermal and hydrodynamical models of high energy nuclear collisions became worldwide fields of interest and helped to successfully interpret bulk features of soft strongly interacting matter phenomena in CERN and BNL experiments.

Chapter 6
The Tale of the Hagedorn Temperature

Johann Rafelski and Torleif Ericson

Abstract We recall the context and impact of Rolf Hagedorn's discovery of limiting temperature, in effect a melting point of hadrons, and its influence on the physics of strong interactions.

6.1 Particle Production

Collisions of particles at very high energies generally result in the production of many secondary particles. When first observed in cosmic-ray interactions, this effect was unexpected for almost everyone[1], but it led to the idea of applying the wide body of knowledge of statistical thermodynamics to multiparticle production processes. Prominent physicists such as Enrico Fermi, Lev Landau, and Isaak Pomeranchuk made pioneering contributions to this approach, but because difficulties soon arose this work did not initially become the mainstream for the study of particle production. However, it was natural for Rolf Hagedorn to turn to the problem.

Hagedorn had an unusually diverse educational and research background, which included thermal, solid-state, particle, and nuclear physics. His initial work on statistical particle production led to his prediction, in the 1960s, of particle yields at the highest accelerator energies at the time at CERN's proton synchrotron. Though there were few clues on how to proceed, he began by making the most of the 'fireball' concept, which was then supported by cosmic-ray studies. In this approach, all the energy of the collision was regarded as being contained within a small space-time volume from which particles radiated, as in a burning fireball.

Adapted from an article which appeared in the September 2003 issue of CERN Courier

JR: Department of Physics, The University of Arizona, Tucson, AZ 85721, USA
TE: CERN-TH, 1211 Geneve 23, Switzerland

[1] Excpected for example by W. Heisenberg, see Z. Phys. **126**, 569 (1949) and references therein. We thank H. Satz for bringing this work to our attention.

Several key ingredients from early experiments helped him to refine this approach. Among these observations, the most noticeable was the limited transverse momentum of the overwhelming majority of the secondary particles. Also, the elastic scattering cross-section at large angles was found to drop exponentially as a function of incident energy. Such behavior strongly suggested an inherently thermal momentum distribution.

However, many objections were raised in these pioneering days of the early 1960s. What might actually be 'thermalized' in a high-energy collision? Applying straightforward statistical mechanics gave too small a yield of pions. Moreover, even if there was a thermalized system in the first place, why was the apparent temperature constant? Should it not rise with incident beam energy?

It is to Hagedorn's great credit that he stayed with his thermal interpretation, solving the problems one after another. His particle-production models turned out to be remarkably accurate at predicting yields for the many different types of secondaries that originate in high-energy collisions. He understood that the temperature governing particle spectra does not increase, because as more and more energy is poured into the system, new particles are produced.

It is the entropy that increases with the collision energy. If the number of particles of a given mass (or mass spectrum) increases exponentially, the temperature becomes stuck at a limiting value. This is the Hagedorn temperature T_{H}. The value of T_{H} is hard to pin precisely, as it depends also on other parameters of the strongly interacting particles still evolving in our understanding today as exact mathematical tools, such as lattice gauge theory, mature. Hagedorn gave its value as $T_{\mathrm{H}} \simeq 150$ MeV, but it may be as low as the mass of the lightest hadron, the pion, $T_{\mathrm{H}} \approx m_{\pi} \approx 140$ MeV and as high as 160 MeV.

The impressive number of distinguishable hadronic states which now have to be considered at the same time leads to a rewriting of equations based on statistical physics, and introducing numerous massive hadron resonances which eventually fragment into less massive ones to yield the observed secondary particles. At the 'bottom line', this solved the problem of the pion yield. The factor $1/n!$, which originated in the quantum indistinguishability of identical particles, had plagued the statistical calculations that focused only on pions. Now it had become unimportant as each one of the many states was unlikely to have a population, n, exceeding 1. At long last agreement between experiment and statistical calculations prevailed.

6.2 The Statistical Bootstrap Model

Once these physical facts had been assembled, explanation of the observed hadron production was at hand. However, the implementation of the model required considerable fine-tuning of parameters and mathematical equations, a situation which Hagedorn in the end did not like. Hagedorn turned his attention to improving the theoretical and conceptual interpretation, in particular to present a natural mecha-

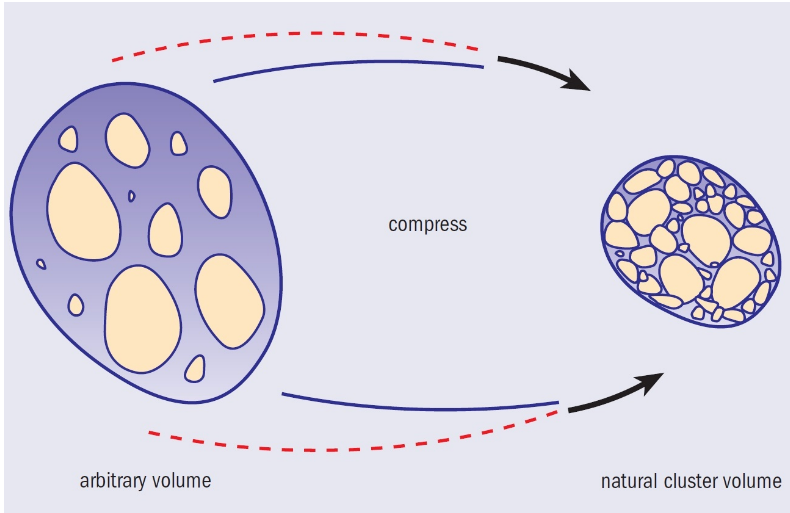


Fig. 6.1 An illustration of the statistical bootstrap idea. When a drop comprising hadronic particles and resonances is compressed to the 'natural volume', it becomes another more massive resonance. This process then repeats, creating heavier resonances, which consist of hadron resonances, which in turn consist of resonances, and so on. *Picture: CERN Courier September 2003 p.30*.

nism generating the numerous hadronic resonances which dominate the scattering cross-section. He proposed the statistical bootstrap model (SBM).

In a nutshell, in the SBM, each of the many resonant states into which hadrons can be excited through a collision is itself a constituent of a still heavier resonance, whilst also being composed of lighter ones. In this way, when compressed to its natural volume, a matter cluster consisting of hadron resonances becomes a more massive resonance with lighter resonances as constituents, as shown in Fig. 6.1. One day in late 1964, one of us (TE) ran into Hagedorn: this must have been soon after he had invented the statistical bootstrap. He gave the impression of a man who had just found the famous philosopher's stone, describing how fireballs turn into fireballs forever and all in a logically consistent way. Visibly, Hagedorn was aware of the importance his ideas. It was interesting to observe how deeply he felt about it from the very beginning.

Using the SBM approach for a strongly interacting system, Hagedorn obtained an exponentially rising mass spectrum of resonant states. As time progresses and new data emerges, experimental results on hadronic level counting reveal ever greater number of cataloged resonances. They agree beautifully with theoretical expectations from the SBM. As our knowledge has increased, the observed mass spectrum has become a better exponential, as illustrated in Fig. 6.2. The solid blue line in Fig. 6.2 is the exponential fit to the smoothed hadron mass spectrum of the present day, which is represented by the short-dashed red line. Note that Hagedorn's long-dashed green line of 1967 was already a remarkably good exponential. One can imagine that the remaining deviation at high mass in the top right corner of the

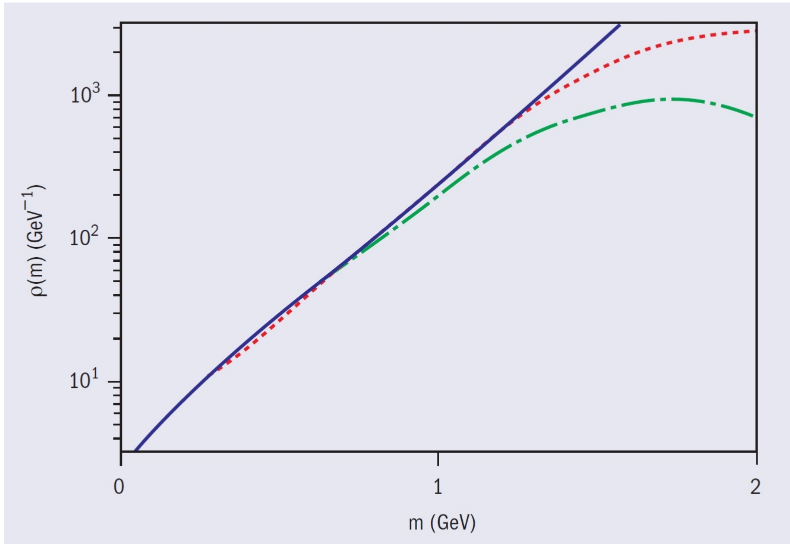


Fig. 6.2 The smoothed mass spectrum of hadronic states as a function of mass. Experimental data: *long-dashed green line* with the 1411 states known in 1967; *short-dashed red line* with the 4627 states of mid '90s. The *solid blue line* represents the exponential fit yielding $T_{\mathrm{H}} = 158$ MeV. Depending on the preexponential factor, a range $T_{\mathrm{H}} = 150 \pm 15$ MeV is possible. *Picture: CERN Courier September 2003 p.30.*

figure originates in the experimental difficulties of discovering all these high mass states, which have a much less obvious experimental signature.

The important physics message of Fig. 6.2 is that the rising slope in the mass spectrum is the same as the falling slope of the particle momentum spectra. The momentum spectra originate in the thermalization process and thus in reaction dynamics; the mass spectrum is an elementary property of strong interactions. The SBM provides an explanation of the relationship between these slopes, and explains why the hadron gas temperature is bounded from above. Moreover, since the smallest building block of all hadronic resonances is the pion, within the SBM one can also understand why the limiting temperature is of the same magnitude as the smallest hadron mass $T_{\mathrm{H}} \approx m_{\pi}$.

As time has passed since the discovery of the limiting temperature, this Hagedorn temperature T_{H} turned into a brand name. The concept of an exponentially rising mass spectrum is part of our understanding of hadron phenomena. However, considering the historical perspective, when first proposed the SBM was viewed with considerable skepticism, even within the CERN Theory Division where Hagedorn worked. Over the years, the understanding of the particle-production process that Hagedorn brought about has grown in significance and his work has become the standard model. Such is the sign of truly original work, of something that really influenced our thinking. Hagedorn's refereed article[2] presented 50 years ago for the first time and which introduced the statistical bootstrap model of particle production and placed the maximum temperature in the vocabulary of particle physics, has found a place among the most cited physics papers.

The accurate description of particle production, through the conversion of energy into matter, has numerous practical implications. Even in the very early days, Hagedorn's insight into the yields and spectra of the produced secondaries showed that neutrino beams would have sufficient flux to allow a fruitful experimental program, and this gave a theoretical basis for the planning of the first neutrino beams constructed at CERN.

6.3 Quark–Gluon Plasma

At the same time that the SBM was being developed, the newly discovered quarks were gaining acceptance as the building blocks of hadrons. While Hagedorn saw a compressed gas of hadrons as another hadron, in the quark picture it became a drop of quark matter. In quark matter at high temperatures gluons should also be present and as the temperature is increased asymptotic freedom ensures that all constituents interact relatively weakly. There seems to be nothing to stop a dense assembly of hadrons from deconfining into a plasma of quarks and gluons. It also seems that this new state of matter could be heated to a very high temperature, with no limit in sight. So what is the meaning of the Hagedorn temperature in this context?

In the SBM as conceived before quarks, hadrons were point particles. A subtle modification is required when considering quarks as building blocks. Hadrons made of quarks need a finite volume that grows with hadron mass. One of us (JR), upon his arrival at CERN at the end of the 1970s and into the early 1980s, worked on this extension of the SBM with Hagedorn. Much of this work is reported in this volume. We discovered that at the Hagedorn temperature, finite-size hadrons dissolve into a quark-gluon liquid. Both a phase transition and a smoother transformation are possible, depending on the precise nature of the mass spectrum.

The most physically attractive alternative was a first-order phase transition. In this case the latent heat is delivered to the hadron phase at a constant Hagedorn temperature T_{H}. A new phase is then reached wherein the hadron constituents – the quarks and the gluons – are no longer confined. The system temperature can now rise again. The presence of a true phase transition including its mathematical properties turned out to be of no deeper relevance to this concept, as long as the actual physical properties of the system change according to the model described. Therefore we chose to speak of 'transformation' of strongly interacting phases of matter.

Within the study of hot hadronic matter today, the Hagedorn temperature is understood as the phase boundary temperature between the hadron gas phase and the

[2] R. Hagedorn, Nuovo Cim. Suppl. **3** 147 (1965)

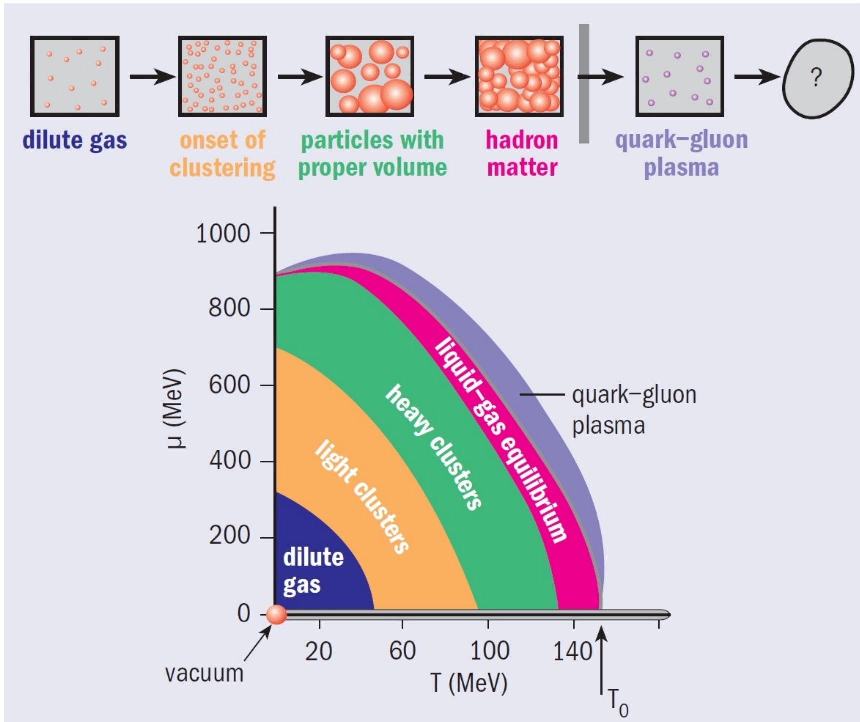


Fig. 6.3 The different regions of the statistical parameter plane (temperature, T, and the baryochemical potential μ), according to the statistical bootstrap model of hadronic matter. *Picture: CERN Courier September 2003 p.30*.

deconfined state of mobile quarks and gluons (see Fig. 6.3). Several experiments involving high-energy nuclear collisions at CERN's Super Proton Synchrotron (SPS), at the Relativistic Heavy Ion Collider (RHIC) at the Brookhaven National Laboratory, and at CERN's Large Hadron Collider (LHC) are testing these new concepts. Nuclei, rather than protons, are used in these experiments in order to maximize the volume of quark deconfinement. This allows a clearer study of the signature of the formation of a new phase of matter, the quark–gluon plasma (QGP).

In past years both CERN and RHIC communities have presented clear evidence for the formation of the deconfined QGP state in which the hadron constituents are dissolved. The current experimental objective is the understanding of the physical properties of this new phase of matter. This requires the use of novel probes, which respond to a change in the nature of the state within the short time available. More precisely, the heating of hadronic matter beyond the Hagedorn temperature is accompanied by a large collision compression pressure, which is the same in magnitude as the pressure in the very early Universe. In the subsequent expansion, a collective flow velocity as large as 60% of the velocity of light is exceeded. The expansion of dense QGP phase occurs on a timescale similar to that needed for light to traverse the interacting nuclei.

In the expansion–cooling process of QGP formed in nuclear collisions, the Hagedorn temperature at which hadrons emerge is again reached after a time that corresponds to the lifespan of a short-lived hadron. A break up – that is, hadronization – then occurs and final-state hadrons emerge. Hagedorn was particularly interested in understanding the hadronic probes of QGP produced in hadronization. He followed closely the initial exploration of the strangeness flavor as a signal of QGP formation.

Looking back, already in February 2000 the totality of intriguing experimental results obtained at the SPS over several years was folded into a public announcement stating that the formation of a new phase of matter was their best explanation[3]. More on this subject is said in in this volume. The key experimental results, including, in particular, strangeness and strange antibaryon enhancement, agreed with the theoretical expectations that were arrived at when one assumes that the QGP state was formed. To this day these signatures are the cornerstone of the QGP discovery.

Other signatures of QGP have been since detected. For example, over the past decade it became evident that this deconfined phase of matter is highly non-transparent to fast quarks. The majority of researchers today are convinced that the deconfined phase has been formed at the SPS, at RHIC and at the LHC. The thrust of current research depends on the range of accessible relativistic heavy ion beams. At lower energy, at SPS, and in special effort at RHIC, research addresses threshold conditions that are necessary for the onset of QGP, and the study of the phase boundary as function of baryon density. At the LHC we seek to understand the initial reaction conditions in dense matter and QGP in conditions similar to those present in the early universe is studied.

In the next few years, the study of hadronic matter near the Hagedorn temperature will also dominate experimental efforts in the field of nuclear collisions, in particular at the new international experimental facility FAIR under construction today at the GSI laboratory in Darmstadt, Germany and at the planned experimental facility NICA in Dubna, Russia. The richness of the physics at hand over the coming years is illustrated in the phase diagram in Fig. 6.3, which was obtained from the study of the SBM. Here, the domain is spanned by the temperature, T, and the baryochemical potential, μ, which regulates the baryon density.

In past 50 years the understanding of the physics related to the Hagedorn temperature has changed. In the beginning it was the maximum temperature seen in proton–proton collisions. It then became the SBM inverse slope of the mass spectrum. Today, it denotes the phase boundary between hadron and quark matter. Moreover, as recent work in string theory has shown, Rolf Hagedorn (Fig. 6.4) will not only be remembered for the physics of hot hadronic matter: his name is already attached to a more general family of elementary phenomena that originate in the methods he developed in the study of strong interaction physics.

---

[3] CERN Courier April 2000: *Harping on About Hadrons*; CERN Courier May 2000: *Opening the Door to the Quark-Gluon Plasma*

Fig. 6.4 Rolf Hagedorn in his garden Fall 1978; *Photo: Johann Rafelski*.

Chapter 7
The Legacy of Rolf Hagedorn: Statistical Bootstrap and Ultimate Temperature

Krzysztof Redlich and Helmut Satz

Abstract In the latter half of the last century, it became evident that there exists an ever increasing number of different states of the so-called elementary particles. The usual reductionist approach to this problem was to search for a simpler infrastructure, culminating in the formulation of the quark model and quantum chromodynamics. In a complementary, completely novel approach, Hagedorn suggested that the mass distribution of the produced particles follows a self-similar composition pattern, predicting an unbounded number of states of increasing mass. He then concluded that such a growth would lead to a limiting temperature for strongly interacting matter. We discuss the conceptual basis for this approach, its relation to critical behavior, and its subsequent applications in different areas of high energy physics.

A prophet is not without honour,
but in his own country.
The New Testament, Mark 6,4.

7.1 Rolf Hagedorn

The development of physics is the achievement of physicists, of humans, persisting against often considerable odds. Even in physics, fashion rather than fact frequently determines judgment and recognition.

When Rolf Hagedorn (Fig. 7.1) carried out his main work, now quite generally recognized as truly pioneering, much of the theoretical community not only ignored it, but even considered it to be nonsense. “Hagedorn ist ein Narr”, he is a fool, was a summary of many leading German theorists of his time. When in the 1990s the

---

KR: Institute of Theoretical Physics, University Wroclaw, Poland
HS: Fakultät fr̈ Physik, Universität Bielefeld, Germany

Fig. 7.1 Rolf Hagedorn (on right) in conversation with Helmut Satz (on left), 30 June 1994. Hagedorn holds a birthday gift from Krzysztof Redlich. *Image credit: CERN Image 1994-06-64-022.*

question was brought up whether he could be proposed for the Max Planck Medal, the highest honor of the German physics community, even then, when his achievements were already known world-wide, the answer was still "proposed, yes. . . ."

At the time Hagedorn carried out his seminal research, much of theoretical physics was ideologically fixed on "causality, unitarity, Poincaré invariance": from these three concepts, from axiomatic quantum field theory, all that is relevant to physics must arise. Those who thought that science should progress instead by comparison to experiment were derogated as "fitters and plotters". Galileo was almost forgotten. . . . Nevertheless, one of the great Austrian theorists of the time, Walter Thirring, himself probably closer to the fundamentalists, noted: "If you want to do something really *new*, you first have to have a *new idea*". Hagedorn did.

He had a number of odds to overcome. He had studied physics in Göttingen under Richard Becker, where he developed a life-long love for thermodynamics. When he took a position at CERN, shortly after completing his doctorate, it was to perform calculations for the planning and construction of the proton synchrotron. When that was finished, he shifted to the study of multihadron production in proton-proton collisions and to modeling the results of these reactions. It took a while before various members of the community, including some of the CERN Theory Division, were willing to accept the significance of his work. This was not made easier by Hagedorn's strongly focused region of interest, but eventually it became generally recognized that here was someone who, in this perhaps similar to John Bell, was developing truly novel ideas which at first sight seemed quite specific, but which eventually turned out to have a lasting impact also on physics well outside its region of origin.

We find that Rolf Hagedorn's work centers on two themes:

- the statistical bootstrap model, a self-similar scheme for the composition and decay of hadrons and their resonances; for Hagedorn, these were the "fireballs".
- the application of the resulting resonance spectrum in an ideal gas containing all possible hadrons and hadron resonances, and to the construction of hadron production models based on such a thermal input.

We will address these topics in the first two sections, and then turn to their roles both in the thermodynamics of strongly interacting matter and in the description of hadron production in elementary as well as nuclear collisions. Our aim here is to provide a general overview of Hagedorn's scientific achievements. Some of what we will say transcends Hagedorn's life. But then, to paraphrase Shakespeare, we have come to praise Hagedorn, not to bury him; we want to show that his ideas are still important and very much alive.

7.2 The Statistical Bootstrap

Around 1950, the physics world still seemed in order for those looking for the ultimate constituents of matter in the universe. Dalton's atoms had been found to be not really *atomos*, indivisible; Rutherford's model of the atom had made them little planetary systems, with the nucleus as the sun and the electrons as encircling planets. The nuclei in turn consisted of positively charged protons and neutral neutrons as the essential mass carriers. With an equal number of protons and electrons, the resulting atoms were electrically neutral, and the states obtained by considering the different possible nucleus compositions reproduced the periodic table of elements. So for a short time, the Greek dream of obtaining the entire complex world by combining three simple *elementary* particles in different ways seemed finally feasible: protons, neutrons and electrons were the building blocks of our universe.

But there were those who rediscovered an old problem, first formulated by the Roman philosopher Lucretius: if your elementary particles, in our case the protons and neutrons, have a size and a mass, as both evidently did, it was natural to ask what they are made of. An obvious way to find out is to hit them against each other and look at the pieces. And it turned out that there were lots of fragments, the more, the harder the collision. But they were not really pieces, since the debris found after a proton-proton collision still also contained the two initial protons. Moreover, the additional fragments, mesons and baryons, were in almost all ways as elementary as protons and neutrons. The study of such collisions was taken up by more and more laboratories and at ever higher collision energies. As a consequence, the number of different "elementary" particles grew by leaps and bounds, from tens to twenties to hundreds. The latest compilation of the Particle Data Group contains over a thousand.

Let us, however, return to the time when physics was confronted by all those elementary particles, challenging its practitioners to find a way out. At this point, in the mid 1960s, Rolf Hagedorn came up with a truly novel idea [1–6]. He was not so much worried about the specific properties of the particles. He just imagined that a

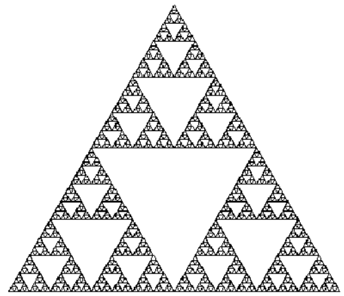


Fig. 7.2 The Sierpinski Triangle

heavy particle was somehow composed out of lighter ones, and these again in turn of still lighter ones, and so on, until one reached the pion as the lightest hadron. And by combining heavy ones, you would get still heavier ones, again: and so on. The crucial input was that the composition law should be the same at each stage. Today we call that self-similarity, and it had been around in various forms for many years. A particularly elegant formulation was written a hundred years before Hagedorn by the English mathematician Augustus de Morgan, the first president of the London Mathematical Society:

Great fleas have little fleas upon their backs to bite'em,
and little fleas have lesser still, and so ad infinitum.
And the great fleas themselves, in turn, have greater fleas to go on,
while these again have greater still, and greater still, and so on.

Hagedorn proposed that "a fireball consists of fireballs, which in turn consist of fireballs, and so on...." The concept later reappeared in various forms in geometry; in 1915, it led to the celebrated triangle, see Fig. 7.2 devised by the Polish mathematician Wacław Sierpinski: " a triangle consists of triangles, which in turn consist of triangles, and so on...," in the words of Hagedorn. Still later, shortly after Hagedorn's proposal, the French mathematician Benoit Mandelbroit initiated the study of such *fractal behavior* as a new field of mathematics.

Hagedorn had recalled a similar problem in number theory: how many ways are there of decomposing an integer into integers? This was something already addressed in 1753 by Leonhard Euler, and more than a century later by the mathematician E. Schröder in Germany. Finally G. H. Hardy and S. Ramanujan in England provided an asymptotic solution [15]. Let us here, however, consider a simplified, easily solvable version of the problem [7], in which we count all possible different ordered arrangements $p(n)$ of an integer n. So we have

1=1	$p(1) = 1 = 2^{n-1}$
2=2 ,1+1	$p(2) = 2 = 2^{n-1}$
3= 3, 2+1, 1+2, 1+1+1	$p(3) = 4 = 2^{n-1}$
4= 4, 3+1, 1+3, 2+2, 2+1+1, 1+2+1, 1+1+2, 1+1+1+1	$p(4) = 8 = 2^{n-1}$

and so on. In other words, there are

$$p(n) = 2^{n-1} = \frac{1}{2}\, e^{n \ln 2} \tag{7.1}$$

ways of partitioning an integer n into ordered partitions: $p(n)$ grows exponentially in n. In this particular case, the solution could be found simply by induction. But there is another way of reaching it, more in line with Hagedorn's thinking: "large integers consist of smaller integers, which in turn consist of still smaller integers, and so on...." This can be formulated as an equation,

$$\rho(n) = \delta(n-1) + \sum_{k=2}^{n} \frac{1}{k!} \prod_{i=1}^{k} \rho(n_i)\ \delta(\Sigma_i n_i - n). \tag{7.2}$$

It is quite evident here that the form of the partition number $\rho(n)$ is determined by a convolution of many similar partitions of smaller n. The solution of the equation is in fact just the number of partitions of n that we had obtained above,

$$\rho(n) = z\ p(n) \tag{7.3}$$

up to a normalization constant of order unity (for the present case, it turns out that $z \simeq 1.25$). For Hagedorn, Eq. (7.2) expressed the idea that the structure of $\rho(n)$ was determined by the structure of $\rho(n)$ – we now call this self-similar. He instead thought of the legendary Baron von Münchhausen, who had extracted himself from a swamp by pulling on his own bootstraps. So for him, Eq. (7.2) became his *bootstrap equation*.

The problem Hagedorn had in mind was, of course, considerably more complex. His heavy resonance was not simply a sum of lighter ones at rest, but it was a system of lighter resonances in motion, with the requirement that the total energy of this system added up to the mass of the heavy one. And similarly, the masses of the lighter ones were the result of still lighter ones in motion. The bootstrap equation for such a situation becomes

$$\rho(m, V_0) = \delta(m - m_0)\ + \sum_N \frac{1}{N!} \left[\frac{V_0}{(2\pi)^3}\right]^{N-1} \int \prod_{i=1}^{N} [dm_i\ \rho(m_i)\ d^3 p_i]\ \delta^4(\Sigma_i p_i - p), \tag{7.4}$$

where the first term corresponds to the case of just one lightest possible particle, a "pion". The factor V_0, the so-called composition volume, specified the size of the overall system, an intrinsic fireball size. Since the mass of any resonance in the composition chain is thus determined by the sum over phase spaces containing lighter ones, whose mass is specified in the same way, Hagedorn called this form of bootstrap "statistical".

After a number of numerical attempts by others, W. Nahm [8] solved the statistical bootstrap equation analytically, obtaining

$$\rho(m, V_0) = \text{const.}\ m^{-3} \exp\{m/T_{\rm H}\}. \tag{7.5}$$

So even though the partitioning now was not just additive in masses, but included the kinetic energy of the moving constituents, the increase was again exponential in mass. The coefficient of the increase, ${T_{\rm H}}^{-1}$, is determined by the equation

$$\frac{V_0 {T_{\rm H}}^3}{2\pi^2}(m_0/T_{\rm H})^2 K_2(m_0/T_{\rm H}) = 2\ln 2 - 1, \tag{7.6}$$

in terms of two parameters V_0 and m_0. Hagedorn assumed that the composition volume V_0, specifying the intrinsic range of strong interactions, was determined by the inverse pion mass as scale, $V_0 \simeq (4\pi/3)m_\pi^{-3}$. This leads to a scale factor $T_{\rm H} \simeq 150$ MeV. It should be emphasized, however, that this is just one possible way to proceed. In the limit $m_0 \to 0$, Eq. (7.6) gives

$$T_{\rm H} = [\pi^2(2\ln 2 - 1)]^{1/3}\, V_0^{-1/3} \simeq 1/r_{\rm h}, \tag{7.7}$$

where $V_0 = (4\pi/3)r_{\rm h}^3$ and $r_{\rm h}$ denotes the range of strong interactions. With $r_{\rm h} \simeq 1$ fm, we thus have $T_{\rm H} \simeq 200$ MeV. From this it is evident that the exponential increase persists also in the chiral limit $m_\pi \to 0$ and is in fact only weakly dependent on m_0, provided the strong interaction scale V_0 is kept fixed.

The weights $\rho(m)$ determine the composition as well as the decay of "resonances", of fireballs. The basis of the entire formalism, the self-similarity postulate – here in the form of the statistical bootstrap condition – results in an unending sequence of ever-heavier fireballs and in an exponentially growing number of different states of a given mass m.

Before we turn to the implications of such a pattern in thermodynamics, we note that not long after Hagedorn's seminal paper, it was found that a rather different approach, the dual resonance model [9–11], see Chapter 8, led to very much the same exponential increase in the number of states. In this model, any scattering amplitude, from an initial two to a final hadrons, was assumed to be determined by the resonance poles in the different kinematic channels. This resulted structurally again in a partition problem of the same type, and again the solution was that the number of possible resonance states of mass m must grow exponentially in m, with an inverse scale factor of the same size as obtained above, some 200 MeV. Needless to say, this unexpected support from the forefront of theoretical hadron dynamics considerably enhanced the interest in Hagedorn's work.

7.3 The Limiting Temperature of Hadronic Matter

Consider a relativistic ideal gas of identical neutral scalar particles of mass m_0 contained in a box of volume V, assuming Boltzmann statistics. The grand canonical partition function of this system is given by

$$\mathscr{Z}(T,V) = \sum_N \frac{1}{N!}\left[\frac{V}{(2\pi)^3}\int d^3p\ \exp\{-\sqrt{p^2+m_0^2}\ /T\}\right]^N, \tag{7.8}$$

leading to

$$\ln \mathscr{Z}(T,V) = \frac{VTm_0^2}{2\pi^2}\, K_2(m_0/T). \tag{7.9}$$

For temperatures $T \gg m_0$, the energy, and the particle density of the system become, respectively

$$\varepsilon(T) = -\frac{1}{V}\,\frac{\partial \ln \mathscr{Z}(T,V)}{\partial\,(1/T)} \simeq \frac{3}{\pi^2}\,T^4, \quad n(T) = \frac{\partial \ln \mathscr{Z}(T,V)}{\partial V} \simeq \frac{1}{\pi^2}\,T^3, \tag{7.10}$$

and so the average energy per particle is given by

$$\frac{E}{N} \simeq 3\,T. \tag{7.11}$$

The important feature to learn from these relations is that, in the case of an ideal gas of one species of elementary particles, an increase of the energy of the system has three consequences. It leads to:

- a higher temperature,
- more constituents, and
- more energetic constituents.

If we now consider an *interacting* gas of such basic hadrons and postulate that the essential form of the interaction is resonance formation, then we can approximate the interacting medium as a non-interacting gas of all possible resonance species [12, 13]. The partition function of this resonance gas is

$$\ln \mathscr{Z}(T,V) = \sum_i \frac{VTm_i^2}{2\pi^2}\,\rho(m_i)\,K_2(m_i/T) \tag{7.12}$$

where the sum begins with the stable ground state m_0 and then includes the possible resonances $m_i, i = 1,2,\ldots$ with weights $\rho(m_i)$ relative to m_0. Clearly the crucial question here is how to specify $\rho(m_i)$, that is how many states there are of mass m_i. It is only at this point that hadron dynamics enters, and it is here that Hagedorn introduced the result obtained in his statistical bootstrap model.

As we had seen above in Eq. (7.5), the density of states then increases exponentially in m, with a coefficient ${T_{\rm H}}^{-1}$ determined by Eq. (7.6) in terms of two parameters V_0 and m_0. If we replace the sum in the resonance gas partition function Eq. (7.12) by an integral and insert the exponentially growing mass spectrum Eq. (7.5), Eq. (7.12) becomes

$$\ln \mathscr{Z}(T,V) \simeq \frac{VT}{2\pi^2}\int dm\, m^2 \rho(m_i)\,K_2(m_i/T) \sim V\left[\frac{T}{2\pi}\right]^{3/2}\int \frac{dm}{m^{3/2}}\,e^{-\left[\frac{m}{T}-\frac{m}{T_{\rm H}}\right]}. \tag{7.13}$$

Evidently, the result is divergent for all $T > T_{\rm H}$: in other words, $T_{\rm H}$ is the highest possible temperature of hadronic matter. Moreover, if we compare such a system with the ideal gas of only basic particles (a "pion" gas), we find:

pion gas	resonance gas
$n_\pi \sim \varepsilon^{3/4}$	$n_{res} \sim \varepsilon$
$\omega_\pi \sim \varepsilon^{1/4}$	$\omega_{res} \sim$ const.

Here n denotes the average number density of constituents, ω the average energy of a constituent. In contrast to to the pion gas, an increase of energy now leads to

- a fixed temperature limit, $T \to T_{\mathrm{H}}$,
- the momenta of the constituents do not continue to increase, and
- more and more species of ever heavier particles appear.

We thus obtain a new, non-kinetic way to use energy, increasing the number of species and their masses, not the momentum per particle. Temperature is a measure of the momentum of the constituents, and if that cannot continue to increase, there is a highest possible, a "limiting" temperature for hadronic systems.

Hagedorn originally interpreted T_{H} as the ultimate temperature of strongly interacting matter. It is clear today that T_{H} signals the transition from hadronic matter to a quark-gluon plasma. Hadron physics alone can only specify its inherent limit; to go beyond this limit, we need more information: we need QCD.

As seen in Eq. (7.5), the solution of the statistical bootstrap equation has the general form

$$\rho(m, V_0) \sim m^{-a} \exp(m/T_{\mathrm{H}}), \tag{7.14}$$

with some constant a; the exact solution of Eq. (7.4) by Nahm gave $a = 3$. It is possible, however, to consider variations of the bootstrap model which lead to different a, but always retain the exponential increase in m. While the exponential form makes T_{H} the upper limit of permissible temperatures, the power law coefficient a determines the behavior of the system at $T = T_{\mathrm{H}}$. For $a = 3$, the partition function Eq. (7.13) itself exists at that point, while the energy density as first derivative in temperature diverges there. This is what made Hagedorn conclude that T_{H} is indeed the highest possible temperature of matter: it would require an infinite energy to reach it.

Only a few years later it was, however, pointed out by N. Cabibbo and G. Parisi [14] that larger a shifted the divergence at $T = T_{\mathrm{H}}$ to ever higher derivatives. In particular, for $4 > a > 3$, the energy density would remain finite at that point, shifting the divergence to the specific heat as next higher derivative. Such critical behavior was in fact quite conventional in thermodynamics: it signaled a phase transition leading to the onset of a new state of matter. By that time, the quark model and quantum chromodynamics as fundamental theory of strong interactions had appeared and suggested the existence of a quark-gluon plasma as the relevant state of matter at extreme temperature or density. It was therefore natural to interpret the Hagedorn temperature T_{H} as the critical transition temperature from hadronic matter to such a plasma. This interpretation is moreover corroborated by a calculation of the critical exponents [16] governing the singular behavior of the resonance gas thermodynamics based on a spectrum of the form Eq. (7.14).

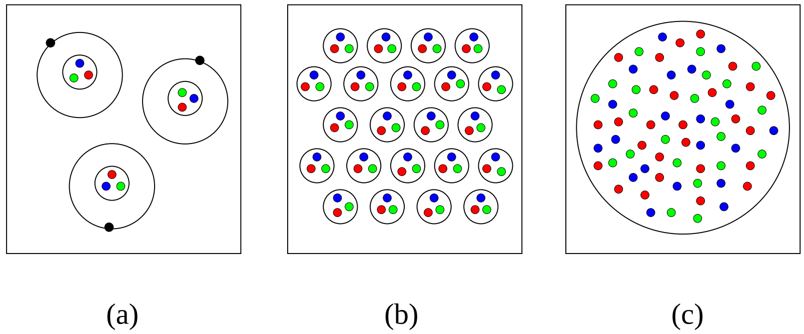


Fig. 7.3 Schematic view of matter for increasing density, from atomic (a) to nuclear (b) and then to quark matter (c).

It should be noted, however, that in some sense $T_{\rm H}$ did remain the highest possible temperature of matter as we know it. Our matter exists in the physical vacuum and is constructed out of fundamental building blocks which in turn have an independent existence in this vacuum. Our matter ultimately consists of and can be broken up into nucleons; we can isolate and study a single nucleon. The quark-gluon plasma, on the other hand, has its own ground state, distinct from the physical vacuum, and its constituents can exist only in a dense medium of other quarks – we can never isolate and study a single quark.

That does not mean, however, that quarks are eternally confined to a given part of space. Let us start with atomic matter and compress that to form nuclear matter, as it exists in heavy nuclei. At this stage, we have nucleons existing in the physical vacuum. Each nucleon consists of three quarks, and they are confined to remain close to each other; there is no way to break up a given nucleon into its quark constituents. But if we continue to compress, then eventually the nucleons will penetrate each other, until we reach a dense medium of quarks. Now each quark finds in its immediate neighborhood many other quarks besides those which were with it in the nucleon stage. It is therefore no longer possible to partition quarks into nucleons; the medium consists of unbound quarks, whose interaction becomes ever weaker with increasing density, approaching the limit of asymptotic freedom predicted by QCD. Any quark can now move freely throughout the medium: we have quark liberation through swarm formation. Wherever a quark goes, there are many other quarks nearby. The transition from atomic to quark matter is schematically illustrated in Fig. 7.3.

We have here considered quark matter formation through the compression of cold nuclear matter. A similar effect is obtained if we heat a meson gas; with increasing temperature, collisions and pair production lead to an ever denser medium of mesons. And according to Hagedorn, also of ever heavier mesons of an increasing degeneracy. For Hagedorn, the fireballs were point like, so that the overlap we had just noted simply does not occur. In the real world, however, they do have hadronic size, so that they will in fact interpenetrate and overlap before the divergence of the Hagedorn resonance gas occurs [17]. Hence now again there will be a transition from resonance gas to a quark-gluon plasma, now formed by the liberation of the quarks and gluons making up the resonances.

At this point, it seems worthwhile to note an even earlier approach leading to a limiting temperature for hadronic matter. More than a decade before Hagedorn, I. Ya. Pomeranchuk [18] pointed out that a crucial feature of hadrons is their size, and hence the density of any hadronic medium is limited by volume restriction: each hadron must have its own volume to exist, and once the density reaches the dense packing limit, it's the end for hadronic matter. This simply led to a temperature limit, and for an ideal gas of pions of 1 fm radius, the resulting temperature was again around 200 MeV. Nevertheless, these early results remained largely unnoticed until the work of Hagedorn.

Such geometric considerations do, however, lead even further. If hadrons are allowed to interpenetrate, to overlap, then percolation theory predicts two different states of matter [19, 20]: hadronic matter, consisting of isolated hadrons or finite hadronic clusters, and a medium formed as an infinite sized cluster of overlapping hadrons. The transition from one to the other now becomes a genuine critical phenomenon, occurring at a critical value of the hadron density.

We thus conclude that the pioneering work of Rolf Hagedorn opened up the field of critical behavior in strong interaction physics, a field in which still today much is determined by his ideas. On a more theoretical level, the continuation of such studies was provided by finite temperature lattice QCD, and on the more experimental side, by resonance gas analysis of the hadron abundances in high energy collisions. In both cases, it was found that the observed behavior was essentially that predicted by Hagedorn's ideas.

7.4 Resonance Gas & QCD Thermodynamics

With the formulation of Quantum Chromodynamics (QCD) as a theoretical framework for the strong interaction force among elementary particles it became clear that the appearance of the ultimate Hagedorn's temperature T_{H}, signals indeed the transition from the hadronic phase to a new phase of strongly interacting matter, the quark-gluon plasma (QGP) [21]. As QCD is an asymptotically free theory, the interaction between quarks and gluons vanishes logarithmically with increasing temperature, thus at very high temperatures the QGP effectively behaves like an ideal gas of quarks and gluons.

Today we have detailed information, obtained from numerical calculations in the framework of finite temperature lattice Quantum Chromodynamics [22, 23], about the thermodynamics of hot and dense matter. We know the transition temperature to the QGP and the temperature dependence of basic bulk thermodynamic observables such as the energy density and the pressure [24, 25]. We also begin to have results on fluctuations and correlations of conserved charges [26–28].

The recent increase in numerical accuracy of lattice QCD calculations and their extrapolation to the continuum limit, makes it possible to confront the fundamental results of QCD with Hagedorn's concepts [2,6], which provide a theoretical scenario for the thermodynamics of strongly interacting hadronic matter [28–30].

In particular, the equation of state calculated on the lattice at vanishing and finite chemical potential, and restricted to the confined hadronic phase, can be directly compared to that obtained from the partition function Eq. (7.13) of the hadron resonance gas, using the form Eq. (7.14) introduced by Hagedorn for a continuum mass spectrum. Alternatively, as a first approximation, one can also consider a discrete mass spectrum which accounts for all experimentally known hadrons and resonances. In this case the continuum partition function of the Hagedorn model is expressed by Eq. (7.12) with $\rho(m_i)$ replaced by the spin degeneracy factor of the i^{th} hadron, with the summation taken over all known resonance species listed by the Particle Data Group [31].

With the above assumption on the dynamics and the mass spectrum, the resonance gas partition function introduced by Hagedorn [2, 6], can be calculated exactly and expressed as a sum of one–particle partition functions Z_i^1 of all hadrons and resonances,

$$\ln Z(T,V) = \sum_i Z_i^1(T,V). \tag{7.15}$$

For particles of mass m_i and spin degeneracy factor g_i, the one–particle partition function Z_i^1, in the Boltzmann approximation, reads

$$Z_i^1(T,V) = g_i \frac{VTm_i^2}{2\pi^2} K_2(m_i/T). \tag{7.16}$$

Due to the factorization of the partition function in Eq. (7.15), the energy density and the pressure of the Hagedorn resonance gas with a discrete mass spectrum, can also be expressed as a sum over single particle contributions

$$\varepsilon = \sum_i \varepsilon_i^1, \qquad P = \sum_i P_i^1, \tag{7.17}$$

with

$$\frac{\varepsilon_i^1}{T^4} = \frac{g_i}{2\pi^2} \left(\frac{m_i}{T}\right)^3 \left[\frac{3\,K_2(\beta m_i)}{\beta m} + K_1(\beta m_i)\right] \tag{7.18}$$

$$\frac{P_i^1}{T^4} = \frac{g_i}{2\pi^2} \left(\frac{m_i}{T}\right)^3 K_2(\beta m_i), \tag{7.19}$$

where $\beta = 1/T$ and K_1 and K_2 are modified Bessel functions. At vanishing chemical potentials and at finite temperature, the energy density ε, the entropy density s and the pressure P, are connected through the thermodynamic relation,

$$\varepsilon = -P + sT. \tag{7.20}$$

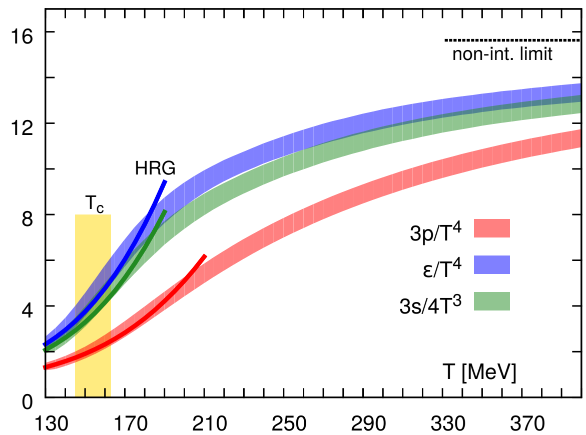


Fig. 7.4 The normalized pressure $P(T)$, the energy density $\varepsilon(T)$ and the entropy density $s(T)$ obtained in lattice QCD calculations as a function of temperature. The dark lines show predictions of the Hagedorn resonance gas for a discrete mass spectrum, Eqs. (7.17)–(7.20). The lattice results are from Ref. [25].

Summing up in Eq. (7.17), the contributions from experimentally known hadronic states; constitute the resonance gas [2,6,32] for the thermodynamics of the hadronic phase of QCD. Taking e.g. contribution of all mesonic and baryonic resonances with masses up to 1.8 GeV and 2.0 GeV, respectively, amounts to 1026 resonances.

The crucial question thus is, if the equation of state of hadronic matter introduced by Hagedorn can describe the corresponding results obtained from QCD within lattice approach.

In Fig. 7.4 we show the temperature dependence of the energy density, pressure and the entropy density obtained recently in lattice QCD studies with physical masses of up, down and strange quarks [25]. The bands in lattice QCD results indicate error bars due to extrapolation to the continuum limit. The vertical band marks the temperature, $T_c = (154 \pm 9)$ MeV, which within an error, is the crossover temperature from a hadronic phase to a quark-gluon plasma [33]. These QCD results are compared in Fig. 7.4 to the Hagedorn resonance gas model formulated for a discrete mass spectrum in Eq. (7.17) and Eq. (7.20).

There is a clear coincidence of the Hagedorn resonance model results and the lattice data on the equation of states. All bulk thermodynamical observables are very strongly changing with temperature when approaching the deconfinement transition. This behavior is well understood in the Hagedorn model as being due to the contribution of resonances. Although Hagedorn's model formulated for a discrete mass spectrum does not exhibit a critical behavior, it nevertheless reproduces remarkably well the lattice results in the hadronic phase. This agreement has now been extended to an analysis of fluctuations and correlations of conserved charges as well.

In summary of this section we note that a remarkably good description of lattice QCD results on the equation of states by the Hagedorn thermal model *justifies, that resonances are indeed the essential degrees of freedom near deconfinement.* Thus, on the thermodynamical level, modeling hadronic interactions by formation and excitation of resonances, as introduced by Hagedorn, is an excellent approximation of strong interactions.

7.5 Resonance Gas & Heavy Ion Collisions

Long before lattice QCD could provide a direct evidence that strong interaction thermodynamics can be quantified by the resonance gas partition function, Hagedorn's concept was verified phenomenologically by considering particle production in elementary and heavy ion collisions [32, 34–37]. In a strongly interacting medium, one includes the conservation of electric charge, baryon number and strangeness. In this case, the partition function of Hagedorn's thermal model depends not only on temperature but also on chemical potential μ, which guarantees, that charges are conserved on an average. For a non vanishing μ, the partition function Eq. (7.15) is replaced by

$$\ln Z(T,V,\mu) = \sum_i Z_i^1(T,V,\mu), \tag{7.21}$$

with $\mu = (\mu_B, \mu_S, \mu_Q)$, where μ_i are the chemical potentials related to the baryon number, strangeness and electric charge conservation, respectively.

For particle i carrying strangeness S_i, the baryon number B_i, the electric charge Q_i and the spin–isospin degeneracy factor g_i, the one particle partition function, reads

$$Z_i^1(T,V,\mu) = \frac{V g_i T m_i^2}{2\pi^2} K_2(m_i/T) \exp\left(\frac{B_i\mu_B + S_i\mu_S + Q_i\mu_Q}{T}\right). \tag{7.22}$$

For $\mu = 0$ one recovers the result from Eq. (7.16).

The calculation of a density n_i of particle i from the partition function Eq. (7.21) is rather straightforward [47]. It amounts to the replacement $Z_i^1 \to \gamma_i Z_i^1$ in Eq. (7.21) and taking a derivative with respect to the particle fugacity γ_i, as

$$n_i = \frac{\langle N_i \rangle^{\text{th}}}{V} = \left.\frac{\partial \ln Z}{\partial \gamma_i}\right|_{\gamma_i=1}, \tag{7.23}$$

consequently, $n_i = Z_i^1/V$ with Z_i^1 as in Eq. (7.22).

The Hagedorn model, formulated in Eq. (7.21), describes bulk thermodynamic properties and particle composition of a thermal fireball at finite temperature and at non vanishing charge densities. If such a fireball is created in high energy heavy ion collisions, then yields of different hadron species are fully quantified by thermal parameters. However, following Hagedorn's idea, the contribution of resonances decaying into lighter particles, must be included [2, 6].

In Hagedorn's thermal model, the average number $\langle N_i \rangle$ of particles i in volume V and at temperature T that carries strangeness S_i, the baryon number B_i, and the

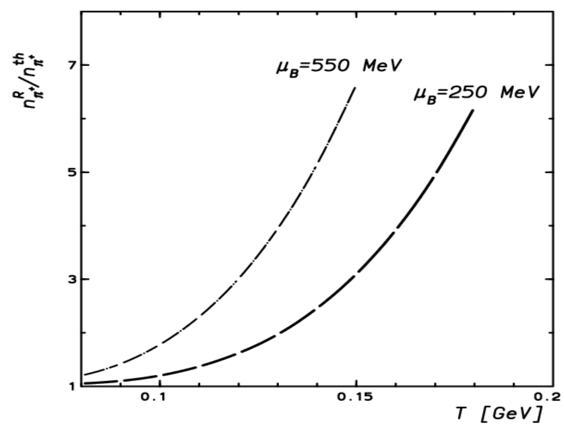


Fig. 7.5 The ratio of the total density of positively charged pions, $n^R_{\pi^+}$ from Eq. (7.24), and the density of thermal pions, $n^{\rm th}_{\pi^+}$ from Eq. (7.23). The calculations are done in the Hagedorn resonance gas model for μ_B =250 MeV and μ_B =550 MeV at different temperatures.

electric charge Q_i, is obtained from Eq. (7.21), see [2,6]

$$\langle N_i\rangle(T,\mu) \;=\; \langle N_i\rangle^{\rm th}(T,\mu)+\sum\nolimits_j \Gamma_{j\to i}\langle N_j\rangle^{th,R}(T,\mu). \tag{7.24}$$

The first term in Eq. (7.24) describes the thermal average number of particles of species i from Eq. (7.23) and the second term describes overall contribution from resonances. This term is taken as a sum of all resonances that decay into particle i. The $\Gamma_{j\to i}$ is the corresponding decay branching ratio of $j \to i$. The multiplicities of resonances $\langle N_j\rangle^{th,R}$ in Eq. (7.24), are obtained from Eq. (7.23).

The importance of resonance contributions to the total particle yield in Eq. (7.24) is illustrated in Fig. 7.5 for charge pions. In Fig. 7.5 we show the ratio of the total number of charge pions from Eq. (7.24) and the number of prompt pions from Eq. (7.23). The ratio is strongly increasing with temperature and chemical potential. This is due to an increasing contribution of mesonic and baryonic resonances. From Eq. (7.5) it is clear, that at high temperature and/or density, the overall multiplicity of pions is mostly due to resonance decays.

The particle yields in Hagedorn's model Eq. (7.24) depend, in general, on five parameters. However, in high energy heavy ion collisions, only three parameters are independent. In the initial state the isospin asymmetry, fixes the charge chemical potential and the strangeness neutrality condition eliminates the strange chemical potential. Thus, on the level of particle multiplicity, we are left with temperature T and the baryon chemical potential μ_B as independent parameters, as well as, with fireball volume as an overall normalization factor.

Hagedorn's thermal model introduced in Eq. (7.24) was successfully applied to describe particle yields measured in heavy ion collisions. The model was compared with available experimental data obtained in a broad energy range from AGS up to

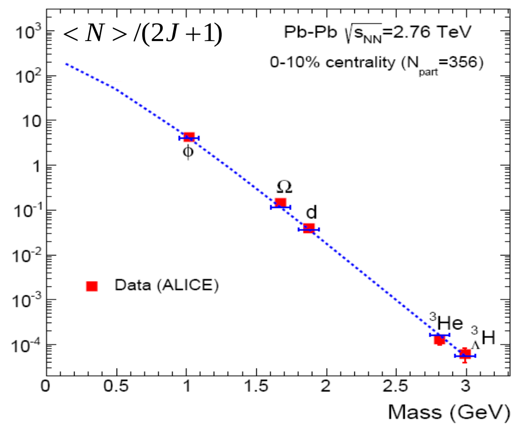


Fig. 7.6 Yields of several different particle species per unit rapidity normalized to spin degeneracy factor as a function of their mass. Data are from ALICE collaboration taken at the LHC in central Pb-Pb collisions. The line is the Hagedorn thermal model result, Eq. (7.25), see Ref. [38].

LHC. Hadron multiplicities ranging from pions to omega baryons and their ratios, as well as composite objects like e.g. deuteron or alpha particles, were used to verify if there was a set of thermal parameters (T, μ_B) and V, which simultaneously reproduces all measured yields.

The systematic studies of particle production extended over more than two decades, using experimental results at different beam energies, have revealed a clear justification, that in central heavy ion collisions particle yields are indeed consistent with the expectation of the Hagedorn thermal model. There is also a clear pattern of the energy, $\sqrt{s}$-dependence of thermal parameters. The temperature is increasing with $\sqrt{s}$, and at the SPS energy essentially saturates at the value, which corresponds to the transition temperature from a hadronic phase to a QGP, as obtained in LQCD. The chemical potential, on the other hand, is gradually decreasing with $\sqrt{s}$ and almost vanishes at the LHC.

In Fig. 7.6 we show, as an illustration, a comparison of Hagedorn's thermal model and recent data on selected particle yields, obtained by ALICE collaboration in central Pb-Pb collisions at midrapidity at the LHC energy [38]. At such high collision energy, particle yields from Eq. (7.24) are quantified entirely by the temperature and the fireball volume.[1] Thus, there is transparent prediction of Hagedorn's model Eq. (7.24), that yields of heavier particles $\langle N_i \rangle$ with no resonance decay contributions, normalized to their spin degeneracy factor $g_i = (2J+1)$, should be quantified by

$$\frac{\langle N_i \rangle}{2J+1} \simeq VT^3 \left(\frac{m_i}{2\pi T}\right)^{3/2} \exp(-m_i/T), \tag{7.25}$$

[1] The chemical potential μ in Eq. (7.24) vanishes, since at the LHC and at midrapidity particles and their antiparticles are produced symmetrically.

where we have used Eq. (7.23) and the asymptotic expansion of the Bessel function, $K_2(x) \sim x^{-1/2}\exp(-x)$, valid for large x.

In Fig. 7.6 we show the yields of particles with no resonance contribution, like ϕ, Ω, the deuteron 'd', ^{3}He and the hypertriton $^3_\Lambda$He, normalized to their spin degeneracy factor, as a function of particle mass. Also shown in this figure is the prediction from Eq. (7.25) at $T \simeq 156$ and for volume $V \simeq 5000$ fm^3 [38]. There is a clear coincidence of data taken in Pb-Pb collisions at the LHC and predictions of the Hagedorn model Eq. (7.25). Particles with no resonance contribution measured by ALICE collaboration follow the Hagedorn's expectations that they are produced from a thermal fireball at common temperature. A similar agreement of Hagedorn's thermal concept and experimental data taken in central heavy ion collisions has been found for different yields of measured particles and collision energies from AGS, SPS, RHIC and LHC [32].

7.6 Particle Yields & Canonical Charge Conservation

The Hagedorn thermodynamical model for particle production, was originally applied to quantify and understand particle yields and spectra measured in elementary collisions – there were no data available from heavy ion collisions.

Initial work on particle production by Hagedorn began in 1957 in collaboration with F. Cerulus when they applied the Fermi phase space model, see Section 25.2. In this microcanonical approach, conservation laws of baryon number or electric charge were implemented exactly. Almost 15 years later the production of complex light antinuclei, such as anti-He^3, preoccupied Hagedorn [2, 6]. He realized and discussed clearly the need to find a path to enforce exact conservation of baryon number to describe the anti-He^3 production correctly within the canonical statistical formulation.

Indeed, applying in *pp* reactions the thermal model without concern for conservation of baryon number overestimates the production of anti-He^3 in proton-proton collisions by seven orders of magnitude [2, 4, 6]. The reason was that when the number of particles in the interaction volume is small, one has to take into account the fact that the production of anti-He^3 must be accompanied by the production of another three nucleons with energy E_N, in order to exactly conserve the baryon number. Thus, in case the production of anti-He^3 is not originating from reservoir of many antiquarks or antinucleons already present in a large volume, but is rather originating from some small volume V_{pp} that is present in *pp* collisions, the abundance of anti-He^3 will not be proportional to the single standard Boltzmann factor, as in Eq. (7.25)

$$n_{\overline{\mathrm{He}}^3} \sim \exp\left(-m_{\overline{\mathrm{He}}^3}/T\right), \tag{7.26}$$

but is accompanied by additional Boltzmann factors that characterize the production of the associated nucleons, needed in order to conserve baryon number [2,6]

$$n_{\overline{\mathrm{He}}^3} \sim \exp\left(-m_{\overline{\mathrm{He}}^3}/T\right)\left[V_{\mathrm{pp}}\int \frac{d^3p}{(2\pi)^3}\exp\left(-\frac{E_N}{T}\right)\right]^3. \tag{7.27}$$

This suppresses the rate and introduces a strong power-law dependence on volume V_{pp} for the anti-He^3 yield.

The problem of exact conservation of discrete quantum numbers in a thermal model formulated in early 1970s by Hagedorn in the context of baryon number conservation remained unsolved for a decade. When the heavy ion QGP research program was approaching and strangeness emerged as a potential QGP signature, Hagedorn pointed out the need to consider exact conservation of strangeness [39]. This is the reason that the old problem of baryon number conservation was solved in the new context of strangeness conservation [40–42], see also Section 27.6. A more general solution, applicable to *all* discrete conserved charges, abelian and non-abelian, was also introduced in Ref. [43] and expanded in [44–49]. Recently, it has become clear that a similar treatment should be followed not only for strangeness but also for charm abundance study in high energy e^+e^- collisions [50, 51].

To summarize this section, we note that the usual form of the statistical model, based on a grand canonical formulation of the conservation laws, cannot be used when either the temperature or the volume or both are small. As a rough estimate, one needs $VT^3 > 1$ for a grand canonical description to hold [40,47]. In the opposite limit, a path was found within the canonical ensemble to enforce charge conservations exactly.

The canonical approach has been shown to provide a consistent description of particle production in high energy hadron-hadron, e^+e^- and peripheral heavy ion collisions [32, 46, 50, 51]. As noted in the context of developing strangeness as signature of QGP, see Section 27.6, such a model also provides, within the realm of assumed strangeness chemical equilibrium, a description of an observed increase of single- and multi-strange particle yields from *pp*, *pA* to *AA* collisions and its energy dependence [41].

7.7 Concluding Remarks

Rolf Hagedorn's work, introducing concepts from statistical mechanics and from the mathematics of self-similarity into the analysis of high energy multiparticle production, started a new field of research, alive and active still today. On the theory side, the limiting temperature of hadronic matter and the behavior of the Hagedorn resonance gas approaching that limit were subsequently verified by first principle calculations in finite temperature QCD. On the experimental side, particle yields as well as, more recently, fluctuations of conserved quantities, were also found to follow the pattern predicted by the Hagedorn resonance gas. Rarely has an idea in physics risen from such humble and little appreciated beginnings to such a striking vindication. So perhaps it is appropriate to close with a poetic summary one of us (HS) formulated some twenty years ago for a Hagedorn-Fest, with a slight update.

HOT HADRONIC MATTER
(A Poetic Summary)

In days of old
a tale was told
of hadrons ever fatter.
Behold, my friends, said Hagedorn,
the ultimate of matter.

Then Muster Mark
called in the quarks,
to hadrons they were mated.
Of colors three, and never free,
all to confinement fated.

But in dense matter,
their bonds can shatter
and they freely move around.
Above T_{H}, their colors shine
as the QGP is found.

Said Hagedorn,
when quarks were born
they had different advances.
Today they form, as we can see,
a gas of all their chances.

Acknowledgements K.R. acknowledges support by the Polish Science Foundation (NCN), under Maestro grant DEC-2013/10/A/ST2/00106.

References

1. R. Hagedorn: "Thermodynamics of distinguishable particles: A key to high-energy Strong Interactions?," see Chapter 19 in this volume, see also commentary by J. Rafelski, Chapter 17.5
2. R. Hagedorn: "Statistical thermodynamics of strong interactions at high-energies," Nuovo Cim. Suppl. **3**, 147 (1965)
3. R. Hagedorn and J. Ranft: "Statistical thermodynamics of strong interactions at high-energies. 2. Momentum spectra of particles produced in *pp* collisions," Nuovo Cim. Suppl. **6**, 169 (1968)
4. R. Hagedorn: "Statistical thermodynamics of strong interactions at high energies. 3. Heavy-pair (quark) production rates," Nuovo Cim. Suppl. **6**, 311 (1968)
5. R. Hagedorn: "Hadronic matter near the boiling point," Nuovo Cim. A **56**, 1027 (1968)
6. R. Hagedorn: *Thermodynamics of Strong Interactions*, CERN-Report 71-12 (1971)

7. Ph. Blanchard, S. Fortunato and H. Satz: "The Hagedorn temperature and partition thermodynamics," Eur. Phys. J. C **34**, 361 (2004)
8. W. Nahm: "Analytical solution of the statistical bootstrap model," Phys. B **45**, 525 (1972)
9. G. Veneziano: "Construction of a crossing - symmetric, Regge behaved amplitude for linearly rising trajectories," Nuovo Cim. A **57**, 190 (1968)
10. K. Bardakci and S. Mandelstam: "Analytic solution of the linear-trajectory bootstrap," Phys. Rev. **184**, 1640 (1969)
11. S. Fubini and G. Veneziano: "Level structure of dual-resonance models," Nuovo Cim. A **64**, 811 (1969)
12. E. Beth and G.E. Uhlenbeck: "The quantum theory of the non-ideal gas. II. Behaviour at low temperatures," Physica **4**, 915 (1937)
13. R. Dashen, S.-K. Ma and H.J. Bernstein: "S Matrix formulation of statistical mechanics," Phys. Rev. **187**, 345 (1969)
14. N. Cabibbo and G. Parisi: "Exponential Hadronic Spectrum and Quark Liberation," Phys. Lett. B **59**, 67 (1975)
15. G.H. Hardy and S. Ramanujan:"Asymptotic formulae in combinatory analysis" Proc.London Math. Soc. **17**, 75 (1918)
16. H. Satz: "Critical Behavior of Hadronic Matter. 1. Critical Point Exponents," Phys. Rev. **D19**, 1912 (1979)
17. M. I. Gorenstein, V. A. Miransky, V. P. Shelest, G. M. Zinovev and H. Satz: "The Physical Content Of The Statistical Bootstrap," Nucl. Phys. **B76**, 453 (1974)
18. I.Ya. Pomeranchuk: "On the theory of multiple particle production in a single collision," Dokl. Akad. Nauk **78**, 889 (1951)
19. G Baym: "Confinement of quarks in nuclear matter," Physica A **96**, 131 (1979)
20. T. Çelik, F. Karsch and H. Satz: "A Percolation Approach To Strongly Interacting Matter," Phys. Lett. B **97**, 128 (1980)
21. See e.g. H. Satz: *Extreme States of Matter in Strong Interaction Physics: An Introduction*, Lect. Notes Phys. **841** (2012), and references therein
22. **a)** L.D. McLerran and B. Svetitsky: "A Monte Carlo Study of SU(2) Yang-Mills Theory at Finite Temperature," Phys. Lett. B **98**, 195 (1981);
b) J. Kuti, J. Polonyi and K. Szlachanyi: "Monte Carlo Study of SU(2) Gauge Theory at Finite Temperature," Phys. Lett. B **98**, 199 (1981);
c) J. Engels, F. Karsch, I. Montvay and H. Satz: "High Temperature SU(2) Gluon Matter on the Lattice," Phys. Lett. B **101**, 89 (1981)
23. F. Karsch: "Lattice QCD at high temperature and density," Lect. Notes. Phys. **583**, 209 (2002)
24. Y. Aoki, G. Endrodi, Z. Fodor, S. D. Katz and K. K. Szabo: "The Order of the quantum chromodynamics transition predicted by the standard model of particle physics," Nature **443**, 675 (2006)
25. A. Bazavov et al. [HotQCD]: "Equation of state in (2+1)-flavor QCD," Phys. Rev. D **90**, 094503 (2014)
26. S. Ejiri, F. Karsch and K. Redlich: "Hadronic fluctuations at the QCD phase transition," Phys. Lett. B **633**, 275 (2006)
27. A. Bazavov, et al., [HotQCD Collaboration]: "Fluctuations and Correlations of net baryon number, electric charge, and strangeness: A comparison of lattice QCD results with the hadron resonance gas model," Phys. Rev. D **86**, 034509 (2012)
28. A. Bazavov, H.-T. Ding, P. Hegde, O. Kaczmarek, F. Karsch, E. Laermann, Y. Maezawa and S. Mukherjee *et al.*: "Additional Strange Hadrons from QCD Thermodynamics and Strangeness Freezeout in Heavy Ion Collisions," Phys. Rev. Lett. **113**, 072001 (2014)
29. **a)** F. Karsch, K. Redlich and A. Tawfik: "Thermodynamics at nonzero baryon number density: A Comparison of lattice and hadron resonance gas model calculations," Phys. Lett. B **571**, 67

(2003) and; **b)** “Hadron resonance mass spectrum and lattice QCD thermodynamics,” Eur. Phys. J. C **29**, 549 (2003)
30. F. Karsch: “Thermodynamics of strong interaction matter from lattice QCD and the hadron resonance gas model,” Acta Phys. Polon. Supp. **7**, 117 (2014)
31. K. A. Olive, et al., [Particle Data Group]: Chin. Phys. C **38**, 090001 (2014)
32. For a review, see eg. P. Braun-Munzinger, K. Redlich, and J. Stachel: “Particle Production in Heavy Ion Collisions”, In Hwa, R.C. (ed.): *Quark Gluon Plasma 2*, World Scientific, (2004) p491
33. T. Bhattacharya, M. I. Buchoff, N. H. Christ, H.-T. Ding, R. Gupta, C. Jung, F. Karsch and Z. Lin *et al.*: “QCD Phase Transition with Chiral Quarks and Physical Quark Masses,” Phys. Rev. Lett. **113**, 082001 (2014)
34. J. Cleymans and H. Satz: “Thermal hadron production in high-energy heavy ion collisions,” Z. Phys. C **57**, 135 (1993)
35. K. Redlich, J. Cleymans, H. Satz and E. Suhonen: “Hadronization of quark - gluon plasma,” Nucl. Phys. A **566**, 391 (1994)
36. P. Braun-Munzinger, I. Heppe and J. Stachel: “Chemical equilibration in Pb + Pb collisions at the SPS,” Phys. Lett. B **344**, 43 (1995)
37. F. Becattini: “A Thermodynamical approach to hadron production in e+ e- collisions,” Z. Phys. C **69**, 485 (1996)
38. **a)** A. Andronic: “An overview of the experimental study of quark-gluon matter in high-energy nucleus-nucleus collisions,” Int. J. Mod. Phys. A **29**, 1430047 (2014); **b)** J. Stachel, A. Andronic, P. Braun-Munzinger and K. Redlich: “Confronting LHC data with the statistical hadronization model,” J. Phys. Conf. Ser. **509**, 012019 (2014)
39. J. Rafelski: private communication
40. J. Rafelski and M. Danos: “The Importance Of The Reaction Volume In Hadronic Collisions,” Phys. Lett. B **97**, 279 (1980).
41. S. Hamieh, K. Redlich and A. Tounsi: “Canonical description of strangeness enhancement from p-A to Pb Pb collisions,” Phys. Lett. B **486**, 61 (2000).
42. J. Cleymans, H. Oeschler and K. Redlich: “Influence of impact parameter on thermal description of relativistic heavy ion collisions at (1-2) A-GeV,” Phys. Rev. C **59**, 1663 (1999).
43. K. Redlich and L. Turko: “Phase Transitions in the Statistical Bootstrap Model with an Internal Symmetry,” Z. Phys. C **5**, 201 (1980).
44. L. Turko: “Quantum Gases With Internal Symmetry,” Phys. Lett. B **104**, 153 (1981).
45. M. I. Gorenstein, V. K. Petrov and G. M. Zinovev: “Phase Transition in the Hadron Gas Model,” Phys. Lett. B **106**, 327 (1981)
46. B. Muller and J. Rafelski: “Role of Internal Symmetry in $p\bar{p}$ Annihilation,” Phys. Lett. B **116**, 274 (1982)
47. R. Hagedorn and K. Redlich: “Statistical Thermodynamics in Relativistic Particle and Ion Physics: Canonical or Grand Canonical?,” Z. Phys. C **27**, 541 (1985)
48. C. Derreth, W. Greiner, H. T. Elze and J. Rafelski: “Strangeness Abundances In Anti-p Nucleus Annihilations,” Phys. Rev. C **31**, 1360 (1985)
49. H. T. Elze, W. Greiner and J. Rafelski: “Color Degrees of Freedom in a Quark - Glue Plasma at Finite Baryon Density,” Z. Phys. C **24**, 361 (1984)
50. F. Becattini, P. Castorina, J. Manninen and H. Satz: “The Thermal Production of Strange and Non-Strange Hadrons in e+ e- Collisions,” Eur. Phys. J. C **56**, 493 (2008)
51. A. Andronic, F. Beutler, P. Braun-Munzinger, K. Redlich and J. Stachel: “Thermal description of hadron production in e+e- collisions revisited,” Phys. Lett. B **675**, 312 (2009)

Chapter 8
The Hagedorn Spectrum and the Dual Resonance Model: an Old Love Affair

Gabriele Veneziano

Abstract In this contribution I recall how people working in the late sixties on the dual resonance model came to the surprising discovery of a Hagedorn-like spectrum, and why they should *not* have been surprised. I will then turn to discussing the Hagedorn spectrum from a string theory viewpoint (which adds a huge degeneracy to the exponential spectrum). Finally, I will discuss how all this can be reinterpreted in the new incarnation of string theory through the properties of quantum black holes.

Preamble

My first (virtual) "encounter" with Rolf Hagedorn dates back, I think, to my University studies when I read his CERN yellow report on relativistic kinematics and phase space. I remember finding his notes particularly clear and instructive. Much later, I had the privilege of being his colleague in the TH-Division at CERN, and of benefiting from his insight into physics for many years, even after his retirement, when he would still attend regularly the theory seminars. But I was particularly pleased by one of our last (real this time) encounters:

Shortly after the 1994 Divonne Conference we met each other at CERN. He thanked me for my contribution and added something like this:

> The dual resonance model (or string theory) gives a microscopic explanation for the spectrum I arrived at using my bootstrap arguments. There is an amusing analogy here to what statistical mechanics does to thermodynamics by providing a microscopic interpretation of entropy.

Center for Cosmology and Particle Physics, Department of Physics, New York University, 4 Washington Place, New York, NY 10003, USA;
Collège de France, 11 place Marcelin Berthelot, 75005 Paris, France; and:
Theory Division, CERN, CH-1211 Geneva 23, Switzerland

8.1 A Surprise That Should Not Have Been One

There is a simple (a posteriori!) physical argument for the necessity of a Hagedorn-like spectrum of excited states in any model that satisfies duality (possibly including QCD itself). Let me remind you that, by definition of (Dolen-Horn-Schmit) duality, the asymptotic behavior of (the imaginary part of) any scattering amplitude should be correctly described by **either** Regge pole exchange in the t-channel **or** by resonance formation and decay in the s-channel.

The Regge-pole description gives a power-like behavior, hence, if we work to exponential accuracy, a constant amplitude at high energy. The resonance description yields instead (say, for an elastic two-body collision):

$$\mathrm{Im}A_{\mathrm{el}} \simeq \frac{1}{E}\sum_{\mathrm{R}} \Gamma_{2\mathrm{b}}^{\mathrm{R}} = \sum_{\mathrm{R}} \frac{\Gamma_T^{\mathrm{R}}}{E} B_{2\mathrm{b}}^{\mathrm{R}} \leq N(E) \cdot \overline{B}_{2\mathrm{b}} \,, \tag{8.1}$$

where $\Gamma_{2\mathrm{b}}^{\mathrm{R}}$ is the partial width of the resonance 'R' in the chosen two-body channel '2b', and we have assumed the total width Γ_T^{R} to be smaller than the mass $M_{\mathrm{R}} \sim E$. This gives a bound on $\mathrm{Im}A_{\mathrm{el}}$ in terms of the number of states $N(E)$ at energy E and of their average branching ratio $\bar{B}_{2\mathrm{b}}$ into the particular two-body channel under consideration. By asking for consistency of the two descriptions we thus find, to exponential accuracy:

$$N(E) \geq \overline{B}_{2\mathrm{b}}^{-1} \tag{8.2}$$

Making now the (reasonable) assumption that $\overline{B}_{2\mathrm{b}}$ behaves like the ratio of two-body phase space to total phase space, and using the fact that the latter becomes exponentially small at high energy, we arrive at the "prediction" of an exponentially growing spectrum for duality-fulfilling resonances. The (crucial) power of E appearing in the exponent is not fixed, however, by this argument.

We may ask why this conclusion was not immediately reached in the very early days of duality. The reason was, I believe, that one was accustomed to associate each resonance with a separate pole in the scattering amplitude; now, the number of poles occurring in the dual-model amplitudes was **not** growing exponentially with increasing energy. All one could see was that the Nth pole contained resonances of spin up to N, a fact that, in itself, could only account for a power-like growth of the number of states, not for an exponential one.

Of course we know very well now (and for at least 45 years!) the answer to this apparent paradox. The exponential growth of the number of states in the dual resonance model is hidden behind an exponential degeneracy! This degeneracy, which is neither predicted by Hagedorn's arguments, nor implied by the duality-based reasoning given above, is the truly new ingredient brought in by the dual resonance model. As I shall now argue this degeneracy is directly related to an underlying string picture for the resonances appearing in dual models.

8.2 From T_{H} to the String

When, in the summer of 1968, I first told Sergio Fubini in Torino about my new ansatz for the scattering amplitude his reaction was: "very nice, but....what about negative norm states?" Two months later, I had barely landed in Boston/MIT that we started counting and labeling states, clearly a necessary preliminary step before we could compute their norm.

By early 1969 we had learned how to count (it took longer to answer Sergio's original question about the norm, but eventually people proved, under certain restrictions, the celebrated no-ghost theorem). The rather unexpected result that Sergio and I (and independently Bardakci and Mandelstam) found was that the individual states were labeled by a set of integers $\{N_1, N_2, \dots\}$ with the mass of the state given by the simple formula:

$$\frac{\alpha' M^2}{\hbar} = N = \sum_{n=1}^{\infty} n \cdot N_n \ , \tag{8.3}$$

where α', the universal Regge-slope parameter, sets the energy scale of the theory at $\Lambda \equiv \sqrt{\hbar/\alpha'}$.

The Hagedorn spectrum then simply comes from the observation that a given (allowed) mass $M = \sqrt{N/\alpha'}$ can be obtained via Eq. (8.3) in as many ways as the number of ways in which the integer N can be written as a sum of integers. This "partitio numerorum" number is known to grow like $\exp(c\sqrt{N})$ (with c a known constant) hence like $\exp(c\sqrt{\alpha'/\hbar E})$. This gives immediately a Hagedorn temperature $T_{\mathrm{H}} = c^{-1}\Lambda$.

In the operator reformulation of the dual resonance model the mass-square operator can be written as

$$\frac{\alpha' M^2}{\hbar} = \sum_{n=1}^{\infty} n a_n^{\dagger} a_n \ , \tag{8.4}$$

where $a_n^{\dagger}, a_n$ are an infinite set of ordinary harmonic oscillator creation and destruction operators satisfying the usual commutation relations:

$$[a_n, a_m^{\dagger}] = \delta_{n,m} \ . \tag{8.5}$$

The high degree of degeneracy of the spectrum obviously comes from the presence of the higher harmonics ($n = 2, 3 \dots$ in Eq. (8.4)), but this is just what characterizes a vibrating string!

We conclude that the combination of the Hagedorn spectrum and of degeneracy leads straight into strings. Conversely, if a string picture is assumed for hadrons, one immediately predicts:

i) duality as implied by drawing (duality) diagrams in which string splitting and joining are the basic processes underlying hadronic reactions.

ii) a linear relation between mass and entropy (another way of defining Hagedorn's spectrum) coming from the fact that the energy stored in the string is proportional to its length ($l = \alpha' \cdot E$), while there is a unit of entropy per bit of length.

The unit (bit) of length, λ_{s}, is a quantum object and is related to α' by:

$$\lambda_s = \sqrt{\alpha' \hbar} \, . \tag{8.6}$$

For the hadronic string λ_s is of the order of 10^{-13} cm: there is about one bit of information for every fermi of string length.

8.3 Crisis, Reinterpretations

One of the main motivations (successes) behind Hagedorn's model was the exponential fall-off of the transverse spectrum of produced particles in high energy collisions:

$$d\sigma/dp_\perp \simeq \exp(-p_\perp/T_H) \, . \tag{8.7}$$

This holds well in a sizable region of $p_\perp$. Not surprisingly, a similar behavior was found to occur in the dual model (in string theory).

The discovery of hard constituents inside the hadron, revealing themselves, e.g., through the power-like drop of jet (or exclusive) cross sections, came as a serious blow to both Hagedorn's model and to the hadronic string.

Amusingly, they both survived, with some reinterpretation, in the emerging new theory of strong interactions, QCD. The Hagedorn temperature was reinterpreted as a deconfining phase transition (rather than as an ultimate) temperature, while strings become an effective description of hadrons as composite systems of quarks which are kept together by a thin tube of chromoelectric field.

Around the time that QCD took over, Joel Scherk and John Schwarz came up with the daring proposal that fundamental strings should be relevant for describing all fundamental interactions (including gravity) at much shorter scales than 10^{-13} cm. The natural scale for the new string is simply the Planck length:

$$\lambda_P = \sqrt{G_N \hbar} \, , \tag{8.8}$$

where G_N is Newton's constant. Numerically, $\lambda_P \sim 10^{-33}$ cm.

Obviously, also the new string has a Hagedorn temperature: it is simply shifted upward by some 18 orders of magnitude to about $10^{17} - 10^{18}$ GeV. In the last part of this talk I shall give one example of some new uses of Hagedorn's temperature in this new context: I shall argue that, in analogy with the reinterpretation of the old T_H as deconfining temperature, the new T_H will play the role of a limiting temperature for Black holes, a kind of gravitational-deconfinement temperature, if you like.

8.4 Many Years Later …

There is a (so far undisproved) conjecture by J. Bekenstein that the entropy S of any physical system of energy E and size R cannot be arbitrarily large, i.e.

$$S \leq S_{\mathrm{BB}} \equiv \frac{E \cdot R}{\hbar} \,. \tag{8.9}$$

This is called the Bekenstein bound. Let us consider now a black hole, i.e. a system of energy E contained in a spherical region of space of radius $R < G_{\mathrm{N}} \cdot E$ (this is just the definition of a collapsed state in gravity).

Let us now compare the entropy of the black hole,

$$S_{\mathrm{BH}} = \frac{G_{\mathrm{N}} E^2}{\hbar} \,, \tag{8.10}$$

with S_{BB} in Eq. (8.9), allowing the maximal size for R, $R = G_{\mathrm{N}} \cdot E$. We see that the bound is precisely saturated, something suggesting that a black hole maximizes the entropy of a system of given energy and spatial extension.

We may now ask if a string of energy E satisfies the Beckenstein bound. As said before

$$S_{\mathrm{string}} = \frac{\alpha' \cdot E}{\lambda_{\mathrm{s}}} \,, \tag{8.11}$$

which satisfies the Beckenstein bound Eq. (8.9) only if the size of the string is larger than λ_{s}, a conclusion that can be reached also by independent considerations.

I have told you that λ_{s} is of the same order as λ_{P}, but the more precise relation is actually

$$\lambda_{\mathrm{P}} = \alpha_{\mathrm{gut}}^{1/2}\, \lambda_{\mathrm{s}} \,, \tag{8.12}$$

expressing the physical fact that, in string theory, gravitational and gauge interactions become identical at the (distance) scale λ_{s}.

We can finally compute the ratio between the string and black hole entropies and find

$$\frac{S_{\mathrm{BH}}}{S_{\mathrm{string}}} = \frac{G_{\mathrm{N}} E \lambda_{\mathrm{s}}}{\alpha' \hbar} = \frac{E}{M_{\mathrm{P}}} \cdot \frac{\lambda_{\mathrm{P}}}{\lambda_{\mathrm{s}}} = \frac{\alpha_{\mathrm{gut}}^{1/2} \cdot E}{M_{\mathrm{P}}} \,. \tag{8.13}$$

This ratio becomes 1 at $E = \alpha_{\mathrm{gut}}^{-1/2} M_{\mathrm{P}}$. At this energy both entropies are of order $\alpha_{\mathrm{gut}}^{-1}$, thus probably between 10 and 100. Above this energy, entropy considerations favor the black hole while below a string state is favored. It is easy to see that such a state is not collapsed at all since its physical size, as we argued above, has to be larger than λ_{s} which, in turn, is larger than the gravitational radius $G_{\mathrm{N}} \cdot E$ of the system.

As Hawking has shown, black holes evaporate, losing mass while increasing their temperature. In ordinary gravity this process would continue, until a curvature singularity (and an infinite temperature) is reached. The arguments given above (and due to several people) suggest that, in string theory, black hole evaporation should stop at a certain point, leaving behind a non-collapsed string system of entropy $\mathscr{O}(\alpha_{\mathrm{gut}}^{-1})$.

The Hawking temperature of the black hole at this point is just the Hagedorn temperature of the string theory under consideration. We can thus say that an interpretation of T_{H} in the new incarnation of string theory is that of a "decollapse" temperature if you allow me to use such a word for the gravitational analogue of deconfinement. On the other hand, the analogue of the quark-gluon-plasma phase of QCD in quantum string gravity is still clouded with mystery (is space-time itself "melting"?).

There could be also a cosmological analogue of the limiting temperature for black holes, as it is known that there is a (Hawking) temperature associated with the event horizon of inflationary cosmologies and easily expressible in terms of the Hubble parameter during inflation. This could lead to new insights in the way string-Hagedorn models deal with the very beginning of the Universe.

Conclusion

The love affair between dual and Hagedorn models is still well and alive after many years: and it seems it will last forever.

Hagedorn's Reincarnation in String Theory

Springer Link

John Ellis

Hot Hadronic Matter

NATO ASI Series Volume 346, 1995, pp 75-82

Abstract

After a review of Hagedorn's first incarnation in the quark-hadron phase transition, universality properties of the state level density in string theory are discussed, as well as the possible transition to a new phase beyond the string Hagedorn temperature. Studies of string black holes suggest that the concepts of space and time are likely to disappear in this reincarnation of Hagedorn.

Fig. 8.1 A love affair to one is reincarnation to another.

Chapter 9
Hadronic Matter: The Moscow Perspective

Igor Dremin

Abstract I describe studies done by the theory group of Lebedev Physical Institute in Moscow and point out the cross-influence of some of our work with that of Rolf Hagedorn, and show how this research continued and evolved up to the present.

9.1 The Beginning

Cosmic rays and Landau

High energy hadron interactions were always one of the main topics of research at the Moscow Lebedev Physical Institute. On the experimental side, cosmic ray studies which started already in the 1930s were quite successful. The interest in high energy studies further increased after construction of first accelerator at the Institute with active participation of V.I. Veksler who soon proposed the autofocusing concept (1944) and moved to Dubna with many collaborators to realize his visions and build accelerators.

In parallel, theoretical work became more intense. Many researchers became excited seeing L.D. Landau paper on the hydrodynamic model of hadron interactions in 1953 [1]. Landau used to say that the work on it was the most hard, and time-consuming compared to all other of his papers. Actually, it was the first one which contained the detailed calculations ascribing the macroscopic features to the microscopic objects along the line of thought initiated by E. Fermi in 1950. At that time, in my 3rd graduate year, I devoted several days to studying Landau's paper in the Moscow Polytechnic library.

Landau's approach was widely discussed by theorists at LPI. Hydrodynamic equations were further analyzed by E.L. Feinberg, D.S. Chernavsky and G.A.

Lebedev Physical Institute, Moscow 119991, Russia; and
National Research Nuclear University "MEPhI", Moscow 115409, Russia

Milekhin. They published a series of papers on this topic in the 1950s. This work appeared in Russian journals and is unfamiliar to Western scientists. The activity was strongly slowed down after the tragic death of Milekhin. At the same time, some experimentalists (e.g., G.T. Zatsepin) in our Institute dealing with high energy particles in cosmic rays insisted – mainly based on intuition – that beside drastic central collisions, as we call them nowadays, they observed another class of interesting events which cannot be described by the hydrodynamic approach.

Multiperipheral collisions

I was impressed by these statements and at the very end of 1958 calculated first the one-pion exchange graph of peripheral truly inelastic collisions where both colliding hadrons get excited and produce new particles. This was just at the time when experimental data from the newly constructed Dubna accelerator about inelastic collisions at the laboratory energy 9 BeV (now GeV) became available and were argued about even though not yet published.

Many leading Russian theorists were gathered at our department by I.E. Tamm for a discussion of experimental results and their description by our model. The publication with D.S. Chernavsky appeared in JETP January 1960 [2] issue and initiated a large series of papers on peripheral and multiperipheral models.

The early contributions of D. Amati et al reviewed in [3] were especially important. However, they got the total cross section decreasing with energy increase. To improve their model we proposed that heavier blobs (which we called fireballs and later - clusters) should be created at each vertex, for a review see [4]. At that time such guesses originated from cosmic ray observations, especially those presented by M. Miesowich from Krakow.

In our model, the cluster decay was treated by statistical laws, and that appealed to R. Hagedorn. I do not remember exactly when I first visited CERN, there were so many visits since then. Surely, it was very soon after Hagedorn published his study of particle production. After my CERN seminar and several discussions at the theory department, Hagedorn invited me to his home and we spent a wonderful long evening chatting on many topics including science, literature, theater, politics, and surely, he proudly showed me his horses!

In my conversations with him I gained the feeling that he was very glad to get support from our group and to be able to discuss with all those who could understand his ideas. His work was at the time still little appreciated. This was also true at the CERN Theory Division.

Nowadays, the concept of clusters could be somewhat related to jets numerously created at the LHC. The cluster concept is revived also by studies of *AA* collisions at RHIC. As we say in Russia “Horses run along the rings” and “People come to use old galoshes sometimes!”

9.2 Hot Hadron Matter

Photons and leptons

E.L. Feinberg actively promoted the statistical and hydrodynamical concepts and at the 1970 Kiev conference he first proposed [5] the photon and dilepton signatures (the so-called direct photons and dileptons) of the early stage of the quark-gluon matter (see his summary of 1976 [6]). Namely, these particles could escape from hadronic medium being in comparison almost non-interacting and, therefore, provide important information about its properties.

Many experimental papers aiming to detect such radiation and to confront its properties with theoretical prediction appeared afterwords. The theory was extended to include parton based processes and production of psions ($c\bar{c} = \Psi$) [7].

E.L. (as we called him) often corresponded with Rolf Hagedorn, and also with Peter Carruthers, discussing with them statistical and hydrodynamic approaches which were out of the mainstream of theory research at that time. One had to wait long for experimental data on *AA* collisions before this approach to strong interactions was recognized.

Quark-gluon plasma

Soon two other papers dealing with properties of hadronic matter appeared independently. The work by Kalashnikov-Klimov [8] contained QCD calculations of quark and gluon interactions in matter. At Lebedev (as LPI is often called in the West) we used to speak about the quark-gluon medium without specifying how free are quarks and gluons inside. Thus the term QGP was not widely used here. Earlier, I.Y. Pomeranchuk of ITEP used to call this yet-to-be-identified state of matter, verbatim translated, “the boiling operator liquid”.

Cherenkov radiation

At the same time in 1979 I proposed [9] the idea about Cherenkov gluons which could produce jets (collimated groups of particles) in high energy collisions. That happened after our experimentalists showed me the emulsion plate with the newly registered cosmic ray event where distinct rings formed by secondary hadrons reminding us of the ordinary Cherenkov rings were easily seen.

That new property could also serve as a signature of the properties of the medium. In analogy to the permittivity of the ordinary medium, the term “chromopermittivity” was coined to describe the hadronic medium containing colored quarks and gluons. Its value was directly determined by the positions of the rings. For further work of our group, and in particular, on comparison with the data of RHIC experiments I refer to papers [10] and [11] where our approach has been reviewed.

Correlations and fluctuations

Since then a lot of work was done on correlations and fluctuations of characteristic distributions of particles created from such a medium. More recently, some work on kinetic properties of the quark-gluon medium has been done as reviewed in [12]. The instabilities and time evolution of the newly created state of matter are the main topics of recent research [13].

Charm

The abundant production of charm related to hadron medium properties was noticed in cosmic rays as an effect of the long-flying cascades (for the review see [14]).

9.3 Open Questions

Another direction is represented by my recent attempts to visualize the geometric picture of hadron collisions. By comparison of ISR and LHC data it appears that protons become more dark (absorbing), and larger in size with an increase of energy. Analysis of elastic *pp* scattering data with use of the irrefutable unitarity condition gives rise to a speculative conclusion about very dense (absolutely black at the center) state of matter created at LHC energies in *pp* collisions [15]. A new critical regime of full absorption and rather wide spatial extension has perhaps been reached at the LHC. This could imply that a limiting temperature regime advocated by Hagedorn has been reached, and spatial expansion with increasing number of degrees of freedom prevails. That could correspond to the constancy of this new Hagedorn temperature.

Further implications for *pA* and *AA* collisions should be studied to learn what kind of possibly new matter is produced in these processes. Highest LHC energies can lead to new completely unexpected features also discussed in Ref. [15].

Appreciation

These remarks are in the spirit of my friends and mentors, Feinberg and Hagedorn, see Fig. 9.1, who always were looking for new frontiers and applying new methods in the area of multiparticle production. They certainly would be strongly involved in this present day research. I am very glad to join E.L., Fig. 9.2, in toasting Hagedorn, and to pay a tribute to these two giants of science who inspired our investigations and paved the path we are walking on today.

Fig. 9.1 Front raw: E.L. Feinberg (on left) with Rolf Hagedorn (right), June 27, 1994. In Second raw behind Feinberg: T.E.O. Ericson, and centered A. Martin of CERN-TH. *CREDIT: CERN Photo 1994-06-063-003*.

Acknowledgements This work was supported by the RFBR grants 12-02-91504-CERN-a, 14-02-00099 and the RAS-CERN program.

References

1. L.D. Landau: "On multiple production of particles at collisions of fast moving particles," Izvestia AS USSR, ser. phys. **17**, 51 (1953)
2. I.M. Dremin, D.S. Chernavsky: "Peripheral interactions of nucleons at 9 BeV," ZhETF **38**, 229 (1960)
3. D. Amati, S. Fubini, A. Stanghellini: "Multiperipheral theory," Nuovo Cimento **26**, 896 (1962)
4. I.M. Dremin, A.M. Dunaevsky: "The multiperipheral cluster theory and its comparison to experiment," Physics Reports **18**, 159 (1975)
5. E.L. Feinberg: a) talk at XV International Conference on High-Energy Physics (Kiev, 1970); b) " Multiple generation of hadrons and statistical theory," Sov. Phys. Uspekhi **14**, 455 (1972)
6. E.L. Feinberg: "Direct production of photons and dileptons in thermodynamical models of multiple hadron production," Nuovo Cimento **34**, 31 (1976)
7. E. V. Shuryak: "Production of Leptons, Photons and Psions," Phys. Lett. B **78**, 150 (1978)

8. O. K. Kalashnikov and V. V. Klimov:
a) "Phase Transition in Quark - Gluon Plasma," Phys. Lett. B **88**, 328 (1979);
b) "Polarization Tensor in QCD for Finite Temperature and Density," Sov. J. Nucl. Phys. **31**, 699 (1980) [Yad. Fiz. **31**, 1357 (1980)]
9. I.M. Dremin: "Coherent hadronic radiation at super-high energies," JETP Lett. **30**, 140 (1979)
10. I.M. Dremin: "Ring-like events: Cherenkov gluons or Mach waves?" Nucl. Phys. A **767**, 233 (2006)
11. I.M. Dremin, A.V. Leonidov: "The quark-gluon medium," Physics Uspekhi **53**, 1123 (2010)
12. I.M. Dremin, M.R. Kirakosyan, A.V. Leonidov: "On collective properties of dense QCD matter," Advances in High Energy Physics **2013**, 706521 (2013)
13. A.V. Leonidov, A.A. Radovskaya: "On formation of equation of state of evolving quantum field," JETP Lett. **101**, 235 (2015)
14. I.M. Dremin, V.I. Yakovlev: "Charm in cosmic rays (The long-flying component of EAS cores)," Astroparticle Physics **26**, 1 (2006)
15. I.M. Dremin: "Interaction region of high energy protons," Uspekhi Fiz. Nauk **185**, 85 (2015)

Fig. 9.2 E.L. Feinberg (on left) toasts Rolf Hagedorn on occasion of his 75th Birthday. Sitting: Tania Fabergé, onlooking J. Rafelski, June 30, 1994. *CREDIT: CERN Photo 1994-06-066-015*.

Chapter 10
Hagedorn Model of Critical Behavior: Comparison of Lattice and SBM Calculations

Ludwik Turko

Abstract The Statistical Bootstrap Model and the related concept of the limiting temperature began the discussion about phase transitions in the hadronic matter. This was also the origin of the quark-gluon plasma concept. We discuss here to which extent lattice studies of QCD critical behavior at non-zero chemical potential are compatible with the statistical bootstrap model calculations.

10.1 Rolf Hagedorn - Some Personal Impressions

"A fireball consists of fireballs, which in turn consist of fireballs, and so on . . . " – that was the leading sentence from the famous CERN Yellow Report 71-12 where Rolf Hagedorn presented in detail the leading ideas and results of his Statistical Bootstrap Model (SBM) [1]. I met this report in the late 1970s having yet some scientific experience both in quantum field theory as well as in the theory of high energy multi production processes.

Starting from the beginning I realized that I was reading something unusual. I was impressed by the elegance and precision of the presentation. It was quite obvious to me that the author had spent a lot of time on discussions to clarify his arguments. Some questions were answered before I could even think about them. All was achieved without overuse of mathematical formalism, although all presentation was mathematically very rigorous. The author, however, used as simple and natural mathematical tools as possible, without going into the complex jungle of formulae and multilevel definitions. It was also clearly visible that the model, all its architecture and equipment was a one man project - Rolf Hagedorn.

And the most important point - a new idea was presented. I was not sure at that time - is this idea right or wrong – but that it was an idea not to be ignored. It was a nice answer to the long-standing question – how to effectively describe the basic

Institute of Theoretical Physics, University of Wrocław
Pl. Maksa Borna 9, 50-204 Wrocław, Poland

structure of matter, i.e. here hadronic matter. We knew the whole hierarchy - nuclei, nucleons, elementary particles, quarks. Any of those ‘levels’ pretended at some time to be the ‘real’ elementary one. The SBM didn’t try to answer the question about basic constituents. It just pointed out that this would be a wrong question.

About two years later I met Rolf Hagedorn at CERN in 1979. I was quite convinced at that time about the idea of statistical bootstrap. I saw SBM as a good topic to explore – at least as a way of thinking, and, I wanted to gain a deeper knowledge about statistical physics which in the domain of strong interactions had been to me a rather obscure subject. I also convinced my PhD student at that time, Krzysztof Redlich, that this mixture of statistical physics and theory of elementary particles could be a very fruitful and interesting subject.

Traveling to CERN I was quite excited to meet the physicist whose papers were giving me not only scientific but also quite aesthetic experience. In short: personal meetings with Hagedorn were even more interesting then reading his papers. He was a man of great general culture, very polite but also expecting well-prepared arguments in discussions. From the other side he was very open to share his reasoning, his calculations – even those that still were at a preliminary level of development. His handwritten notes were famous – in an almost calligraphic script, nicely written formulae, alternative arguments. He handed those notes to collaborators – it was just like received a chapter of an advanced textbook.

After two years our relationship rapidly changed. The martial law, introduced in Poland in December 1981, not only made impossible my stay at CERN expected in Spring 1982, but also put me first in an internee camp, then in jail. I was not the only scientist at that time who found himself in such an unexpected surrounding. And it was Rolf Hagedorn, who without any delay in the very first days of martial law, co-initiated at CERN a campaign to free interned or jailed physicists in Poland. Posters with photos and names were posted on walls of TH division, signatures of protest were collected, and letters of protest were sent to Polish officials.

When we met again in 1989 we still kept our relationship, not only on a scientific but also on a friendly level. Looking now back I must admit that Rolf Hagedorn was among those who shaped my profile - not only as a scientist, but also as a man. He was definitely worth following - in any respect. I am happy I had the possibility to be close to such an exceptional scientist and an exceptional man. A man of honor.

10.2 Critical Behavior of Hadronic Matter

Quantum chromodynamics (QCD) gauge theory is an excellent tool for description of single hadronic events in vacuum. However, for the dense and hot hadronic matter the most reliable theoretical results, based on first principles, can be obtained only through lattice gauge theory calculations. In particular, a phase transition or crossover phenomena are expected. This critical behavior is related to peculiarities in standard lattice QCD quantities as the Polyakov loop or susceptibilities.

From the other side, a surprisingly simple resonance gas model provides a good description of particle yields in the relativistic heavy ion collisions in the broad energy range [2, 3]. The clue to this result is in the exponential-like behavior of the particle mass spectrum. This model, slightly extended [4], also reproduces results of the transition between a hadron resonance gas phase and the quark gluon plasma obtained in course of QCD lattice simulation.

The concept of the limiting temperature of the hadronic matter has appeared in the SBM [5–8]. An introduction of the baryonic chemical potential transforms this critical temperature into the critical curve, see Chapter 23. Higher internal symmetries lead to an appearance of the critical surface [9, 10]. The hadronic matter, above the critical curve, is interpreted nowadays as a QGP phase.

I will compare here calculations of critical curves obtained in the $\mu - T$ plane from lattice Monte Carlo simulations with analogous critical curves obtained with the same input from the SBM. It was shown [11, 12] that a gas of non-interacting resonances provides a good description of the low temperature phase of lattice QCD. As the hadronic mass spectrum is similar to the exponential mass spectrum expected from the SBM, it is interesting to check if the critical behavior obtained from the SBM resembles Monte Carlo results of lattice QCD.

We are interested here in the region of the (μ_B, T) plane covered by ultrarelativistic heavy ion collisions where the phase transition is expected i.e. the high temperatures and low baryonic densities. The efficient method of lattice simulation proceeds here via a Taylor expansion with respect to the baryonic chemical potential at $\mu_B = 0$ [13, 14]. This lattice technique, supplemented with other technical tools specific for QCD lattice simulations, was used to obtain the phase transition curve $T_c(\mu)$ for 2-flavor and 3-flavor QCD [15–19].

Critical curve from the lattice calculations

In order to compare the SBM with lattice results one should take into account that the latter are not obtained from calculations performed with the physically realized quark mass spectrum. One finds [20–22] that the quark mass dependence is well parametrized through the relation

$$(m_H a)^2 = (m_H a)^2_{phys} + b(m_\pi a)^2 \,, \tag{10.1}$$

where $(m_H a)_{phys}$ denotes the physical mass value of a hadron expressed in lattice units and $(m_H a)$ is the value calculated on the lattice for a certain value of the quark mass or equivalently a certain value of the pion mass.

The lattice constant a can be treated as a specific ultra-violet regularization which is removed in the continuous limit $a \to 0$. The value of the critical temperature T_c is dependent on the pion mass [17]. Pion here is understood as the lowest pseudoscalar mesonic state $q\bar{q}$ of the mass m_{PS}. This mass decreases to its physical pion mass $m_\pi = 0.140$ GeV in the continuous limit along with the critical temperature.

Critical curves for 2-flavor and 3-flavor QCD were obtained at some assumed quarks masses (in lattice constant a units): $m_q = 0.1$ on the left-hand figure and $m_q = 0.1,\ m_q = 0.005$ on the right-hand plots. Corresponding m_{PS} masses were 0.770 GeV, 0.190 GeV and 0.170 GeV, respectively.

Critical curve from the Statistical Bootstrap Model

Let us start from the bootstrap equation taken for the system with pions and nucleons taken as basic constituents. The bootstrap input function is given as, compare Eq. (27.19) on page 343

$$\varphi_{n_\pi,n_N}(\mu,T) = 2H\pi T\left[n_\pi m_\pi K_1\left(\frac{m_\pi}{T}\right) + n_N m_N K_1\left(\frac{m_N}{T}\right)\cosh\left(\frac{\mu_B}{T}\right)\right], \quad (10.2)$$

I restricted the set of input particles for a given valence quark input to the lightest mesonic and baryonic states respectively. n_π and n_N are their numbers, spin degeneration and antibaryons are taken into account here. They form an input for the SBM. So for two quark flavors there are $n_\pi = 3$ mesonic states and $n_N = 8$ baryonic states. For three quark flavors with the threefold quark mass degeneracy one gets $n_\pi = 8$ and $n_N = 32$, respectively.

The bootstrap constant H is written as

$$H = A\frac{2m_\pi m_N}{(2\pi)^3}\frac{1}{B} \quad (10.3)$$

where $B^{1/4} \approx 0.190$ MeV is the bag constant to reproduce critical energy density $\varepsilon \approx 0.6$ GeV/fm ^{3} and the parameter A is chosen so to get the critical temperature T_c at $\mu_B = 0$ from the corresponding QCD lattice simulation.

The Statistical Bootstrap Model used on the QCD lattice system has its basis components such as they appear in lattice QCD simulation. It means, particularly, nucleon mass expressed by pion mass (all in GeV) as

$$m_N(m_\pi) = 0.94 + \frac{m_\pi^2}{0.94} \quad (10.4)$$

The critical curve for incompressible hadrons is obtained directly from the bootstrap equation

$$2\Phi = \varphi + e^{\Phi} - 1\,, \quad (10.5)$$

which is meaningful only for

$$\varphi \leq \ln 4 - 1\,.$$

So the critical curve $T_c(\mu)$ is given on the $\mu_B - T$ plane by the condition

$$\varphi_{n_\pi,n_N}(\mu,T) = \ln 4 - 1\,. \quad (10.6)$$

Comparison of SBM and lattice-QCD

I consider the lattice bootstrap system as described above. The exact result is shown in left panel of Fig. 10.1, corresponding to Hagedorn's result Fig. 25.3 on page 296. This result should be compared to the 2-flavor QCD lattice simulation, indicated with vertical error line domain in the right panel in Fig. 10.1.

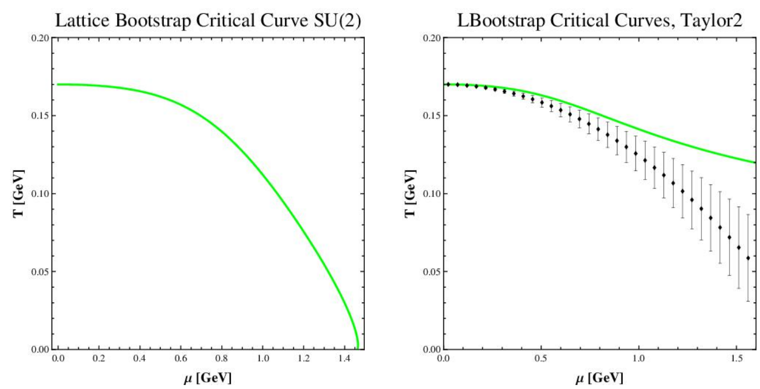


Fig. 10.1 The transition temperature T_c, as a function of baryonic chemical potential for 2-flavor lattice, bootstrap system (left-hand figure). The critical temperature 2-flavor lattice bootstrap system, compared with result of corresponding lattice simulation (right-hand figure), both considered in power law approximation.

As the critical curves $T_c(\mu_q)$ from the QCD lattice calculations were obtained up to $\mathscr{O}((\mu_q/T_c(0))^2)$ term, so the similar approximation should be used for the critical curves obtained from Eq. (10.6). This means that the expression $\cosh[\left(\frac{\mu_B}{T}\right)]$ in Eq. (10.2) should be replaced by the corresponding Taylor expansion truncated to the first two terms. The result of this procedure is presented on the Fig. 10.1 - right panel.

10.3 Conclusions

Results presented on Fig. 10.1 show that the Statistical Bootstrap Model reproduces at least qualitatively basic properties of the critical curve obtained in the course of QCD lattice simulation. We have quantitative agreement for smaller values of baryonic chemical potential, not exceeding 0.7 GeV. This is rather natural taking into account that the method used in the simulations was based on the idea of analytical continuation in the chemical potential variable, starting from the point $\mu = 0$.

The statistical bootstrap model, created by Rolf Hagedorn half a century ago, at a time when quarks were still a bold hypothesis, remains a very inspiring research tool of hadronic matter. Based on Hagedorn's deep knowledge and great intuition, the Statistical Bootstrap Model has still some unknown and unexpected properties, waiting to be discovered.

Acknowledgements I acknowledge the stimulating discussions with F. Karsch and K. Redlich. This work has been supported by the Polish National Science Center under grant no. DEC-2013/10/A/ST2/00106.

References

1. R. Hagedorn: "Thermodynamics of strong interactions", CERN Report 71-12 (1971)
2. J. Cleymans and K. Redlich: Phys. Rev. Lett. **81**, 5284 (1998)
3. P. Braun-Munzinger, D. Magestro, K. Redlich and J. Stachel: Phys. Lett. B **518**, 41 (2001)
4. D. Blaschke, A. Dubinin and L. Turko: "Mott-hadron resonance gas and lattice QCD thermodynamics," arXiv:1501.00485 [hep-ph] (2015).
5. R. Hagedorn: Suppl. Nuovo Cimento **3**, 147 (1965); R. Hagedorn and J. Ranft: Suppl. Nuovo Cimento **6**, 169 (1968)
6. R. Hagedorn, J. Rafelski: in *Statistical Mechanics of Quarks and Hadrons* pp 237-272, ed. H.Satz, (North Holland 1981)
7. S. Frautschi: Phys. Rev. D **3**, 2821 (1971)
8. R. Hagedorn: *The Long Way to the Statistical Bootstrap Model*, Chapter 17 in this volume
9. K. Redlich and L. Turko: Z. Phys. C **5**, 201 (1980)
10. L. Turko: Phys. Lett. B **104**, 153 (1981)
11. F. Karsch, K. Redlich and A. Tawfik: Eur. Phys. J. C **29**, 549 (2003); and Phys. Lett. B **571**, 67 (2003)
12. F. Karsch: Prog. Theor. Phys. Suppl., **153** 106 (2004)
13. R.V. Gavai and S. Gupta: Phys. Rev. D **64**, 074506 (2001); and Phys. Rev. D **65**, 094515 (2002);
    R.V. Gavai, S. Gupta and P. Majumdar: Phys. Rev. D **65**, 054506 (2002)
14. S. Gottlieb *et al.*, Phys. Rev. D **38**, 2888 (1988)
15. C. R. Allton, S. Ejiri, S. J. Hands, O. Kaczmarek, F. Karsch, E. Laermann, C. Schmidt and L. Scorzato: Phys. Rev. D **66**, 074507 (2002)
16. F. Karsch, E. Laermann and C. Schmidt: Phys. Lett. B **520**, 41 (2001)
17. F. Karsch, E. Laermann and A. Peikert: Phys. Lett. B **478**, 447 (2000); and Nucl. Phys. B **605**, 579 (2001)
18. F. Karsch, C. R. Allton, S. Ejiri, S. J. Hands, O. Kaczmarek, E. Laermann and C. Schmidt: Nucl. Phys. Proc. Suppl. **129**, 614 (2004)
19. K. Redlich, F. Karsch and A. Tawfik: J. Phys. G **30**, S1271 (2004)
20. D. G. Richards *et al.* [LHPC Collaboration]: Nucl. Phys. Proc. Suppl. **109A**, 89 (2002)
21. M. Göckeler *et al.*: Phys. Lett. B **532**, 63 (2002)
22. J. M. Zanotti *et al.* [CSSM Lattice Collaboration]: Phys. Rev. D **68**, 054506 (2003)

Chapter 11
Hagedorn's Hadron Mass Spectrum and The Onset of Deconfinement

Marek Gaździcki and Mark I. Gorenstein

Abstract A brief history of the observation of the onset of deconfinement – the beginning of the creation of quark-gluon plasma in nucleus-nucleus collisions with increasing collision energy - is presented. It starts with the measurement of the hadron mass spectrum and Hagedorn's hypothesis of the limiting temperature of hadronic matter (the Hagedorn temperature). Then the conjecture that the Hagedorn temperature is the phase transition temperature was formulated with the crucial Hagedorn participation. It was confirmed by the observation of the onset of deconfinement in lead-lead collisions at the CERN SPS energies.

Marek: Goethe-University, Frankfurt, Germany; and Jan Kochanowski University, Kielce, Poland
Mark: Bogolyubov Institute for Theoretical Physics, Kiev, Ukraine; and Frankfurt Institute for Advanced Studies, Frankfurt, Germany

11.1 Hadron Mass Spectrum and the Hagedorn Temperature

A history of multi-particle production started with the discoveries of hadrons, first in cosmic-ray experiments and soon after in experiments using beams of particles produced in accelerators. Naturally, the first hadrons, discovered in collisions of cosmic-ray particles, were the lightest ones, pion, kaon and Λ. With the rapid advent of particle accelerators new particles were uncovered almost daily. About 1000 hadronic states are known so far. Their density in mass $\rho(m)$ increases approximately exponentially as predicted by Hagedorn's Statistical Bootstrap Model [1] formulated in 1965:

$$\rho(m) \ = \ \mathrm{const}\, m^{-a} \exp(bm) \ . \tag{11.1}$$

In the case of point-like hadron states this leads to a single-particle partition function:

$$Z(T,V) \ = \ \frac{V}{2\pi^2} \int_{m_\pi}^{\infty} dm \int_0^{\infty} k^2 dk \exp\left(- \frac{\sqrt{k^2+m^2}}{T} \right) \rho(m) \ , \tag{11.2}$$

where V and T are the system volume and temperature, respectively. The m-integral exists only for $T < 1/b$. Thus, the hadron gas temperature is limited from above. Its maximum temperature $T_{\mathrm{H}} = 1/b$ (the so-called Hagedorn temperature) was estimated by Rolf Hagedorn based on 1965 data to be $T_{\mathrm{H}} \cong 160$ MeV. More recent estimates of the Hagedorn temperature(s) can be found in Ref. [2], for further discussion see Chapters 20 and 21.

The first statistical model of multi–hadron production was proposed by Fermi [3] in 1950. It assumes that hadrons produced in high energy collisions are in equilibrium and that the energy density of the created hadronic system increases with increasing collision energy. Soon after, Pomeranchuk [4] pointed out that hadrons cannot decouple (freeze-out) at high energy densities. They will rather continue to interact while expanding until the matter density is low enough for interactions to be neglected. Pomeranchuk estimated the freeze-out temperature to be close to pion mass, $\approx$150 MeV. Inspired by this idea Landau [5] and his collaborators formulated a quantitative hydrodynamical model describing the expansion of strongly interacting hadronic matter between the Fermi's equilibrium high density stage (the early stage) and the Pomeranchuk's low density decoupling stage (the freeze-out). The Fermi-Pomeranchuk-Landau picture serves as a base for modeling high energy nuclear collisions up to now [6].

Hagedorn's conjecture concerning the limiting temperature was in contradiction to the Fermi-Pomeranchuk-Landau model in which the temperature of hadronic matter created at the early stage of collisions increases monotonically with collision energy and is unlimited.

11.2 Discovery of the Onset of Deconfinement

The quark model of hadron classification proposed by Gell-Mann and Zweig in 1964 starts a 15 years-long period in which sub-hadronic particles, quarks and gluons, were discovered and a theory of their interactions, quantum chromodynamics (QCD) was established. In parallel, conjectures were formulated concerning the existence and properties of matter consisting of sub-hadronic particles, soon called the QGP and studied in detail within the QCD [7].

Ivanenko, Kurdgelaidze [8], Itoh [9] and Collins, Perry [10] suggested that quasi-free quarks may exist in the centre of neutron stars. Many physicists started to speculate that the QGP can be formed in nucleus–nucleus collisions at high energies and thus it may be discovered in laboratory experiments. Questions concerning QGP properties and properties of its transition to matter consisting of hadrons were considered since the late 70s.

Cabibbo, Parisi [11] pointed out that the exponentially increasing mass spectrum proposed by Hagedorn may be connected to the existence of the phase in which quarks are not confined. Then Hagedorn and Rafelski [12], see Chapter 23, Gorenstein, Petrov, and Zinovjev [13] suggested that the Hagedorn massive states are not point-like objects but quark-gluon bags. These picture leads to the interpretation of the upper limit of the hadron gas temperature, the Hagedorn temperature, as the transition temperature from the hadron gas to a quark-gluon plasma. Namely, at $T > T_{\mathrm{H}}$ the temperature refers to an interior of the quark-gluon bag, i.e., to the QGP.

In the mid-1990s the Statistical Model of the Early Stage (SMES) was formulated [14] as an extension of Fermi's statistical model of hadron production. It assumes a statistical production of confined matter at low collision energies (energy densities) and a statistical QGP creation at high collision energies (energy densities). The model predicts a rapid change of the collision energy dependence of hadron production properties, that are sensitive to QGP, as a signal of a transition to QGP (the onset of deconfinement) in nucleus–nucleus collisions. The onset energy was estimated to be located in the CERN SPS energy range.

Clearly, the QGP hypothesis and the SMES model removed the contradiction between Fermi's and Hagedorn's statistical approaches. Namely, the early stage temperature of strongly interacting matter is unlimited and increases monotonically with collisions energy, whereas there is a maximum temperature of the hadron gas, $T_{\mathrm{H}} \approx 160$ MeV, above which strongly interacting matter is in the QGP phase.

Rich data from experiments at the CERN SPS and LHC as well as at the BNL AGS and RHIC clearly indicate that a system of strongly interacting particles created in heavy collisions at high energies is close to, at least local, equilibrium. At freeze-out the system occupies a volume which is much larger than a volume of an individual hadron. Thus, one concludes that strongly interacting matter is created in heavy ion collisions [6].

The phase transition of strongly interacting matter to the QGP was discovered within the energy scan program of the NA49 Collaboration at the CERN SPS [15, 16]. The program was motivated by the predictions of the SMES model. The discovery was based on the observation that several basic hadron production

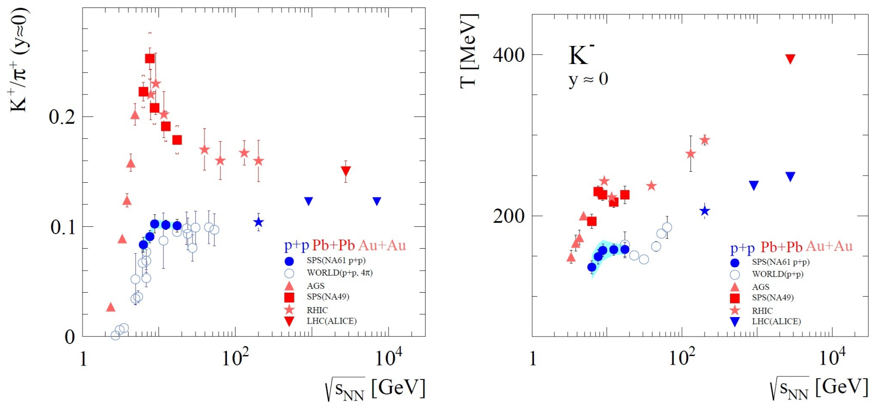


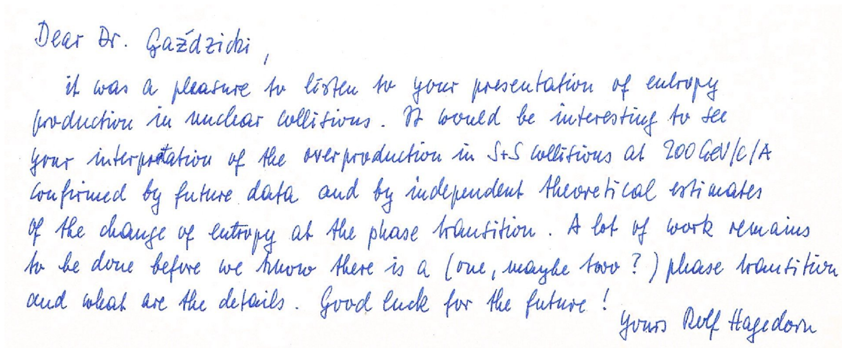
Fig. 11.1 Recent results on the observation of the phase transition in central Pb+Pb (Au+Au) collisions [18]. The horn (left) and step (right) structures in energy dependence of the K^+/π^+ ratio and the inverse slope parameter of K^- $m_\perp$ spectra signal the onset of deconfinement located at the low CERN SPS energies.

properties measured in heavy ion collisions rapidly change their dependence on collisions energy in a common energy domain [17], see Fig. 11.1.

The first ideas which resulted in formulation of the SMES model were presented by one of us [19] (Fig. 11.2) at the Workshop on *Hot hadronic matter: Theory and experiment*, which took place in Divonne, France in June 1994. The workshop was dedicated to 75th birthday of Rolf Hagedorn.

Fig. 11.2 Marek Gaździcki (facing to right off center) at Hagedorn Divonne Fest, June 30, 1994. *Photo: Lucy Carruthers*.

Hagedorn's letter on the presentation is reprinted in Fig. 11.3 in lieu of a summary:



Dear Dr. Gaździcki,

it was a pleasure to listen to your presentation of entropy production in nuclear collisions. It would be interesting to see your interpretation of the overproduction in S+S collisions at 200 GeV/c/A confirmed by future data and by independent theoretical estimates of the change of entropy at the phase transition. A lot of work remains to be done before we know there is a (one, maybe two?) phase transition and what are the details. Good luck for the future!

Yours Rolf Hagedorn

Fig. 11.3 The letter of Rolf Hagedorn to Marek Gaździcki commenting the first talk on the onset of deconfinement in nucleus-nucleus collisions at the low CERN SPS energies [19] presented in June 1994 at the Divonne workshop dedicated to Rolf Hagedorn on occasion of his 75th birthday.

Acknowledgements This work was supported by the German Research Foundation under grant GA 1480/2-2 and by the Program of Fundamental Research of the Department of Physics and Astronomy of NAS Ukraine.

References

1. R. Hagedorn: "Statistical thermodynamics of strong interactions at high-energies," Nuovo Cim. Suppl. **3**, 147 (1965)
2. W. Broniowski, W. Florkowski and L. Y. .Glozman: "Update of the Hagedorn mass spectrum," Phys. Rev. D **70**, 117503 (2004)
3. E. Fermi: "High-energy nuclear events," Prog. Theor. Phys. **5**, 570 (1950)
4. I. Y. .Pomeranchuk: "On the theory of multiple particle production in a single collision," Dokl. Akad. Nauk Ser. Fiz. **78**, 889 (1951)
5. L. D. Landau: "On the multiparticle production in high-energy collisions," Izv. Akad. Nauk Ser. Fiz. **17**, 51 (1953)
6. W. Florkowski: *Phenomenology of Ultra-Relativistic Heavy–Ion Collisions* (World Scientific, 2010)
7. E. V. Shuryak: "Quantum Chromodynamics and the Theory of Superdense Matter," Phys. Rept. **61**, 71 (1980)
8. D. D. Ivanenko, D. F. Kurdgelaidze: "Hypothesis concerning quark stars," Astrophysics **1**, 251 (1965) [Astrofiz. **1**, 479 (1965)].
9. N. Itoh, "Hydrostatic Equilibrium of Hypothetical Quark Stars," Prog. Theor. Phys. **44**, 291 (1970).
10. J. C. Collins, M. J. Perry: "Superdense Matter: Neutrons Or Asymptotically Free Quarks?," Phys. Rev. Lett. **34**, 1353 (1975).
11. N. Cabibbo and G. Parisi: "Exponential Hadronic Spectrum and Quark Liberation," Phys. Lett. B **59**, 67 (1975)

12. R. Hagedorn, J. Rafelski, "From Hadron Gas To Quark Matter. 1.," In Bielefeld 1980, Proceedings, *Statistical Mechanics Of Quarks and Hadrons*, pp.237–251 and CERN-TH-2947; R. Hagedorn, I. Montvay and J. Rafelski: "Thermodynamics Of Nuclear Matter From The Statistical Bootstrap Model," CERN-TH-2605 (1978); see also J. Rafelski and R. Hagedorn, "Thermodynamics of Hot Nuclear Matter in the Statistical Bootstrap Model," Chapter 23 in this volume
13. M. I. Gorenstein, V. K. Petrov, G. M. Zinovjev: "Phase Transition in the Hadron Gas Model," Phys. Lett. B **106**, 327 (1981)
14. M. Gazdzicki, M. I. Gorenstein: "On the early stage of nucleus-nucleus collisions," Acta Phys. Polon. B **30**, 2705 (1999)
15. S. V. Afanasiev *et al.* [The NA49 Collaboration]: "Energy dependence of pion and kaon production in central Pb + Pb collisions," Phys. Rev. C **66**, 054902 (2002)
16. C. Alt *et al.* [NA49 Collaboration]: "Pion and kaon production in central Pb + Pb collisions at 20-A and 30-A-GeV: Evidence for the onset of deconfinement," Phys. Rev. C **77**, 024903 (2008)
17. M. Gazdzicki, M. Gorenstein, P. Seyboth: "Onset of deconfinement in nucleus-nucleus collisions: Review for pedestrians and experts," Acta Phys. Polon. B **42**, 307 (2011); "Recent developments in study of deconfinement in relativistic nucleus-nucleus collisions," Int. J. Mod. Phys. E **23**, 1430008 (2014).
18. N. Abgrall *et al.* [NA61/SHINE Collaboration]: "Report from the NA61/SHINE experiment at the CERN SPS," CERN-SPSC-2014-031, SPSC-SR-145 (2014).
19. M. Gazdzicki: "Pions, baryons and entropy in nuclear collisions," in: *Hot Hadronic Matter: theory and experiment* Divonne 1994, J. Letessier, H.H. Gutbrod and J. Rafelski, edts. NATO-ASI **346**, .pp 215-222, (Plenum Press, New York 1995)

Chapter 12
Begin of the Search for the Quark-Gluon Plasma

Grazyna Odyniec

Abstract LBL has been the cradle where relativistic heavy ion physics began, and where the Hagedorn Statistical Model was first connected to relativistic heavy ion physics. The early program of research at the Bevalac and its development into an international program at CERN, paying tribute to the seminal effort by Howel Pugh, is described.

12.1 The Beginning

Bevalac and ISR

By the early 1970s the Lawrence Berkeley Laboratory (LBL, now LBNL - Lawrence Berkeley National Laboratory) became the leading force in the nascent field of relativistic heavy ion physics, with the Bevalac providing beams of relativistic heavy ions for fixed target experiments. At that time very little was known about quark-gluon plasma. The word "quark" does not even appear in any document related to the Bevalac. Experimental interest was focused on ultra dense nuclear matter proposed by Lee and Wick and on the related possibility to create a pion condensate. Instead, abundant hot particle production was observed which entailed interest in Hagedorn's physics, see Chapter 13.

It was generally agreed that in the near future more energetic nuclear beams would be used in the search for new physics. Quarks soon came onto the menu of these new plans. The potential of relativistic heavy ion collisions became fully apparent and very important to the nuclear physics community. These collisions required at least 10 times higher center of mass (CM) energy than was available at the Berkeley Bevalac, which at that time was the world's highest energy heavy ion machine. Several laboratories in the world showed interest in this program.

LBNL, Berkeley, CA 94720, USA

CERN had instrumental capability and interest at the time and became the first center of this new physics. At CERN, the synergy of nuclear physics and high-energy physics was a key factor in these rapid developments. This synergy was entirely absent in other places, where particle physics did not have the Hagedorian soft hadron physics tradition.

The path to the heavy ion physics program at CERN was, however, not easy.

In 1975, Howel Pugh (then at the University of Maryland) was advocating putting nuclear beams into the CERN Intersecting Storage Rings (ISR) to reach 10 times the Bevalac CM energy, so that experiments could investigate the hypothetical quark deconfinement matter. The ISR, a proton-proton collider, could accelerate nuclear beams at about CM energy of 30 GeV per nucleon pair for light nuclei and about 12 GeV for the typical case of the heavier ones (Au, Pb, U).

The experimental program involving nuclear collisions at ISR was limited to alpha particles (^{4}He), not for lack of interest, see also Chapter 28, but because no appropriate heavy ion source was as yet available at CERN. Howel Pugh and James Symons of LBL became involved in the ^{4}He program. Results were, however, rather disappointing as also the experimental capabilities were limited, and, ^{4}He was simply too light a nucleus to demonstrate collective fireball-like effects at even the top ISR energy.

When the ISR was set to be closed in 1983, the future of the heavy ion program at CERN became uncertain. These early initiatives, however, in the end and with the help of the CERN DG Herwig Schopper (1981-1988), paved the way for a strong SPS Heavy Ion Program at CERN in the coming decades, see Chapter 29.

SPS and RHIC programs take shape

In 1979 Howel Pugh arrived at LBL to assume the post of Bevalac Scientific Research Director. He was attracted by the unique physics opportunities of the Bevalac, with which he was familiar from regularly attending Bevalac Program Advisory Committee meetings in his capacity as an NSF official. He saw LBL as the best place to push the ultra-relativistic heavy ion physics program forward to higher collision energies.

The next years were marked by a vigorous development of several proposals at LBL, all meant to reach the higher energies required for the formation of quark-gluon plasma. They began to appear on the agendas of the various committees. Howel Pugh was the principal author of the VENUS (Variable Energy Nuclear Synchrotron) proposal, which envisioned a large synchrotron plus stretcher ring that would fill LBL's entire hillside. This facility was never built, but the VENUS proposal served as the prototype for the future RHIC collider concept.

The next development, also led by Howel Pugh, was the TEVALAC, a fixed target program which would increase the energy of the Bevalac. This idea followed somewhat similar proposals, which had been in discussion at CERN since 1973. Finally Howel turned his attention to the design of the "Mini-Collider", which would take the heavy ion beams from the Bevalac and collide them in a circular accelerator. In a way it was a stepping-stone for future projects at RHIC and LHC. If it had been funded, this “Mini-Collider” would have been the shortest path to the discovery of QGP.

During this period, CERN moved forward as well. The next step, after ISR, was establishing the heavy ion program to the SPS, a proton synchrotron that would provide nuclear beams at 20 GeV CM energy. In 1984 a heavy ion CERN high energy SPS scientific program was approved. External resources needed to be found to generate the required heavy ion beams and to realize the multi-purpose usage of PS-SPS synchrotron as both, LEP injector and heavy ion accelerator. GSI and LBL agreed to build a special heavy ion injector for the CERN synchrotron. LBL was to build the novel RFQ pre-accelerator. Howel Pugh was the driving force behind LBL participation in the SPS program.

At the same time, the availability of the half-built and abandoned ISABELLE *pp* collider civil structure at BNL generated a strong push for the development of a heavy ion collider there. This project replaced the heavy ion options considered at LBL. However, the decision to start with heavy ions afresh at BNL meant that one had to develop heavy ion beam transfer line to the AGS, train a new generation of experts, and carry out within this new environment the collider technology development of what ultimately became RHIC and its experiments. All this was going to take much precious time – gambling that SPS energy was too low to matter in the search for, and discovery of, the new phase of matter, quark-gluon plasma.

12.2 Quark-Gluon Plasma Discovered

New instrumentation

Moving to the higher energies required not only new accelerators but also new, modern, high-capability detectors. Howel Pugh proposed using a time projection chamber electronic detector (TPC) and recognized that such modern detectors would make a large difference for the Bevalac experiments. A TPC for heavy collisions was a new and quite revolutionary concept at the time and many were skeptical that this could be done. It had the advantage of allowing a 3-dimensional analysis of complex heavy ion collisions.

Pugh with three LBL collaborators (G. Odyniec, G. Rai and P. Seidel) provided all the necessary calculations and simulations (“EOS: A Time Projection Chamber for the Study of nucleus-nucleus collisions at the Bevalac”, LBL-22314, UC-34C). He named this new detector EOS TPC (after Equation Of State, but also after the Greek goddess of the dawn). Working on this project was the most exhilarating experience available to a young physicist. We were astonished by the amount of “impossible” or “unsolvable” problems we encountered. They seemed like brick walls, but, in fact, they were only temporary. Under Howel’s direction we were able to overcome each of them.

EOS TPC was built in LBL under H. Wieman's leadership, and installed in 1991 into the HISS magnet to be used in the Bevalac heavy ion program. It was a tremendous success and the credit goes to Howel Pugh's vision of the program necessary to pursue the physics of heavy ion collisions. The EOS TPC opened a totally new way of analyzing data and was a precursor for the future CERN SPS-NA49 TPC, BNL RHIC-STAR TPC and CERN LHC-ALICE TPC.

One can say that this contribution was the preeminent legacy of Howel Pugh in the field of relativistic heavy ion physics.

Experiments

The BNL-RHIC did not become operational for many years. In the interim, from 1986 to 1993, LBL teams worked at CERN SPS in the experiments NA35, NA36 and WA80. The initial success of these experiments catalyzed a wider group of new international CERN experiments. The earlier beams consisted of Oxygen (^{16}O) and Sulfur (^{32}S) nuclei, which are rather light. They were not massive enough to assure QGP formation and the CERN Heavy Ion Program needed to proceed to a mass 200 nuclei. An injector complex for Lead ions (^{208}Pb) was completed in 1994. For these beams a totally new TPC experiment, NA 49, based on the LBL experience with the EOS TPC, was built. There was also a highly developed spectrometer facility, the Ω-spectrometer that was readied to take data, see Chapter 15. CERN had a wide and diverse relativistic heavy ion physics program.

In the mid-1990s the early results from this successful program, including the NA35/49 experiment in which I participated, indicated most interesting changes in the energy dependence of hadron production, particularly hadrons containing strange quarks, in Pb+Pb collisions, see Chapter 11. Within the statistical model, which was extensively developed by Rolf Hagedorn and his collaborators, the observed changes could be interpreted as the onset of the deconfinement of the phase transition to the quark-gluon plasma state.

In a locally thermalized fireball of particles created in the collisions, the apparent temperature is related to the thermal motion of the particles and their collective expansion velocity. From the composition of hadrons resulting from the decay of the fireball, the temperature at which the transition takes place can be estimated to be below $T \simeq 1.8 \times 10^{12}\,\mathrm{K} = 155$ MeV, a value near to the limiting temperature, i.e. the temperature where hadronic matter dissolves into quark matter. Introduced earlier by Rolf Hagedorn, the limiting temperature following from the exponential slope of the mass spectrum is $T_{\mathrm{H}} \simeq 155$–$160\,\mathrm{MeV}$, see Chapter 21.

This observation of the QGP formation at SPS in strange particle production, see Chapters 11 and 15, was followed by strong and irrefutable results confirming the quark-gluon plasma in heavy ion collisions at the higher RHIC energies, see Chapter 14, and the present day results at yet much higher energies at the LHC agree.

Chapter 13
The Path to Heavy Ions at LHC and Beyond

Hans H. Gutbrod

Abstract My appreciation of Rolf Hagedorn motivates me to look back at my more than 40 years of trial and error in relativistic heavy ion physics. More than once, wise colleagues helped me move forward to new and better understandings. Rolf Hagedorn was one of these important people. At first, I met him anonymously in the mid 1970s when reading his 1971 Cargèse Lectures in Physics, and later in person for many years in and around CERN. I wonder what this modest person would say about his impact on physics in this millennium. As he is not here to answer, I and others give our answers in this book. I focus my report on the beginning of the research program with relativistic heavy ions, the move to CERN-SPS and the development of the heavy ion collaboration at the CERN-LHC

13.1 Work at the Bevalac

For more than four decades, I have studied relativistic collisions of heavy nuclei with the goal to create matter at extreme density and temperature, as it may exist in Supernovae implosions, or in the Early Universe. It began in 1975, when I was working at the Lawrence Berkeley National Laboratory (LBL) Bevalac accelerator complex, using beams of 0.2-2GeV/c medium heavy ions.

Would nuclei be dense enough to create a compressed fireball in relativistic nuclear collisions, or would they just pass through each other, producing nuclear shock waves in each other? In 1974, an experiment with AgCl detectors claimed to have seen the shock waves, although with very low statistics and little particle identification for the reaction products.

My group wanted to measure nuclear shock waves employing electronic detectors of many sorts, but we did not find them. Instead we found coalescence of nucleons forming light clusters due to high density in phase space of the collisions. The inclusive proton spectra looked very thermal. This led to the formulation of the

FAIR Russia Research Centre, NRC "Kurchatov Institute" FSBI "SSC RF ITEP", B. Cheremushkinskaya, 25, Moscow, 117218, Russia; and GSI, Helmholtzzentrum für Schwerionenphysik, Planckstrasse 1, 64291 Darmstadt, Germany

nuclear fireball model, a very simple one. For a while our little model was the best in describing the experimental inclusive spectra.

Coming from low energy physics, the fireball was for me at first just a hot 'compound nucleus', but I learned quickly that at these relativistic energies particle and resonance production had to be included. This was when I 'met' Rolf Hagedorn through his 1971 Cargèse Lectures in Physics and that is how this story began. Our Bevalac collaboration drew the attention of Léon Van Hove, who was then director of CERN. In 1977, he visited me at LBL and listened to me, the youngster, explaining to him our experiment and our understanding at that time. I was proud to tell him that at the Bevalac we knew the work of a CERN theorist, named Rolf Hagedorn.

However, by 1977/78, I was also very frustrated: We had not seen any experimental proof that heavy ion collisions were more than just an ensemble of nucleon-nucleon collisions, as described in intra-nuclear cascade models of that time. I recognized fairly quickly that our small detector system was only able to measure a fraction of the particles coming from one collisions and that we needed a 4π coverage to measure each event fully to see possible collective phenomena in these nuclear collisions.

With beams of ions of mass up to 40, the measured signals of the first experiments were not much different from proton-proton collisions and an added value for the complexity and difficulties in the acceleration of heavy ions was not clear. Later, in HeliumHelium (mass 4+4) collisions at the ISR collider at CERN, no new physics was discovered either. All beginnings are difficult but this situation was outright catastrophic.

Then I came on invitation of Johann Rafelski and Rolf Hagedorn in early 1978 to visit CERN. Johann, a young nuclear theorist educated in Walter Greiner's group in Frankfurt, worked at CERN with Rolf Hagedorn and under the tutelage of Leon van Hove, Maurice Jacob and John S Bell. I saw that he recognized the striking opportunity that the CERN system of accelerators offered to the field of relativistic heavy ion physics. He was also the first nuclear theorist I met who recognized the paradigm shift of the new quark-gluon plasma (QGP) physics, carrying with his ideas his teacher and mentor of statistical and thermal physics, Rolf Hagedorn.

Through his Frankfurt-GSI (the heavy ion laboratory in Darmstadt) connection, Johann heard of my work and wanted to learn more about the experimental potential of Bevalac and the results obtained in my first experiments. During this visit, I learned about Hagedorn-Rafelski's recent work, the theory of finite size hadron gas. I did not yet in full realize that a revolution of the established particle physics wisdom was in the making. It certainly lifted my morale to learn that someone in a major laboratory of particle physics was paying attention to work by a few nuclear experimentalists who were struggling to find anything interesting in their Bevalac data.

Not anything interesting? I should note that at that time one of our Bevalac experimental competitors did announce a large entropy production in these nuclear collisions, which would have meant production of some new phase of matter produced at these low energies. However, my group could resolve this entropy puzzle by our detection of large amount of light nuclear clusters like deuterons, tritons,

^{3}He, and ^{4}He, which lowered the entropy production to 'normal' values. So, yes, at that time, not interesting.

Fortunately, thanks to generous support from our division leader Rudolf Bock at our home institute at GSI Darmstadt, my Bevalac collaboration could move quickly, reaching new heights. From 1978 to 1981, I constructed the "Plastic Ball" experiment at the Bevalac, the first electronic detector in relativistic nuclear physics with 4π coverage and particle identification. In 1982-84, Bevalac employed beams of Niobium (mass 93) and later of Gold ions (mass 197) at fixed-target energies from 200 MeV to 1 GeV per nucleon.

We promptly discovered two nuclear matter flow phenomena in the emission pattern of these collisions: the 'side-splash' and the 'squeeze-out.' The side splash was in the reaction plane, whereas the squeeze out was perpendicular to it. Inelastic scattering of spectator particles determined the reaction plane. These discoveries gave evidence for collective phenomena in relativistic heavy ion collisions, which could not be explained by standard nucleon-nucleon cascades in the collision.

Today these phenomena are called directed flow ($v1$) and elliptical flow ($v2$). These flow phenomena support the assumption that in the collision zone the nuclear participants interact with each other, building up pressure and thus, one surmises, matter of high density and temperature has been formed. These experimental data allowed us to extract first information of the equation of state of nuclear matter when compressed to near 2 to 3 times nuclear densities. In these initial experiments at the Bevalac, the energy available was comparable to Supernovae explosions, although our collisions were also generating a lot of heat, and thus, we believe, the compressed nuclear matter in these nuclear collisions was much hotter than in most extreme Supernovae.

Furthermore, the pion, and strange hadron, kaon and lambda production changed in these heavy ion collisions compared to proton-proton collisions, pointing to an ongoing hadron-chemistry inside this hot matter. At low energies, nucleons play the dominant part in the collision, therefore one talks of a 'nuclear fireball'. Perhaps the key lesson learned in these experiments was that one needs very large, massive heavy ions to form a dense nuclear fireball.

These results had established the relativistic heavy ion physics as a field where nuclear collisions produced collective features not known from *pp* collisions. Hot compressed matter had been discovered and it was clear that very heavy nuclei were needed to form and study its behaviour. These results opened the path to go to higher energies; that is, higher compression and temperatures. Only at a bit higher collision energies, Hagedorn and Rafelski suggested, quarks and gluons inside the nucleons start participating in the collision. This is how the actual push towards the experimental discovery of the QGP came to be – in my opinion this happened at CERN and one of the important forces was this theoretical work in the late 1970s.

Questions arose where and how to get heavy ion beams of much higher energies than at the Bevalac. We developed new ideas for accelerators at GSI, Darmstadt and at LBL, Berkeley. At Berkeley, we came up with the VENUS collider proposal, where head on collisions of 20 + 20 AGeV gold beams would go to very high densities and temperatures. This project was cancelled when the ISABELLE 200 +

200GeV p+p collider made space for the SSC project, the superconducting super collider which itself fell victim of US-national politics. However at BNL, the ready circular tunnel plus superconductor cooling system were left for further use. This ultimately became the collider project RHIC in 1983.

While all this was going on, Rudolf Bock, Reinhard Stock and I proposed to the German government to build a synchrotron with 100 Tesla meter rigidity (SIS 100) at GSI, similar in design and specifications to the AGS at Brookhaven and the PS at CERN. When looking back, one must say that fortunately for the field of relativistic heavy ions, our proposal was turned down and GSI built just a 'modern Bevalac'. I say fortunately since our physics objective, the discovery and study of QGP, meant that the SIS 100 was too small, and its demise meant that we were encouraged to seek elsewhere the required experimental capability.

13.2 . . . and at the SPS

With Reinhard Stock and 40 others I proposed a heavy ion program at the CERN accelerator complex. I offered to move our collaboration and our detector, the Plastic Ball, from LBL to CERN, and Reinhard Stock offered to bring his Bevalac Streamer Chamber group to CERN. In 1982 a memorandum of understanding was signed by the GSI, CERN and LBL to get heavy ions to CERN. The terms were that GSI promised to bring an electron cyclotron resonance (ECR) ion source, LBL a radio frequency quadrupole (RFQ) linear accelerator to the CERN site. Rudolf Bock (GSI), Herrmann Grunder (LBL), Reinhard Stock (Uni Marburg), and many others including myself, we proposed at first experiments with heavy ions at the CERN PS.

Our view was strongly supported by the director of CERN non-LEP research programs, Robert Klapisch, who, however, saw the higher energy potential of the SPS as much more adequate. A decision had to be made by CERN at the time that LEP program constraints were already putting the entire plan for the CERN heavy ion program at risk. In the end a miracle happened and we got heavy ions at the SPS. This was an excellent development, offering us access to the high energy range, and clearly separated our program from the AGS program at Brookhaven, which was proposed shorty after ours at CERN.

At the beginning of the SPS heavy ion program (with Oxygen and Sulphur beams) very few scientists expected to see similar matter flow features like at the Bevalac. Again the question was if nucleons in nuclei would be dense enough to allow the formation of a QGP. In fact quite a few thought that at SPS energies there would be transparency in nucleon flow, ruling out collective effects and the formation of a quark-gluon fireball. This thinking was of course not what Hagedorn and Rafelski saw as the most likely outcome of heavy ion collisions at SPS.

The first runs with Oxygen ions in SPS took place in Fall of 1986. One year later, we had Sulphur ions and more than 400 scientists participated in six experiments at CERN. There were several predictions for exciting physics discoveries: a) Johann Rafelski's strangeness enhancement, for which initially three experiments (NA35, NA36 and WA85) were specifically equipped, b) high temperatures, which my experimental collaboration WA80 took on to measure via direct photon spectroscopy, c) J/psi suppression which NA 38 set out to measure, d) Experiment NA34 addressed spectra of dileptons, a signature of dense hadronic matter.

All experiments had to master measuring the global event character, like impact parameter, multiplicity of charged particles, transverse energy flow, etc., which some did only after many years of trial and error. Hope was that we should find in the QGP temperatures above 'Hagedorn's limiting temperature', recognized as the freeze-out temperature for hadrons from a cooling down QGP. In nearly all SPS experiments, Rolf Hagedorn's limiting temperature was found in various measured hadron spectra.

The strangeness QGP signature was particularly successful. Strangeness and more specifically strange antibaryons were recognized and developed by J. Rafelski, at times in collaborations involving R. Hagedorn, B. Muller, M. Danos, as the key to the QGP discovery. In the early years, both QGP and abundant strangeness, were very "exotic" topics. As an example, Johann's strangeness presentation was relegated to the "exotica" section of the LBL conference proceedings in 1983 in company of 'Anomalons', a long forgotten false discovery. With a strong experimental program and clear objectives at SPS, very strong and diverse evidence for QGP was discovered in study of strange hadrons, in particular strange antibaryons.

My WA93 collaboration discovered flow phenomena in 200 AGeV S+Au collisions at a level of 5 to 10 times weaker than at the Bevalac. As mentioned earlier, this became then a real industry of $v1$ and $v2$ measurements, now extended to much higher orders.

My group's experimental series WA80/93/98 was keen to measure direct photons, developed exquisite technologies and methods to do so. Some predicted QGP-temperatures of up to 1 GeV in early stages of the SPS collisions. In retrospect we can say that this was impossible, as this would require huge compressed energy densities and very short thermalisation times. This is indeed impossible to achieve even at the LHC. In our initial optimism we hoped to measure these extreme conditions, but were realistic enough to prepare for low thermal photon yields of $\gamma_{\mathrm{thermal}}/\pi^0$ ratios of a few percent only. However, the extreme values never showed up, and we could only measure an upper limit for thermal photons from a plasma with a temperature of about 220 MeV. At RHIC, our WA98 photon spectrometer was employed again and could measure indeed direct photons telling a temperature of about 280 MeV of the QGP.

13.3 How Heavy Ions Got into LHC and the ALICE Was Born

I must admit that I like to create and build new things when physics asks for it and no other existing device can be scavenged or reused. So, as the research program at SPS evolved towards the announcement of the QGP discovery, some of my interests were already focused on future opportunities. As always, chance helps at the beginning: In 1983 at the relativistic heavy ion meeting at Brookhaven, I discussed with Carlo Rubbia topics of the future heavy ion collider at BNL, later called RHIC, when he told me: ‘You will get your collider at CERN, with enough energy for your physics case’. During all this discussion we were walking very swiftly back and forth through the corridors of BNL physics department, as swiftly as Carlo’s thoughts were rushing in.

At first, I did not understand what he was talking about. He argued that the quark physics signals became much clearer at the high energies of the S$p\bar{p}$S collider compared to the smaller collider at CERN, the ISR. He kept his promise once he was Director of CERN and installed the heavy ion option into the LHC project from early on, among many things by insisting on a two-in-one magnet solution for the LHC allowing matter on matter collisions instead of a cheaper $p\bar{p}$ mode that is matter-antimatter beams sharing only one vacuum chamber.

Fast forward to the Aachen meeting in 1990: a small group (Ch. Fabian, H. Gutbrod, H. Specht, W. Willis et al.) sketched a detector concept comprising a large solenoid, coupled with one dipole at each end with full particle tracking. This was similar to the smaller 4π detector the group had proposed several years earlier for the BNL collider.

From 1991 on, a small group of initially about 20 persons met at CERN regularly to work on a proposal for a dedicated heavy ion experiment at the LHC. In parallel we had to build and run our lead beam experiments at the SPS. From the start of discussing the concept of a dedicated heavy ion experiment at the LHC we had two concepts, one having a silicon tracker inside of a superconducting thin solenoid, pushed by Jürgen Schukraft, and one having all detectors inside of a large solenoid, pushed by myself.

Our consensus was that the large magnet solution required a much larger budget, but that the small magnet risked producing unwanted tracks in the detectors in the outside field free region with worse resolution due to multiple scattering in the magnet material. I must add further, that we had little guidance about the multiplicity of produced particles. Theoretical predictions ran from $dN/dy = 2000$ to 8000 charged particles and of course as history has shown these as function of energy and collision centrality were only a bit off the actual result. So we needed to be prepared for the worst, ten thousand particles to be recognized and identified. I did not hesitate to push for the more expensive solution: At Berkeley I learned that one had to make adequate investments on the experimental side in order to make proper use of the costly beam time of the accelerator system itself.

At the LHC meeting at Evian-les-Bains in March of 1992, Jürgen Schukraft presented the small magnet project. The omission of the big magnet solution was due to the simple fact that there was no money to create a big detector from scratch. In the following months, I tried to find a cost effective solution: seeking a large magnet I looked first at the DEPLPHI solenoid magnet, which was unfortunately too small for the track density we anticipated in nuclear collisions.

Then, one late Spring day in 1992 in the CERN cafeteria when having lunch with Johann Rafelski I outlined to him in the presence of Paulo Giubellino and Lars Leistam and a few others what I saw as the real opportunity, the implementation of heavy ions in the L3 experiment. Its huge magnet was perfect for our purpose. At that time Sam Ting was proposing his own L3P experiment, i.e. an upgraded L3 setup for the LHC program. It was to me obvious to name my proposed set-up L3 H.I., hoping to interest the L3 team in the H.I. program from the beginning.

On July 2nd 1992, in one of our 'H.I. at LHC' proto-collaboration meetings at CERN, the question was raised: Should we have two parallel simulations, one for the small magnet and one for the large magnet? The decision was 'yes', we should follow both paths, and I was preparing the large magnet project, interacting with L3. Sam Ting asked me to present this proposal only after a decision was clear about L3P. In the Fall of 1992, the fate of L3P was clear, and Sam invited me to give a talk to his collaboration. I started my presentation saying: 'I feel somewhat awkward coming here wanting to steal your beautiful experiment.' From then on we got strong support from the L3 team as well as from the CERN management.

On February 1, 1993 the small magnet scenario was dropped and the L3 H.I. concept was adopted. The transfer of ownership of the L3 magnet and the infrastructure of the L3 site from the L3 collaboration to the H.I. collaboration was performed at a seafood dinner hosted by Jürgen Schukraft in a restaurant at Ferney-Voltaire and in March 1993, the Letter of Intent for 'A Large Ion Collider Experiment' (ALICE, see Fig. 13.1) was submitted as CERN/LHCC/93-16 LHCC/I4 on the basis of L3 H.I. . Jürgen Schukraft pushed it ahead towards completion and to many years of successful operation with striking new results and discoveries, published in a large number of publications.

One must note with delight, that the heavy ion physics at LHC has seen a tremendous strengthening due to the successful participation of the two *pp* experiments ATLAS and CMS in the heavy ion runs. Competition in science is always good!

13.4 Future Facilities

Complementary to the scientific programs at RHIC and LHC, where the thrust is to create a fireball with vanishing baryon chemical potential, a new scientific push has been undertaken to study in detail the equation of state of highly compressed baryon-rich matter. In highly compressed 'cold' nuclear matter - as it may exist in the interior of neutron stars - the baryons lose their identity and dissolve into quarks and gluons. The critical density at which this transition occurs, however, is not known. The same is true for the entire high-density area of the phase diagram. At very high densities and low temperatures, beyond the deconfinement transition, a new phase is expected: the quarks are correlated and form a color superconductor. At the "critical point" the deconfinement/chiral phase transition is predicted to change its character.

I can say that Rolf Hagedorn's work on dense nuclear and hadron matter is pursued today with more vigor than ever before. Starting about 3 years ago, two new facilities have been under construction, and we are witnessing renewed interest in this energy range at CERN-SPS:

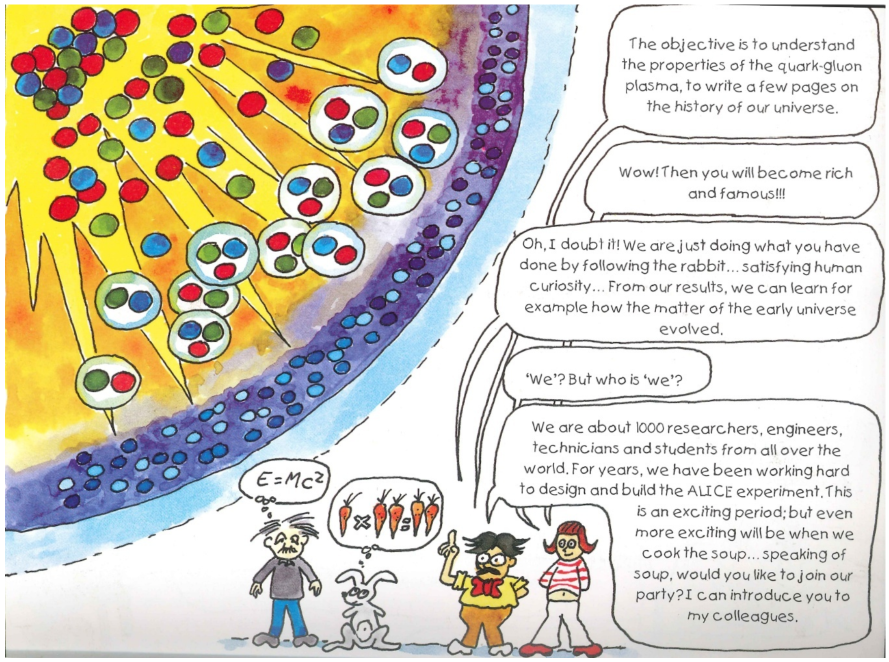


Fig. 13.1 ALICE at CERN explains itself in the 2004 outreach brochure – the backpage. Cartoonist: Jordi Boixader; Scenario and text: Federico Antinori, Hans de Groot, Catherine Decasse, Yiota Foka, Yves Schutz and Christine Vanoli.

- At Darmstadt, Germany, the international FAIR facility is focusing on hadrons in compressed baryonic matter in the CBM experiment, and on hadron structures in proton-anti-proton reactions in the PANDA experiment. High precision measurements will allow the determination of the lifetime and mass of hadrons to new precisions. (Additional FAIR scientific programs are in nuclear structure and astrophysics, atomic physics, plasma physics and applied physics. I had the pleasure to lead this project until 2008).
- At Dubna, Russia, the project NICA is going to study baryon rich systems in a fixed target experiment BMN and in an ion collider experiment MPD at NICA.
- At CERN SPS a pilot program uses the full available low energy range exploring the discovery potential of the above two new experimental facilities.
- We recall the AGS at BNL before RHIC came on-line nearly reached this energy domain, albeit with limited instrumental detector capacity and very limited beam time.

The heavy-ion collisions at FAIR and NICA energies permit the exploration of the "terra incognita" of the QCD phase diagram in the region of high baryon densities. At both facilities FAIR and NICA, fully dedicated research with nuclear collisions of highest intensities will allow us to create highest baryon densities, to explore the properties of super-dense nuclear matter, to search for in-medium modification

Fig. 13.2 Forefront Standing: left Rolf Hagedorn received his 75th anniversary gift from the organizers of the Divonne Conference, represented by Hans H. Gutbrod standing on right. Next to Hagedorn's right standing Maurice Jacob. Left of Hagedorn sitting at image edge: Luigi Sertorio, *Image credit: CERN Image 199406-067-014*

of hadrons, the confinement of quarks in hadrons, to shed light on the restoration of chiral symmetry, to give insight into the origin of hadron masses, to look for the transition from dense hadronic matter to quark-gluon matter, and for the critical endpoint in the phase diagram of strongly interacting matter, and finally to provide 'hopefully' understanding of the structure of neutron stars and the dynamics of core-collapse supernovae.

13.5 Epilogue

In 1993/94 I worked with Johann Rafelski to organize in Divonne, not too far from CERN, the fest to celebrate Hagedorn's 75th Birthday (Fig. 13.2 and Fig. 13.3), with his consent. Johann invited me to join forces with the otherwise theoretical team he formed with other friends and admirers of Hagedorn: Steven Frautschi, Jean Letessier and Helmut Satz. He explained, he wants the meeting and the dedicated book volume to have the subtitle "Theory and Experiment" in the spirit of Hagedorn. After a splendid celebration week at the end of June 1994, we worked together to publish 'Hot Hadronic Matter', a thick 550 pages volume dedicated to Rolf Hagedorn with the cornerstone observables discussed in depth. In fact the first extensive discussion of the design of the LHC ALICE experiment is presented in this volume on my behest.

I have spent the last 20 years at three laboratories directing, leading, and building many instruments that today form the backbone of the world-wide effort to study and explore the QGP phase of matter. I have seen the first step of the heavy ion program of research at CERN completed with the announcement of the discovery of the new state of matter in early 2000. As predicted by Carlo Rubbia, the high energies at LHC have given through ALICE, ATLAS and CMS, deep insight into this new deconfined phase of matter.

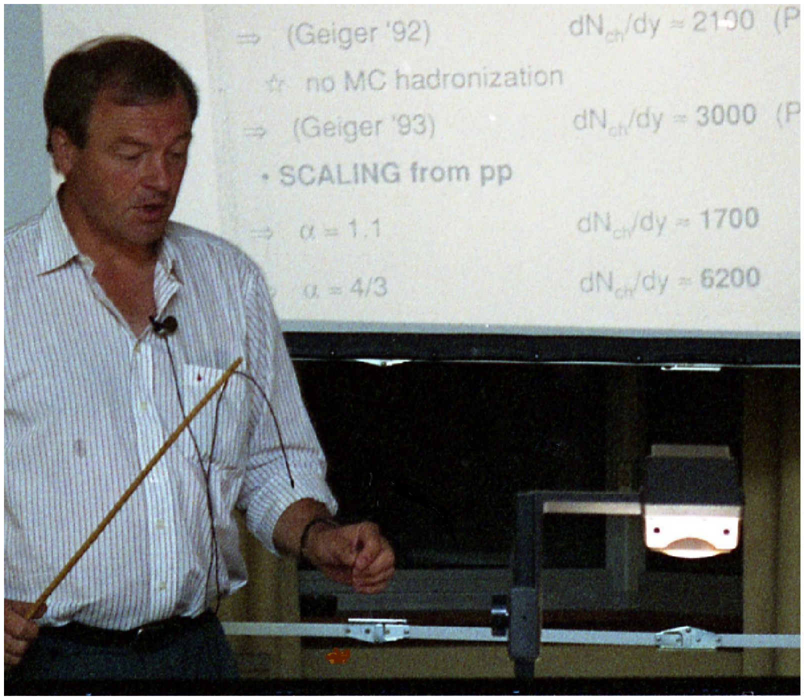


Fig. 13.3 Hans Gutbrod lecturing at Divonne on future high energy collider *AA* experiments. We note on the screen the results of Klaus Geiger, see Fig. 14.2. Credit: CERN Image 1994-06-068-008

Every year exciting results appear due to instrumental advances: We gain a greater capability to observe and evaluate the large and diverse particle multiplicity. Thus we can address today more precisely and convincingly the established observables of the plasma phase and in doing this we begin to understand plasma properties, evolution history, and the mechanisms governing the hadronization process. We seek also to understand in detail the thresholds in volume size and energy that govern formation of the new deconfined state of matter.

It is abundantly clear to me that the program of research as it is constructed and executed presently relies on principles and ideas that were recognized in the first years at CERN, when the SPS program was developed, and which relies in a great measure on the legacy of Rolf Hagedorn. It is a great pleasure for me to have contributed a little bit to establishing a future for young scientists to go much further than we could do in the past. Rolf Hagedorn has built the base for this physics.

Chapter 14
A New Phase of Matter: Quark-Gluon Plasma Beyond the Hagedorn Critical Temperature

Berndt Müller

Abstract I retrace the developments from Hagedorn's concept of a limiting temperature for hadronic matter to the discovery and characterization of the quark-gluon plasma as a new state of matter. My recollections begin with the transformation more than 30 years ago of Hagedorn's original concept into its modern interpretation as the critical temperature separating the hadron gas and quark-gluon plasma phases of strongly interacting matter. This was followed by the realization that the QCD phase transformation could be studied experimentally in high-energy nuclear collisions. I describe here my personal effort to help develop the strangeness experimental signatures of quark and gluon deconfinement and recall how the experimental program proceeded soon to investigate this idea, at first at the SPS, then at RHIC, and finally at LHC. As is often the case, the experiment finds more than theory predicts, and I highlight the discovery of the perfectly" liquid quark-gluon plasma at RHIC. I conclude with an outline of future opportunities, especially the search for a critical point in the QCD phase diagram.

14.1 From Hagedorn to Quark-Gluon Plasma

Deconfinement of quarks and gluons

While successfully describing many features of multiparticle production at the energies accessible in the late 1960s, Hagedorn's Statistical Bootstrap Model [1] with its exponentially growing mass spectrum of hadrons posed a quandary for cosmology [2]. The discovery of the cosmic microwave background in 1965 had provided unambiguous evidence for the hot Big Bang model. By tracing back the cosmic evolution to very early times it was possible to conclude that the universe must have

Department of Physics, Duke University, Durham, NC 27708-0305, USA
Brookhaven National Laboratory, Upton, NY 11973, USA

experienced temperatures in excess of 200 MeV at times less than 10 μs after the initial Big Bang. But what was the structure of the matter that filled the Universe at such early times? What was its equation of state?

An exponential mass spectrum implied that the equation of state of hadronic matter has a singularity at the Hagedorn temperature, with empirical values in the range 150 MeV $< T_{\mathrm{H}} <$ 200 MeV. Asking what the structure of matter at temperatures greater than T_{H} is was meaningless in the Statistical Bootstrap Model. The resolution of this quandary began with Collins and Perry's observation [3] in early 1975 that the asymptotic freedom of QCD implies that quarks are weakly interacting at short distances and therefore matter at very large quark densities should be composed of unconfined quarks. However, although they note that this argument should apply to matter in the early universe, their discussion is mostly focused on cold QCD matter.

Later in the same year, Cabibbo and Parisi [4] proposed an interpretation to the singularity in the equation of state of Hagedorn's hadronic resonance gas as the point where strongly interacting matter changes from a gas of hadrons to a colored plasma of quarks and gluons. The Hagedorn temperature thus acquired the meaning of the critical temperature T_c at which the composition of strongly interacting matter undergoes a discontinuous transition.[1] Quantitative predictions were impossible in the 1970s because of the lack of reliable mathematical or numerical techniques to solve QCD.

Lattice QCD results

Starting in the early 1980s, Monte-Carlo simulations of the partition function of lattice QCD, first for the pure gauge theory and later for full QCD, made it possible to calculate the equation of state of strongly interacting matter *ab initio*. These calculations, which have recently converged to a definitive result [5–7], showed that matter composed of hadronic resonances is not separated from the quark-gluon plasma by a discontinuous phase transition in the absence of a baryon excess. However, a quasi-critical temperature $T_c \approx 155$ MeV can be defined as the temperature at which the chiral susceptibility – the susceptibility associated with the scalar quark density $\langle\bar{\psi}\psi\rangle$ – peaks. The smooth cross-over is expected to turn into a first-order phase transition in the traditional sense of statistical physics for matter with a large baryon excess.

The lattice simulations showed that Hagedorn's model of a hadron resonance gas with an exponentially growing mass spectrum describes the equation of state of QCD matter and many other observables very well for temperatures below T_c. The precision of the lattice QCD simulations is now good enough to distinguish between the equation of state of a hadron gas made up of the resonances tabulated in the

[1] We now know that the exponentially growing mass spectrum of QCD is not related to a second order phase transition, as Cabibbo and Parisi surmised, but connected with the fact that QCD has an (approximate) string dual. In fact, lattice QCD has conclusively shown that the equation of state of QCD at zero or small net baryon density does not exhibit a singularity.

Particle Data Book or that of a Hagedorn resonance gas. The numerical results point to a continuation of the exponential growth of the hadron mass spectrum beyond the reach of direct detection of resonances and thus support Hagedorn's hadronic bootstrap model [8]. The implied existence of many unknown hadron resonances may also be present in the strange baryon sector [9].

Above T_c the density of states grows much less rapidly and eventually approaches that of a perturbatively interacting quark-gluon plasma composed of massive quasi-particles, confirming the notion that the Hagedorn temperature signals the transition from a hadron resonance gas to a new state of matter.

Hot nuclear matter

The next critical step was the realization, arising most prominently from discussions in the CERN Theory Division[2], that temperatures in the range of T_c and even beyond could be created in the laboratory by colliding heavy atomic nuclei at sufficiently high energies.

The experimental study of relativistic heavy ion collisions with stationary targets had commenced at the Bevalac in the mid-1970s, but the energies available there were recognized to be insufficient to reach T_c. The CERN SPS could provide much higher energies, and back-of-the-envelope calculations suggested that temperatures near and above T_c would be reached if the nuclear matter in the colliding nuclei thermalized rapidly. Hagedorn and Rafelski extended the Statistical Bootstrap Model to matter with a baryon excess and found that under certain assumptions the equation of state exhibited a first-order phase transition [12].

I had the good fortune of meeting Hagedorn during several visits with Johann at CERN during this formative period in the late 1970s. My conversations with them inspired my own interest in hot QCD and soon thereafter resulted in our joint work on the thermal properties of the QCD vacuum [13] and on particle production with exact symmetry in proton-antiproton annihilation [14]. What impressed me most on these occasions was Hagedorn's willingness to share his thoughts with a young scientist without imposing on him. One puzzling aspect of the experimental observation of thermal particle emission that is still occupying theorists today – how a large fraction of the kinetic energy carried by the incident particles could be thermalized within a time of order 1 fm/c – led to my interest in the chaotic properties of non-abelian gauge theories. I vividly recall Hagedorn's excitement after he listened to my talk about our numerical studies of dynamical chaos of the Yang-Mills field at the workshop in Divonne [15], see Fig. 14.1.

---

[2] What distinguished these discussions from other theoretical speculation in the mid-1970s was that the focus was on *thermal* properties of strongly interacting matter, rather than properties of compressed baryonic matter (see e.g. Lee [10, 11].)

Fig. 14.1 Berndt Müller at Divonne Hagedorn Fest 1994. *Photo: Lucy Carruthers.*

14.2 Path to Discovery of the QGP

QGP observables

The biggest challenge on the way to discovery was finding signatures that could provide evidence that nuclear matter had made the transition to a quark-gluon plasma for a brief period during the collision. One either had to look at penetrating probes, such as photons and lepton pairs [16], that could escape from the hot fireball, or at probes that retained their identity under the action of the strong interactions in the final state, such as quark flavor.

Shuryak took the matter further by evoking quark and gluon degrees of freedom in *pp* reactions and focusing on electromagnetic probes and charm quarks as signatures for the formation of a thermal QCD plasma [17, 18]. Rafelski, in collaboration with Hagedorn, Danos, and myself, focused on strange quarks whose mass is sufficiently low for them to be produced thermally in the quark-gluon plasma [19] (see Chapters 27, 31–33).

The strangeness argument was not simply that strange quarks and antiquarks would be produced abundantly at temperatures above T_c, but that baryons containing multiple strange quarks would be produced copiously and in chemical equilibrium when the quark-gluon plasma hadronizes by recombination of the deconfined quarks into hadrons. A calculation of thermal strange quark pair production in the quark-gluon plasma [20] confirmed that flavor equilibrium could, indeed, be reached on the time scales of a relativistic heavy ion collision and showed that thermal gluons played a crucial role in the flavor equilibration process.

Following on the recognition of the abundant strangeness in quark-gluon plasma, Johann and I embarked on the task of developing a bulk hadronization model that would enable us to make quantitative predictions for the strange antibaryon signature of the quark-gluon plasma, see Fig. 33.5. Our effort grew over two years, in collaboration with Peter Koch, into a Physics Reports article [21]. Among the highlights of this work is the development of the recombination and fragmentation-recombination models of quark-gluon hadronization that in slightly modified form remain in use today [22]. We enforced conservation laws, assured increase of entropy, and quantified the production of strange (anti-)baryons with their strangeness content. These developments set clear experimental goals for the forthcoming SPS strangeness experiments which are further discussed below and in the contribution of Emanuele Quercigh, Chapter 15.

SPS results

The heavy ion experiments at the SPS, which commenced in 1986/87, impressively confirmed these ideas. The chemical composition of the hadrons emitted from the collisions can be well described by a chemical near equilibrium gas at a temperature close to T_c and a baryon chemical potential that varies strongly with the collision energy [23]. The strong enhancement and full chemical equilibration of baryons and anti-baryons containing multiple strange quarks [24, 25] could only be explained if hadrons containing valence quarks of all three light flavors were born into thermal abundances [26–28].

However, the SPS data did not provide other corroborating evidence for the existence of a thermal phase of matter at temperatures above T_c from which these hadrons formed by statistical emission. The (unpublished) CERN announcement of a new state of matter [29] in 2000 was thus greeted with skepticism by many physicists. Experiments with heavy ion collisions at much higher energies were needed to resolve this issue.

Experiments at RHIC

Commencing at RHIC in year 2000, these experiments allowed access to a new kinematic domain, in which the interactions among quarks and gluons contained in the colliding nuclei (see: Ref. [30], Fig. 14.2) cause the produced matter to be imprinted from the start with a nearly boost invariant longitudinal flow profile. An analytical solution of relativistic hydrodynamics for this initial condition had been found by Bjorken [31], and it provided the basis for a systematic investigation of the collective properties of the matter formed in the nuclear collisions [32–35]. The fact that the transverse geometric profile of the reaction zone and the initial energy density fluctuations from event to event could be correlated with the patterns observed in the collective flow of the emitted hadrons made it possible to pin down the transport properties of the expanding matter, which was shown to have an extraordinarily low shear viscosity, relative to its entropy density [36–38]. The matter was thus shown to be a liquid at temperatures well above T_c.

Fig. 14.2 Klaus Geiger (on right) in conversation with Berndt Müller, at the Divonne Hagedorn Fest, June 1994. Klaus had the bad luck to take on Wednesday, 2 September 1998, the Swissair flight 111 from JFK to Geneva. The aircraft crashed into the Atlantic Ocean. Shuttling between BNL and CERN, Klaus gave his life to heavy ions, hadrons and QGP. *Photo Credit: Lucy Carruthers*.

A detailed study of the subtle variations of the flow profile between different hadron species revealed that these variations disappeared when all hadrons were assumed to be formed by recombination of deconfined, collectively flowing quarks when the matter cooled below T_c [39]. Together, these observations provided strong evidence for the notion that the matter formed in nuclear collisions at RHIC is, indeed, a plasma of deconfined quarks and gluons, which behaves as a nearly inviscid liquid and decays by the emission of hadrons in chemical and thermal equilibrium. Because the matter is already expanding very rapidly when the transition to a hadron gas occurs, many observables are nearly unaffected by final-state interactions among hadrons. The low viscosity of the liquid quark-gluon plasma implies that the interactions among quarks and gluons contained in it are strong. Other observations, such as the strong suppression of high-momentum hadrons and of charmonium, support this conclusion (for early reviews, see: [40, 41]).

Experiments at LHC

Experiments at even higher energies at the LHC have impressively confirmed the nature of QCD matter above T_c as a strongly coupled, liquid quark-gluon plasma [42]. A careful analysis of the LHC data revealed that the average strong coupling at the higher energy density reached at LHC is slightly weaker than at RHIC [43], in accordance with the running of α_s with temperature. The reduced coupling is also reflected in a somewhat larger shear viscosity-to-entropy density ratio [44].

In addition to consolidating the insights gained at RHIC, the much higher energy available at LHC permit more detailed studies of the event-by-event fluctuations of the collective flow pattern, which reflect the quantum fluctuations of the initial energy density distribution. Enabled by the design of the LHC detectors, the higher energy also allows for precise studies of the phenomenon of jet quenching that was first discovered at RHIC. And finally, the large yield of primordially produced charm quarks at LHC results in abundant late-stage recombination of charm-anticharm quark pairs into charmonium, providing additional evidence for the deconfinement of quarks in the QCD plasma phase.

Beam energy scan at RHIC

How far down in beam energy does the phenomenology discovered and established at RHIC persist? Where is the threshold below which no quark-gluon plasma is formed? Did the SPS experiments produce a quark-gluon plasma? In order to address these open questions, RHIC has recently collided heavy ions at lower energies, down to $\sqrt{s_{\mathrm{NN}}} = 7.7$ GeV. An extensive analysis of the data gathered in this beam energy scan is now available [45, 46]. It shows that the matter produced in collisions down to the top SPS energy, $\sqrt{s_{\mathrm{NN}}} = 19.6$ GeV, exhibits some of the same characteristics as that produced at the top RHIC energy, $\sqrt{s_{\mathrm{NN}}} = 200$ GeV.

However, there are noticeable differences. Matter produced at the lower beam energies contains a larger excess of baryons resulting in a different chemical composition of the emitted hadrons; energetic hadrons are no longer suppressed at lower energies; and no direct photon signal has been observed. Thus it is quite likely that the CERN experiments succeeded in breaking through the thermal barrier of the Hagedorn temperature, but it is still unclear what kind of baryon-rich matter they produced and whether it exhibited collective behavior at the parton level. Theoretical models that can more reliably describe nuclear reactions at these lower energies will be needed to finally address this issue.

Next steps

Where do we go from here? Two major questions remain to be answered: (1) Is there a critical point in the phase diagram of QCD matter where the cross-over from hadron resonance gas to the quark-gluon plasma turns into a true phase transition, and where is it located in T and μ? (2) What are the effective constituents of the liquid quark-gluon plasma?

The first question will be addressed in a second, high statistics beam energy scan that is planned to be carried out in 2019–20 at RHIC after a luminosity upgrade of the collider at low beam energies. Physicists will then look for telltale signs of a phase transition, including critical fluctuations in baryon number or large event-by-event fluctuations caused by spinodal decomposition of the matter at the phase boundary. A recent discussion of the theoretical and experimental challenges of locating the QCD critical point can be found in [47].

Addressing the second question requires probes that are sensitive to the structure of the quark-gluon plasma at shorter than thermal length scales. Two such probes are heavy quarks and jets. The experiments at LHC and now at RHIC are equipped with powerful vertex detectors that can identify hadrons containing heavy quarks. They will study the transport of c and b quarks in the plasma in great detail and hopefully detect clues to its internal structure. Jets explore multiple length scales as they develop inside the matter after the initial hard scattering event. Extensive jet measurement programs, which are already underway at the LHC, are planned for RHIC in the decade ahead [48].

14.3 Outlook and Conclusions

Our understanding of the structure and properties of hadronic matter at high energy density has made tremendous progress since the days when the question first arose in full urgency in the late 1960s, and remarkable discoveries have been made along the way. We have established that Hagedorn's gas of hadron resonances turns into a liquid quark-gluon plasma when heated above 155 MeV, quite an extraordinary phenomenon in itself. We have discovered a liquid that comes very close to the quantum bound on the shear viscosity imposed by unitarity. And we have learned that the statistical and collective properties of the flowing quark-gluon plasma get imprinted onto the emitted hadrons in a characteristic way that makes it possible to experimentally determine the thermal and chemical properties of the QCD phase boundary. Rolf Hagedorn would surely be satisfied to witness that the questions he helped pose fifty years ago have proved to be so extraordinarily fertile.

Acknowledgements I thank W. A. Zajc, J. Rafelski, and U. Heinz for constructive comments. This work was supported by Grant no. DE-FG02-05ER41367 from the U. S. Department of Energy.

References

1. R. Hagedorn: Statistical thermodynamics of strong interactions at high-energies, Nuovo Cim. Suppl. **3**, 147 (1965)
2. K. Huang and S. Weinberg: Ultimate temperature and the early universe, Phys. Rev. Lett. **25**, 895 (1970)
3. J. C. Collins and M. J. Perry: Superdense Matter: Neutrons Or Asymptotically Free Quarks?, Phys. Rev. Lett. **34**, 1353 (1975)
4. N. Cabibbo and G. Parisi: Exponential Hadronic Spectrum and Quark Liberation, Phys. Lett. B **59**, 67 (1975)

5. S. Borsanyi, G. Endrodi, Z. Fodor, A. Jakovac, S. D. Katz, S. Krieg, C. Ratti and K. K. Szabo: The QCD equation of state with dynamical quarks, JHEP **1011**, 077 (2010)
6. S. Borsanyi, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg and K. K. Szabo: Full result for the QCD equation of state with 2+1 flavors, Phys. Lett. B **730**, 99 (2014)
7. A. Bazavov *et al.* [HotQCD Collaboration]: Equation of state in (2+1)-flavor QCD, Phys. Rev. D **90**, no. 9, 094503 (2014)
8. A. Majumder and B. Müller: Hadron Mass Spectrum from Lattice QCD, Phys. Rev. Lett. **105**, 252002 (2010)
9. A. Bazavov, H.-T. Ding, P. Hegde, O. Kaczmarek, F. Karsch, E. Laermann, Y. Maezawa and S. Mukherjee *et al.*: Additional Strange Hadrons from QCD Thermodynamics and Strangeness Freezeout in Heavy Ion Collisions, Phys. Rev. Lett. **113**, no. 7, 072001 (2014)
10. *Report of the workshop on BeV/nucleon collisions of heavy ions – how and why*, Bear Mountain, New York, Nov. 29-Dec. 1, 1974 (BNL-AUI, 1975).
11. T. D. Lee: Abnormal Nuclear States and Vacuum Excitations, Rev. Mod. Phys. **47**, 267 (1975)
12. R. Hagedorn and J. Rafelski: Hot Hadronic Matter and Nuclear Collisions, Phys. Lett. B **97**, 136 (1980)
13. B. Müller and J. Rafelski: Temperature Dependence of the Bag Constant and the Effective Lagrangian for Gauge Fields at Finite Temperatures, Phys. Lett. B **101**, 111 (1981)
14. B. Müller and J. Rafelski: Role of Internal Symmetry in $p\bar{p}$ Annihilation, Phys. Lett. B **116**, 274 (1982)
15. B. Müller: Colored Chaos, in: *Hot Hadronic Matter: Theory and Experiment*, Divonne 1994, J. Letessier, H.H. Gutbrod and J. Rafelski, eds. NATO-ASI **346**, pp. 171-180, (Plenum Press, New York 1995)
16. E. L. Feinberg: Direct Production of Photons and Dileptons in Thermodynamical Models of Multiple Hadron Production, Nuovo Cim. A **34**, 391 (1976)
17. E. V. Shuryak: Theory of Hadronic Plasma, Sov. Phys. JETP **47**, 212 (1978) [Zh. Eksp. Teor. Fiz. **74**, 408 (1978)]
18. E. V. Shuryak: Quark-Gluon Plasma and Hadronic Production of Leptons, Photons and Psions, Phys. Lett. B **78**, 150 (1978)
19. J. Rafelski and R. Hagedorn: From Hadron Gas to Quark Matter. 2., pp. 253-272 In *Bielefeld 1980, Statistical Mechanics Of Quarks and Hadrons* H. Satz, edt. (Amsterdam 1980)
20. J. Rafelski and B. Müller: Strangeness Production in the Quark-Gluon Plasma, Phys. Rev. Lett. **48**, 1066 (1982) [Erratum-ibid. **56**, 2334 (1986)]
21. P. Koch, B. Müller and J. Rafelski, Strangeness in Relativistic Heavy Ion Collisions, Phys. Rept. **142**, 167 (1986).
22. R. J. Fries: Quark Recombination in Heavy Ion Collisions, PoS CERP **2010**, 008 (2010)
23. F. Becattini, M. Gazdzicki, A. Keranen, J. Manninen and R. Stock: Chemical equilibrium in nucleus nucleus collisions at relativistic energies, Phys. Rev. C **69**, 024905 (2004)
24. E. Andersen *et al.* [WA97 Collaboration]: Strangeness enhancement at mid-rapidity in Pb Pb collisions at 158-A-GeV/c, Phys. Lett. B **449**, 401 (1999)
25. F. Antinori *et al.* [WA97 Collaboration]: Strangeness enhancement at mid-rapidity in Pb Pb collisions at 158-A-GeV/c: A comparison with VENUS and RQMD models, Eur. Phys. J. C **11**, 79 (1999)
26. J. Letessier and J. Rafelski: Evidence for QGP in Pb Pb 158-A-Gev collisions from strange particle abundances and the Coulomb effect, Acta Phys. Polon. B **30**, 153 (1999)
27. R. Stock: The parton to hadron phase transition observed in Pb + Pb collisions at 158-GeV per nucleon, Phys. Lett. B **456**, 277 (1999)
28. U. W. Heinz: Primordial hadrosynthesis in the Little Bang, Nucl. Phys. A **661**, 140 (1999)
29. U. W. Heinz and M. Jacob: Evidence for a new state of matter: An Assessment of the results from the CERN lead beam program, (CERN February 2000 press release)

30. K. Geiger and B. Müller, Dynamics of parton cascades in highly relativistic nuclear collisions, Nucl. Phys. B **369**, 600 (1992)
31. J. D. Bjorken: Highly Relativistic Nucleus-Nucleus Collisions: The Central Rapidity Region, Phys. Rev. D **27**, 140 (1983)
32. I. Arsene *et al.* [BRAHMS Collaboration]: Quark gluon plasma and color glass condensate at RHIC? The Perspective from the BRAHMS experiment, Nucl. Phys. A **757**, 1 (2005)
33. K. Adcox *et al.* [PHENIX Collaboration]: Formation of dense partonic matter in relativistic nucleus-nucleus collisions at RHIC: Experimental evaluation by the PHENIX collaboration, Nucl. Phys. A **757**, 184 (2005)
34. B. B. Back *et al.* [PHOBOS Collaboration]: The PHOBOS perspective on discoveries at RHIC, Nucl. Phys. A **757**, 28 (2005)
35. J. Adams *et al.* [STAR Collaboration]: Experimental and theoretical challenges in the search for the quark gluon plasma: The STAR Collaboration's critical assessment of the evidence from RHIC collisions, Nucl. Phys. A **757**, 102 (2005)
36. D. Teaney: The Effects of viscosity on spectra, elliptic flow, and HBT radii, Phys. Rev. C **68**, 034913 (2003)
37. P. Romatschke and U. Romatschke: Viscosity Information from Relativistic Nuclear Collisions: How Perfect is the Fluid Observed at RHIC?, Phys. Rev. Lett. **99**, 172301 (2007)
38. H. Song, S. A. Bass, U. Heinz, T. Hirano and C. Shen: 200 A GeV Au+Au collisions serve a nearly perfect quark-gluon liquid, Phys. Rev. Lett. **106**, 192301 (2011) [Erratum-ibid. **109**, 139904 (2012)]
39. R. J. Fries, B. Müller, C. Nonaka and S. A. Bass: Hadronization in heavy ion collisions: Recombination and fragmentation of partons, Phys. Rev. Lett. **90**, 202303 (2003) and Hadron production in heavy ion collisions: Fragmentation and recombination from a dense parton phase, Phys. Rev. C **68**, 044902 (2003)
40. M. Gyulassy, The QGP discovered at RHIC, in: *Structure and Dynamics of Elementary Matter* (Plenum Press, Springer 2004) [Proceedings, NATO Advanced Study Institute, Camyuva-Kemer, Turkey, September 22-October 2, 2003, p. 159, W. Greiner et al (Eds.)]
41. B. Müller and J. L. Nagle: Results from the relativistic heavy ion collider, Ann. Rev. Nucl. Part. Sci. **56**, 93 (2006)
42. B. Müller, J. Schukraft and B. Wyslouch, First Results from Pb+Pb collisions at the LHC, Ann. Rev. Nucl. Part. Sci. **62**, 361 (2012)
43. B. Betz and M. Gyulassy: Examining a reduced jet-medium coupling in Pb+Pb collisions at the Large Hadron Collider, Phys. Rev. C **86**, 024903 (2012)
44. C. Gale, S. Jeon, B. Schenke, P. Tribedy and R. Venugopalan: Event-by-event anisotropic flow in heavy-ion collisions from combined Yang-Mills and viscous fluid dynamics, Phys. Rev. Lett. **110**, 012302 (2013)
45. P. Sorensen: Beam Energy Scan Results from RHIC, J. Phys. Conf. Ser. **446**, 012015 (2013).
46. N. R. Sahoo [STAR Collaboration]: Recent results on event-by-event fluctuations from the RHIC Beam Energy Scan program in the STAR experiment, J. Phys. Conf. Ser. **535**, 012007 (2014)
47. R. V. Gavai: QCD Critical Point: The Race is On, Pramana **84**, 757 (2015)
48. A. M. Sickles *et al.* [PHENIX Collaboration]: The Physics of sPHENIX, J. Phys. Conf. Ser. **446**, 012050 (2013)

Chapter 15
Reminscenses of Rolf Hagedorn

Emanuele Quercigh

Abstract This is a personal recollection of the influence that Rolf Hagedorn had on the launch of the CERN heavy-ion program and on the physics choices made by my colleagues and myself in that context.

15.1 Many Years Ago

In 1964, as a CERN Fellow I started doing research on hadron physics, using at first bubble chambers and then electronic detectors. Like many other Fellows I was able to benefit from the vigorous CERN academic training program and from its teachers, all of whom were excellent physicists. There I met Rolf Hagedorn for the first time and enjoyed his lectures as well as his "Yellow Reports". His lectures were deep and clear. His reasoning was precise and very rigorous, yet he was patient with us and had a sense of humor. For example, once at the beginning of a lecture, he told us about a competition between ethologists of various nationalities for the best essay about "the elephant". While all the others described some facet of the elephant‘s personality, such as its character, its mental and physical capabilities as well as its elegance or its love-life, the German competitor‘s essay was entitled: "On the definition of the elephant". Hagedorn then continued: "at the end of this lecture, you will not have the slightest doubt about my nationality!" . Fifteen years later, during a discussion on a possible heavy ion experiment, I reminded him of the elephant's joke; he smiled and forgave a somewhat imprecise definition of mine.

In the sixties, Hagedorn developed a statistical approach to describe particle production which led to the concept of a finite limiting temperature for hadronic matter – the Hagedorn temperature – and to the formulation of the statistical bootstrap model [1] in which the exponentially rising hadron mass spectrum occurred naturally. This major discovery, however, had to wait a few years before being fully

CERN, 1211 Geneva 23, Switzerland

appreciated, since at the time there was no fundamental theory of the strong interaction - and no consensus on how to construct one.

The first clear formulation of the theory that we know today as QCD appeared in 1973. Two years later Cabibbo and Parisi [2] were the first to take up the challenge of the Hagedorn's limiting temperature and pointed out that it could be a critical temperature, at which hadronic matter could turn into a new state of unbound quarks and gluons. Such a possibility raised considerable interest since head-on collisions of high energy nuclei appeared then as a way to obtain such a new state in the laboratory, albeit for a very short time. It was discussed in several meetings (Erice 1978, Bielefeld 1980) and workshops (LBL Berkeley 1979, GSI Darmstadt 1980).

By 1980, Hagedorn and his close collaborator Johann Rafelski introduced a finite size for hadrons in the Statistical Bootstrap Model, and were able to show that the limiting temperature marked indeed a phase transition from hadronic matter to a quark-gluon plasma phase (QGP) [3]. At the same time Rafelski was the first, together with Hagedorn, to suggest that an excess of strangeness in the hadronic fireball from a nucleus-nucleus collision would be a natural signature of the formation of a de-confined phase [4]. His idea was then explored and developed with Berndt Müller [5]. The key prediction was that the onset of QGP should enhance, with respect to the case of proton-proton collisions, the final state abundance of the rare multistrange hadrons on account of the relatively higher phase space density of strangeness in the plasma. Detailed predictions - such as an increase of strange baryon and antibaryon enhancements with their strangeness content - were later published in a Physics Reports [6]. These predictions prompted many people, myself included, to start thinking of the possibility of detecting the decay of strange and multistrange hadrons amongst the large number of tracks produced in high energy heavy ion collisions.

15.2 The Heavy Ion Era at CERN Begins

In 1980, the time was ripe for action! A Letter of Intent [7] to study Ne-Pb reactions at the CERN Proton-Synchrotron, was submitted by a GSI-LBL Collaboration. This initiative triggered a long and eventually successful approval process [8], that resulted in a new CERN program involving ion beams at the CERN-SPS, at energies much larger therefore than those initially envisaged. However, a few years went by before the ion beams from the SPS became available!

Maurice Jacob, head of the CERN Theory Division from 1982 to 1988 and a strong supporter of ion beam experiments at CERN, played an important role in orchestrating interest among, particle and nuclear physics groups to work together in this new field. In preparation of the possible SPS program, Maurice organized, together with Torleif Ericson, Helmut Satz and Bill Willis, the Quark Matter meeting in Bielefeld 10-14 May 1982. All key participants from both sides of the Atlantic attended and the meeting prepared in six working groups the future CERN experiments. More on this topic is reported in Chapter 29.

At the time, CERN's top priority was to build LEP with a constant yearly budget. At the initiative of Robert Klapisch, nominated in 1981 Director of Research for all Non-LEP activities, a Workshop on the Future of Fixed Target Physics at CERN was held in December 1982: A group “Nuclear Beams and Targets” was convened by Bill Willis and summarized by Mike Albrow [9]. While initially the idea to use the PS energy range was explored, the greater opportunity both in terms of experimental capability as well as higher energy offered by CERN SPS became evident. Hence, the SPS community began to take an active interest in heavy-ion physics.

As a result in 1983, a collaboration between CERN, the GSI nuclear-physics laboratory in Darmstadt and the US Lawrence Berkeley Laboratory, started a pilot program at CERN to accelerate in the SPS oxygen nuclei and then sulfur nuclei, up to energies of 200 GeV per nucleon. These beams arrived in 1986 and 1987 respectively (Fig. 15.1). Following an upgrade of the accelerator complex by a collaboration between researchers from CERN, the Czech Republic, France, Germany, India, Italy, Sweden and Switzerland, a fully fledged CERN-SPS program with lead beams - up to 158 GeV per nucleon - arrived in 1994 (Fig. 15.2).

The concrete possibility of nuclear beams at the CERN-SPS, raised much interest and several experimental proposals were submitted to the CERN Committee. Two of them, NA35 and WA80, being the direct descendants of the 1980 Letter of Intent. The atmosphere was one of enthusiasm despite the severe budgetary constraint, which did not permit any large investment in the building of new detectors. Experiments had then to be assembled by recycling existing detectors and magnets. For an overview of the CERN heavy-ion experiments active from 1986 to 2006, see for example Ref. [10], while the four experiments on which I shall focus here (WA85, WA94, WA97, NA57) are summarized in Ref. [11].

15.3 Experiments WA85 – WA94 – WA97 – NA57

My collaborators and I decided to use the Omega Prime Spectrometer [12] which we had already used for hadron spectroscopy. However, in order to analyze events of unprecedentedly high track multiplicity we had to upgrade its Multi-Wire Proportional Chambers. These could only handle up to about fifteen tracks per event and not hundreds as expected for experiments with high energy sulfur beams. Thus we modified all of them into the so-called ” butterfly chambers”, only sensitive to particles emitted in a restricted phase space region at central rapidity. Later, to cope with the even larger event multiplicities expected in lead-beam experiments, we built the first telescope of silicon pixel detectors. This development began in the framework of the CERN-LAA RD program and continued in the CERN RD19 project [13]. Such a telescope allowed us to determine the space points on a track directly and, because of its high granularity, it could be placed near the target, thus easing the detection of the short-lived strange baryons.

Of course, the beginnings were not simple. Apart from the delicate hardware modifications needed and people's fear that these could permanently damage the

Fig. 15.1 Hadronic collisions family picture October 1988: The first report from WA85 experiment on strange antibaryon production was presented by Emanuele Quercigh at the Tucson *Hadronic Matter in Collision* workshop, October 1988; This picture was taken on this occasion. Those appearing in the book are in bold, all from left: back raw: **M. Danos, M. Gaździcki**, J. Whitmore, **E. Quercigh**, F. Navach, G. Zinoviev, M. Kalelkar, T. Awes, B. Barrett, D. Lodwick, R. Hwa, W. Geist; middle raw: D. Slansky, I. Sarcevic, S. Stampke, M. Tannenbaum, **R. Glauber**, R. Thews (covered), M. Shupe, **H. Gutbrod**, D. Harley, **M. Gorenstein**, K.B. Luk, **B. Muller**, J. Sunier, S. Oh, **W. Greiner, M. Jacob**, T. Carey, S. Frenkel; front raw: A.R. White, H. Eggers, T. Tranh Van, K. Goulianos, E. Friedlander, C. Quigg, I. Derado, **P. Carruthers**, W. Walker, J. Pancheri, **J. Rafelski** (who activated photo self timer), J. Rutherford, **L. Van Hove**, W. Busza, P. Stevenson, **P. Koch**, C. Chiu. Rolf Hagedorn was invited but could not come for personal reasons. *Photo: Johann Rafelski*

Fig. 15.2 Divonne 1994: During presentation of CERN DG Chris Llewellyn-Smith at Hagedorn's 75th birthday: Front row from left to right E.L. Feinberg, J. Rafelski (leaning forward), to right recessed the leaders of Omega Prime Spectrometer Experiments: E. Quercigh, F. Antinori, K. Safarik; 2nd raw: R. Bock, R. Hagedorn (behind Rafelski). *Credit: CERN Image 199406-068-024*.

Omega chambers, there were several open physics questions. We needed to guess what the events from high energy nucleus-nucleus interactions would look like. What, also, would the multiplicities of the secondaries be? And how would these be distributed in phase space? Furthermore, how could we recognize the existence of a QGP during the interaction and how valid would an enhancement of strange-particle production be as a diagnostic tool for QGP? Which effects could distort the measurement and simulate a phase transition?

At that point we went for advice to Hagedorn, who had studied those subjects [14]. He patiently discussed these matters with us and gave much useful advice. For the questions about strangeness, however, Hagedorn suggested that we contact directly Johann Rafelski who, he said, would be delighted to discuss that issue with us! This echoed the advice we got from Léon Van Hove, a former CERN director general, also a strong supporter of the new research program. Indeed, Johann was delighted and this was the start of a long and friendly collaboration.

Our first two experiments, WA85 and WA94, took data at 200 A GeV in a sulfur beam, using the "'butterfly chambers". They were followed by two lead-beam experiments WA97 and NA57. The latter was a North-Area experiment with a new spectrometer and a new spokesperson: Federico Antinori. Both the latter two experiments made use of the Silicon Pixel telescope as their main tracking device.

The experiments confirmed our hopes. We found that the abundances of multistrange baryons and anti-baryons produced in heavy-ion collisions were indeed enhanced [15]. Moreover, these enhancements increased with the strangeness content of the produced baryon [15, 16]. For example, in central lead-lead collisions, the rare Ω^- particles carrying three units of strangeness were enhanced by a factor twenty! A behavior expected to ensue from the appearance of a deconfined phase during the interaction [6]. Similar results were subsequently obtained by many other experiments. These results constituted one of the main pieces of evidence for the formation of a new state of matter at the CERN-SPS energies, which CERN announced in a press release in February 2000. More on this topic is reported in Chapter 33

Another interesting finding, suggesting a thermal production for s and $\bar{s}$ quarks [17] was the similarity of the slopes of the transverse mass spectra between strange baryons and corresponding antibaryons [18]. An observation which did indeed please Hagedorn! With this last example, I conclude my brief review of the influence that Rolf Hagedorn, together with his disciples and continuators, had on the CERN heavy-ion program and on our physics choices.

15.4 The Other Hagedorn

There is, however, another aspect of Hagedorn's activity which should not go unmentioned, namely his involvement in the defense of human rights. I here cite only the case of Yuri Orlov, a founder of the Moscow group set up to monitor the Helsinki Accords, who was arrested in 1978. Hagedorn, together with several other physicists working at CERN, took up his case and founded the Yuri Orlov Committee to campaign on the matter. This the Committee did consistently, even directly approaching the governments of all CERN member states. Finally, during the Gorbachev years, Orlov was able to leave the Soviet Union for the United States and, in 1991 spent one year at CERN as a guest professor. As many of us know, however, Hagedorn's involvement in Orlov's defence was only one example of his readiness to help people whom he felt to be unfairly discriminated against!

References

1. a) R. Hagedorn, Nuovo Cim. Supp. **3**, 147 (1965)
   b) R.Hagedorn, Nuovo Cim. A **52**, 1336 (1967)
   c) R.Hagedorn, Nuovo Cim. A **56**, 1336 (1968)
2. N. Cabibbo and G. Parisi, Phys. Lett. B **59**, 67 (1975)
3. R. Hagedorn and J. Rafelski, in *Statistical mechanics of qurks and hadrons*, Bielefeld Symposium 1980, H. Satz editor p. 237 (North Holland 1981)
4. J. Rafelski and R. Hagedorn, in *Statstical mechanics of quarks and hadrons*, Bielefeld Symposium 1980, H. Satz editor p. 253 (North Holland 1981)
5. J. Rafelski and B. Muller, Phys. Rev. Lett. **48**, 1066 (1982)
6. P. Koch, B. Muller, J. Rafelski, Phs. Rep. **142**, 167 (1986)
7. GSI-LBL Collaboration, CERN/PSCC/80-129; PSCC/I-56 (1980)
8. R. Klapisch, Symposium in honor of A. Poskanzer, preprint LBL30-5, Berkeley 2001
9. M.G. Albrow, in: Workshop on SPS Fixed-Target Physics in the Years 1984-1989, I. Mannelli ed., CERN Yellow rep. 83-02, Vol. 2, 462 (1983)
10. E.Quercigh, K.Safarik, "CERN–THE SECOND 25 YEARS" Phys. Rep. **403-404**, 51 (2004)
11. E.Quercigh, Strangeness in Quark Matter, Cracow 2011. Acta Phys. Pol. B **43**, 771 (2011)
12. W. Beusch, CERN/SPSC/84-62/P204 (1984)
13. E.H.M. Heijne et al., R&D Proposal RD19, CERN DRDC/90-81 (1990)
14. a) R. Hagedorn, in *Future Relativistic Heavy Ion Experiments* Darmstadt, October 7-10, 1980, GSI-81-6, pp.236–281, see Chapter 26 in this volume
    b) R. Hagedorn, Riv. Nuovo Cim. **6**, 1 (1983)
15. WA85 & WA94:
    a) S. Abatzis et al, Phys.Lett. B **259**, 508 (1991)
    b) S. Abatzis et al., Nucl. Phys. A **590**, 317c (1995)
    c) F. Antinori, in: The CERN Omega Spectrometer: 25 Years of Physics, M. Jacob, E. Quercigh, eds., CERN Yellow Rep. 97-02, 43 (1997). Compilation of a), b)
16. WA97 & NA57:
    a) E. Andersen et al., Physics Letters B **449**, 401 (1999)
    b) F. Antinori et al., Nuclear Physics A **661**, 130c (1999)
    c) F. Antinori et al., J. Phys. G **32**, 427 (2006)
17. J.Rafelski Phys.Lett. B **262**, 333 (1991)
18. a) D. Di Bari et al., Nucl. Phys. A **590**, 307c (1995)
    b) L. Sandor et al., Nucl. Phys. A **661**, 481c (1999)
    c) R.A. Fini et al., J. Phys. G **27**, 375 (2001)
    d) K. Safarik et al., J. Phys. G **27**, 579 (2001)

Part II
The Hagedorn Temperature
edited by Johann Rafelski

Contributions by:

Rolf Hagedorn, Johann Rafelski

Part II addresses properties of hot hadronic gas (HG) matter and the proposal and characterization of the phase transformation between HG and quark-gluon plasma (QGP).

The opening Chapter 16 is a long-lost review, appearing for the first time in English. It describes the meaning of limiting (Hagedorn) temperature T_{H}, the Statistical Bootstrap Model (SBM), and its role in the Big Bang and Universe evolution. Chapter 16 can be read by a general science-versed reader. Hagedorn's comprehensive technical 1995 retrospective of the experimental and theoretical developments that compelled introduction of T_{H} and SBM follows in Chapter 17.

Chapter 17.5 is a commentary on Chapter 19, Hagedorn's first unpublished 1964 paper introducing T_{H} and the exponentially growing mass spectrum $\rho(m)$. Chapter 20 presents the experimental 1968 data for $\rho(m)$, and Chapter 21 offers a contemporary discussion of this central result. Chapter 22 is Hagedorn's unpublished 1972 guide to SBM literature.

Chapter 23 is a 1979 unpublished conference paper which presents SBM in its covariant form, introducing finite sized hadrons, and allowing for finite baryon density characterized by a chemical potential. This work shows the transformation from hadron gas to a collapsed single fireball drop that we call QGP today.

This phase transformation is made mathematically more precise in the following Chapter 24. This is Hagedorn's 1981 unpublished resolution of a critisism of Chapter 23 as extended with the concept of the available volume, discussed further in the following Chapter 27. Chapter 25 is Hagedorn's 1984 retrospective about development of the SBM leading on to our work on the phase transition to quark-gluon plasma. Hagedorn explains in plain language and resolves many questions that arise in the study of the material of this book. Noteworthy for Part II are the two paragraphs below Eq. (25.16) which discuss the relation of the phase limit temperature with a limiting temperature.

A short quote from Chapter 16 explains this further: Hagedorn draws the parallel between boiling hadronic matter and boiling water: "... with increasing temperature, it becomes ever easier for a molecule to free itself from the liquid, and when the temperature approaches the boiling point, it is so easy for them to leave, they all want out and actually escape in a rapid manner. They absorb all the heat made available and leave the molecules still remaining behind no energy to increase their temperature." Hagedorn places emphasis on the fact that water cannot get hotter but vapor in principle, could. However the 1968 view was: "... boiling HG matter can never overcook, *because it is the supplied energy itself* which materializes and so ensures that more new particles are always being born. Therefore there can never arise the process corresponding to the continued heating the water vapor. ... $\mathbf{T_{\mathrm{H}} = 1.8 \times 10^{12}\,K}$ **is the highest ever possible temperature** in a stationary thermodynamic equilibrium."

This position evolved with the development of the nuclear bootstrap model for the gas phase, incorporating a finite hadron volume, see Chapter 23. With the rise of QGP as the new phase of matter, the meaning of T_{H} expands to be the phase transformation condition. The new phase, QGP, can be heated – quark and gluon tempeature rises without limit, $T > T_{\mathrm{H}}$.

Chapter 16
Boiling Primordial Matter – 1968

Rolf Hagedorn

Abstract This introductory article presents in popular language how the view of the early Universe was evolving through 1968 under the influence of than new and recent insights about the thermodynamic properties of strongly interacting matter. (by JR, editor).

16.1 The Large and the Small in the Universe

Even though no one was present when the Universe was born, our current understanding of atomic, nuclear and elementary particle physics, constrained by the assumption that the Laws of Nature are unchanging, allows us to construct models with ever better and more accurate descriptions of the beginning. We begin to understand the composition and abundance distribution of nuclei, and we understand the origin of the energy which drives the Sun and countless other stars. We would have never understood these things if we had not advanced on Earth the fields of atomic and nuclear physics.

To understand the great, we must descend into the very small. The objects, which will be discussed here, are incomprehensibly different in their size. In our daily lives a centimeter-sized object is a visible and reasonable magnitude; our direct experience ranges from "very thin" – a sheet of cellophane (10^{-3} cm) – to one hundred

Published as "Siedende Urmaterie" in *Verhandlungen der Schweizerischen Naturforschenden Gesellschaft*; **148** (1968) pp 51-63, Translated from German by Victoria A. Grossack, October 2014; translation edited by Johann Rafelski.

(deceased) CERN-TH, 122 Geneva 23, Switzerland

meters (10^4 cm); below and above these limits we no longer experience lengths directly through our senses, but indirectly with the assistance of our intellect – for example we imagine 100 km as one hour on the freeway. Even with these tricks we can only go so far, because in order to express how small an elementary particle is and how large the currently observable part of the Universe is, we must use numbers that are again beyond our direct comprehension. There are as many protons in a centimeter as there are for example centimeters in the diameter of Earth's solar orbit, and as another example consider the many Earth orbit's diameters needed to reach from here to the furthest visible spiral nebula – that is to say, somewhere between 10^{13} to 10^{14}.

Who can comprehend the number 10^{13}? With an effort I can have a feeling for one million, 10^6: a million teaspoons of water is about one cubic meter. But even 10^9 – a billion– is difficult. Do you want to be a billionaire? Put aside a Swiss Franc every second for 32 years – then you'll be one. One million years yields 3×10^{13} seconds. String protons together, one each second – in a million years you'll have a chain barely 3 centimeters long; string together centimeter-sized pearls, one each second, and in a million years the chain will reach from here to the sun. Lay together an Earth orbit every second, and after a couple million years you will reach the furthest visible spiral nebula (or to be precise, where that spiral nebula was a couple billion years ago, when its light started in our direction). And a last example, which we all know: on a distant island is a diamond mountain, and every hundred years a bird sharpens its beak on the mountain. When the mountain has been whetted away, the first moment of eternity will be finished. Mont Blanc would be whetted away after 10^{40} seconds (the Milky Way is only 10^{17} seconds old!) and for just as long must one lie proton next to proton – one each second – to reach the furthest spiral nebula.

After this attempt, to make the incomprehensible more comprehensible, I propose my assertion:

> *In order to explore the enormously gigantic (*10^{14} *diameters of the earth's orbit), we must apply our knowledge of the extremely small (*10^{-13} *centimeters).*

In large things the Universe follows the laws of macrophysics: mechanics, electrodynamics, thermodynamics, relativity and hydrodynamics. For most part we encounter conditions that differ vastly from those surrounding us. They are more akin to those present in a nuclear experiment carried out at a cosmic scale. How can the inner structure of matter – the extremely small – be the building principle of the Universe, determining for the large part the emergence of galaxies and stars and the course of their lives? All this originates and depends nevertheless on these so unusual circumstances to which matter is subjected – or perhaps one should say, conventional conditions, a statement allowing for the fact that the conditions under which *we* live are extraordinary.

Under these circumstances one can anticipate that each new step in understanding the extremely small develops new relationships in the extremely large and leads us further on the way, which we hope, succeeds in bringing us to a new theory capable to explain simultaneously the functioning of the Universe in both the very large and the very small.

The most recent step into the very small began a few years ago, and it leads today to few if any consequences for our conceptual understanding of the Universe; I believe, however, that these will come soon. With the last step I am referring to the field of high energy physics. Those who prefer precise wording might criticize the use of the expression "last step" because it could be easily misunderstood: namely as the last possible instead of the latest accomplished, as I meant to say here.

But there is no mistake in expression. I meant both and especially the last possible step. Instead of an error of style it has to do with a hypothesis, which is being described by this lecture. It appears that we have reached, in elementary particle physics a completely new paradigm, a kind of a terminal situation, in which the question about the composition of matter receives an unforeseen and satisfying answer. This is actually surprising, because we still can't overcome the old difficulties. Whenever someone says to me, that he has now found the true atom, the building block of all matter, I always ask him, then what is this thing made from? One can just read Kant, to see in what sort of cul-de-sac that leads. And now I claim that high energy physics – perhaps! – offers a final solution to this dilemma? I do not want to be misunderstood: first, I am making a claim, which is not accepted by all of my colleagues, and second, I do not claim that we are about to understand everything about elementary particles. But this new approach seems – at least from a particular perspective – to offer us the view, which could be used to take the picture.

The New Situation: Multiparticle Production in High Energy Physics

I want to show you first why the situation is new.

The question, "How is matter created?" is a challenge for scientists studying nature. This also invites them to take 'it' apart, to study the building blocks and the forces binding these building blocks together, to apply the already known laws of physics as much as possible, to postulate new laws only when unavoidable and to attempt to bring everything together consistently. The importance of conceptual theoretical insight is that this lets us understand how the whole may be more than the sum of the parts, remembering that the first and the last word is spoken by experiment. To study this question, this is what the experiment dictates: break apart particles into their building blocks and measure the forces acting between them that do so for sub building blocks, then break down the sub sub building blocks and again study the forces and so forth, without end. Without end?

We want to follow this continuing decomposition and pay attention to how much energy we must use, in order to break down a given material into its components. The "new situation" will become clearest when we compare the requisite energy with the total energy that is stored within the given material.

Relativity teaches us in that a piece of material with mass m contains the energy equivalent $E = mc^2$ (c is the speed of light).

This proposition has been confirmed experimentally. The energy $E = mc^2$ is enormously large in comparison to familiar energy scales. We will see that soon.

We consider some everyday matter – some cooking salt – and break it down into its elementary building blocks and with each step compare the energy released in the decomposed material to the energy in the material as a whole. So let's take a piece of cooking salt (NaCl), about the size of a fist. How do we decompose it? First we let it fall to the Earth; with a hard floor and a falling distance of about a meter, it breaks into about a hundred smaller pieces – but those splintered pieces are still cooking salt. In order to break apart the smallest piece of salt – a molecule of NaCl – into sodium Na and chloride Cl elements, we must turn to chemical processes.

For centuries, the futile efforts of alchemists demonstrated that one could not go beyond the decomposition of NaCl into Na and Cl. The belief set in that atomic elements are truly the indivisible elementary building blocks. Yet the question remained: why are there 90 different atomic species? If they are different, then their structures must be different, so they must have subparts.

Soon we found a way to break elements apart too: one throws them on the floor – but this time somewhat harder – or rather one bombards them with very fast projectiles. From this we learned that atoms are composed of three different building blocks: protons and neutrons, which are the nuclear building blocks, and electrons, which are needed to create the atomic shells. The very weakly bound electrons are responsible for chemical processes, for which the tightly bound nuclei can have nothing in common – hence the failure of alchemy. Only the energy rich projectiles, which modern particle accelerators shot at the nuclei being studied, enabled these nuclei to be broken apart. When this was accomplished, one attempted the next step, breaking apart the nucleons (the shared name for protons and neutrons, which are similar to each other) with a collision using another nucleon – and this approach failed – but in a way suggesting that something fundamentally new happened.

Now we turn to take a look at a chart which shows what fraction of the energy is required to break mater down into components:

- Mechanical decomposition of a cooking salt crystal into fragments by letting it fall from a height of one meter: 1×10^{-16} of the total energy of the crystal.
- Chemical decomposition NaCl→Na+Cl: 7×10^{-10} of the total energy of a NaCl molecule.
- Nuclear decomposition Na→23 nucleons: 8×10^{-3} of the total energy of the Na-nucleus.
- Decomposition of the nucleon? $5\times$ the total energy does not suffice!

These numbers show how enormous the binding forces become, when the decomposed objects become smaller. To achieve the chemical binding energy of the cooking salt crystal I need to throw it 7000 km high (assuming that Earth's gravity remains the same). However, the energy in the nuclei is still 10 million times higher – and yet this is but barely 1% of its total stored energy as shown in $E = mc^2$.

With so comparatively tiny – albeit growing – fractions of the total energy, we can break down all the known substances into their electron and nucleon building blocks.

It was foreseeable that one would have to bombard the nucleon with an even larger fraction of its total energy in order to get the nucleon to break down into its components. Therefore we have built high energy particle accelerators and smashed nucleons together with higher and higher energies so that now, at the most recently built Soviet accelerator (70 GeV) we are not achieving just a fraction, but five times the energy $E = mc^2$ contained in a nucleon.

The nucleons remain intact!

When the highest energy cosmic rays hit an atom, the collision energy up to a few hundred times greater than that in the nucleon is achieved – and so far there is no evidence that this will break apart the nucleons. Just the opposite. In such experiments a large number of new material particles including even nucleons (and antinucleons) are created. Most of these newly generated particles are certainly unstable – they decay in an unbelievably brief time, nevertheless slowly enough, that one can experiment with them.

I do not want to go now into the detailed properties of these particles nor to describe the astounding way in which their properties can be classified in a simple scheme. What this scheme suggests is that the nucleons as well as all the other newly formed material particles are composed of only a few fundamental building blocks, the so-called quarks. Quarks have never been observed as free particles and might not exist in this form. These insights have been described in a manner understandable for non-specialists in many other popular-scientific articles, thus I do not dwell further on this matter.

My objectives are different. First, I will try to make clear that the above finding suggests that something radically new is really present; and second, let me explain why I believe that we are in a 'final' situation, which nevertheless does not signify an ending of our search for the ultimate building blocks of matter.

First: Imagine that through decomposing and decomposing and decomposing, the matter is finally pushed to small, incredibly hard spheres, say the size of a pea, which can neither be destroyed nor differentiated from each other in any manner. We collide such spheres onto one another and thereby expend energies that were greater than the mass energy of the spheres. However, instead of breaking up, they divided into four such peas (including an anti-pea) – *each just as big, just as heavy and just as hard as the two originals* – therefore two brand new peas were created. In the process appeared also a lot of splinters and sparks of a previously unknown material, all of which almost instantly shattered with a bang and disappeared, while adding some more peas to the type of peas described above. Such a situation should be correctly viewed as a new phenomenon.

For physicists this was not however unexpected: relativity and quantum physics have long taught that energy and mass are equivalent and can spontaneously change into each other; set energy free with an impact, it can reappear as matter, subject only to the constraint that the amount of energy is greater than the mass energy equivalent $E = mc^2$ of the particle to be generated. Other conservation laws such as that of baryon number deserve mention here as well.

Black Body Radiation

I now need to introduce another concept that has played an influential role by undermining classical physics. This idea forced Planck to postulate the quantum hypothesis initiating a radical conceptual change which culminated in the formulation of quantum theory. Arguably, there has not been anything of comparable importance discovered since. I present to you "Black Body Radiation."

If you place a completely empty box – a cavity – in a heat bath of temperature T, it does not remain empty; it fills with electromagnetic radiation, whose spectral distribution, i.e. the composition of different wavelengths (radio waves, heat, light, ultraviolet light, X-rays), is described accurately by Planck's radiation law. This spectral distribution is a function of temperature; in fact, we measure temperature of very hot and/or far and distant bodies (stars), by studying the radiation spectral distribution. Aside of the spectral distribution dependence on the temperature, the intensity of the radiation is also temperature dependent. Namely, the total radiated energy is proportional to T^4. Or said differently, the way I prefer: the temperature is proportional to the fourth root of the radiation energy content. When the temperature just doubles, the radiation energy is increased 16 times.

From daily life experience, by and large, (that is, apart from chemical and phase changes, such as melting, boiling), we are accustomed to thermal energy being approximately proportional to temperature increases; that is, 16 times the thermal energy also means 16 times the temperature. This is because heat is nothing more than the random motion of molecules and that, as their number (usually) remains constant, all energy supplied again finds itself as heat and the temperature increases proportionally: Temperature is defined as a measure of the average kinetic energy per molecule. However, in the radiation field – also called photon gas – the number of molecules," that is to say, the number of photons, is not at all constant: ever more and more of them are created as the temperature is increased, as I supply ever more energy. This larger number of photons, many more than were originally available, must share the newly supplied energy; therefore each photon takes only a minor portion for itself, than it would have received, had their number been constant. The temperature = average energy per photon rises more slowly than in the case of constant particle number; in consideration that a large part has just been invested in the creation of new photons. In a more careful evaluation we find the Stefan-Boltzmann law which I introduced, the temperature is proportional to the fourth root of energy density: $T = \text{Const.} \times \sqrt[4]{E}$

What does this have to do with our indestructible nucleons and the newly created particles?

All we need is to generalize the concept of black body radiation: who says that the radiation must consist only of photons? There is no law in physics prohibiting material particles forming from radiation. In fact, relativity and quantum theory claim it outright: If $E \geq mc^2$, a particle of mass m can arise spontaneously (there are certain constraining conservation laws, but in principle this detail changes nothing). So if we increase the temperature of our box on and on, it is inevitable that in principle within the cavity particle radiation of any sort of matter (and antiparticles) sometimes occur. Admittedly, the probability of finding a particle of mass m, decreases extremely rapidly with increasing mass (that is, exponentially).

Considering very high energy collision processes quantitatively one finds that the newly created particles have just the same energy and angular distribution, which they would have if they were emitted by a black body source of a very high temperature as first argued by Enrico Fermi. Although not of immediate interest in our present context, the black body radiation cavity source is also in motion. As argued, we can measure the source temperature by generalizing the Planck's radiation law to include the radiation of material particles. To each Planck's spectral energy distribution corresponds a certain temperature value T. All we need to do is to measure the energy spectral distribution of the newly generated particles in a given collision process to learn which temperature was reached in the collision between the two projectiles.

By this procedure we can deduce the temperature that prevailed during the incredibly short collision time (10^{-23}) sec in the incredibly small domain of space (10^{-13}) cm –in the time (10^{-23}) sec the light travels the distance (10^{-13}) cm. Using the same method we can make an equally reliable statement about the temperature of the surface of Sirius or in the interior of a blast furnace. As the collision energy is a multiple of the mass energy of the colliding particles, it is not surprising that the temperatures measured in these collisions far surpass all the temperatures known on Earth and in the sky above. Created daily at CERN in billions of collisions these temperatures are of the order 10^{12} K. To imagine this number, consider this: A furnace that becomes hotter by one degree every second, would bring water to a boil in 1.5 minutes; and after 1.5 hours it will be as hot as the surface temperature of the sun; after a year we would reach the interior temperature of the Sun but only after 100,000 years would we reach the temperature of which we speak in high energy physics!

16.2 HighestTemperature= The Boiling Point of Primordial Matter?

I claim that it is not surprising that the temperature seen in high energy collisions is that high – in fact, one would have expected it to be much higher and in particular that it should grow with the energy of the colliding particles. Namely, as one knows from the black body radiation law – and that is what we are dealing with here – temperature should grow at about the fourth root of the energy. Instead, it remains a simple constant, apart from some not yet quite understood exceptions. More precisely, as the particle collision energy grows, the temperature T_0 approaches a finite limit of 1.8×10^{12} K corresponding to 160 MeV.

It appears that this fact is extremely significant indicating that in the decomposition of matter, we have reached an unexpected end, which is, nevertheless, not an end.

Namely: the temperature of ordinary black body radiation only grows with the fourth root of energy, that is, relatively slowly, because a large part of the available energy is used to produce new photons instead of being used to increase the temperature. Considering the case of material particle black body radiation present in high energy physics we have available not only photons but all new types of particles. Each type of particle demands a part of the available energy. Each particle component needs this energy to participate fully. The more such particle fractions are present, that is the more different *types* of such "elementary particles" are present – the less energy that can be vested in each type of component and thus less energy remains available to raise the temperature.

In fact today there are many different types of particles that can be produced in a high energy collision – one already knows about 100 new "elementary particles" – and all these have distinct mass. Thus we are led to, and we need to characterize the concept of the mass spectrum. To this end I would like to introduce a seemingly absurd but valid comparison, namely books. There are many different titles, each with a fixed price (if two have the same price, one can introduce another distinguishing property). In this approach let me compare the book title with a particle type, and book price with particle mass; the print number with the probability of finding this sort of particle. Even without looking at the content of the books we can generate a spectral price distribution by asking: How many books are there in each price interval (such as between Fr 10 and Fr 11 or between Fr 31.50 and Fr 36.75). Similarly, one can arrange the various types of elementary particles without considering their individual properties – by specifying how many species there are in each mass interval. This distribution we call mass spectrum, just as one speaks of the price distribution counting books.

Clearly, the radiation equilibrium within our black body source will now depend on material particle mass spectrum. The more different particle types there are, the less is the temperature rise given the same input energy. The precise terms "mass spectrum" and "radiative equilibrium in cavity" permit a precise mathematical treatment of the problem.

The outcome is that if the mass spectrum of the participating "elementary particles" increases immensely strongly and in a very specific way, the temperature may never grow beyond a pre-established limiting value. This limiting temperature T_0 emerged as a characteristic constant in the mathematical description of the mass spectrum: Each equal length mass steps $\Delta m = 2.4 \times T_0$ moving up the mass spectrum, brings into the picture 10 times more new types of particles as compared to all previous steps taken together. It is said that the mass spectrum grows exponentially as e^{m/T_0}.

This we can verify experimentally: In high energy experiments for a temperature characterized by the limiting value T_0 one would further experimentally observe new types of "elementary particles" that can be sorted into a mass spectrum from which it is possible to read off the constant T_0 again. Of course it is possible to study a small mass spectrum domain of the low-mass to mitigate the effect that for the larger masses few particles are produced; that is, in our book example at high price only "Limited Editions" are produced which limits the printed number of copies; this reduction is again exponential. One finds in such a study:

> *The nearly fully known mass spectrum grows in exactly the way that is required for the existence of a limiting temperature, and the constant T_0 is numerically consistent with the upper bound of the temperatures measured in high energy collisions.*

Now, a few limiting temperatures are familiar to us from our daily lives, perhaps the best known being the boiling point of water: No matter how hot I make the stove, at normal atmospheric pressure water boils at exactly 100^0 C. Why? Because all of the additional heat energy is used to lift water molecules out of the liquid. Generally, any additional energy is divided between two competing mechanisms: increase in temperature, and evaporation. Since molecules do not have a sharp temperature controlled energy but a distribution, some can cross over from liquid into vapor at practically any temperature. However, with increasing temperature, it becomes ever easier for a molecule to free itself from the liquid, and when the temperature approaches the boiling point, it is so easy for them to leave, they all want out and actually escape in a rapid manner. They absorb all the heat made available and leave the molecules still remaining behind no energy to increase their temperature.

The limiting temperature appears in the high-energy collisions in analog fashion. You have only to replace the words "leave the liquid" with "make the leap from non-being into being." To make this transition a particle of mass m needs the energy $E = mc^2$, and when there are as many different particle types as described above, then the all-particle birth rate will eventually be so great with increasing temperature, and the many required mc^2 amounts will use up all energy supply such that already-born particles will have nothing left to increase their common temperature. Because of this analogy I speak of "boiling primordial matter."

Of course, once all the water has evaporated, additional energy will further increase the temperature of the steam. Moreover, all the water can boil away, given that a fixed amount of water has a fixed number of molecules. Our boiling primordial matter can never overcook, *because it is the supplied energy itself* which materializes and so ensures that more new particles are always being born. Therefore there can never arise the process corresponding to the continued heating the water vapor.

> *If these considerations are correct: that is, we were not lured by nature into a trap of following the correspondence between the experimental limiting temperature T_0 and the shape of the growing mass spectrum (which in principle can never be ruled by these experiments), then $T_0 = 1.8 \times 10^{12}$ K is the highest ever possible temperature in a stationary thermodynamic equilibrium. Occasional exceedances of T_0 likely correspond to the familiar phenomenon of superheating leading to an increased boiling point.*

16.3 Is the Question about the “Final Building Block” Meaningless?

There is the final question that remains: Suppose, that everything were correct; there is an infinite number and an exponential mass spectrum of new types of particles and a corresponding limiting temperature – what does that have to do with the here presented end situation, which nevertheless does not mean an end? Here we enter into a theoretical construction wherein one abstracts a general rule from a limited number of experimental data, which is then tentatively postulated as a universal principle. This introduces us to the usual practical circumstance of theoretical physics: We have a model whose other properties are analytically derived using established methods of mathematics and the assumptions that generally apply to the already known laws of nature. In this way we obtain experimentally testable predictions as derived from known or later verifiable behavior. Agreement of these predictions with the facts is necessary, but not sufficient, to ensure that the theoretical model is correct. This applies especially to the model I will now describe.

In order to introduce the model in words, I will characterize the situation far less exactly than the technical tools of theoretical physics would allow me to do this. I proceed in this way as I seek at all cost to avoid technical jargon.

In a high-energy collision new material particles are copiously produced (events with a multiplicity of a hundred or more have been observed). In our terminology, these particles emerge from the collision-produced boiling primordial matter. In a certain and physically quite precise sense they were all contained in this piece of boiling primal matter. Taking one of these newly generated particles under the microscope (which is not easy: lifespan $\simeq 10^{-23}$ sec), we observe that it behaves itself as boiling primordial matter; namely it can decay further into many particles. The greater its mass, the greater is this tendency. Such a particle with a large mass thus has a dual nature: on the one hand, it can be used as an “elementary particle” contributing to radiative energy equilibrium, on the other hand it can itself create other “elementary particles” which contribute to the radiative energy equilibrium. Seen from this perspective, none of these produced particle types can be viewed as an elementary particle, given that other particles can emanate from any of the produced particles, which are again no more elementary since each can be simultaneously created out of the other, and in this way all these particles have undetermined building block composition.

Nothing in this picture changes if one day quarks should be confirmed as the primordial building blocks. In our approach they would play a preferential role, being the stuff from which “everything is built.” As an aside, it is the virtue of our approach that the statement “composed of” does not characterize the number and the character of the fundamental building blocks. The composition and nature of the source of produced particles can remain cloaked in mystery; it can remain undetermined.

The model aims to overcome the limited number of presently known types of particles by continuing the observed behavior of the mass spectrum at low mass to higher mass, (where we experimentally know nothing yet). Once this is done, much of what follows can be found ready to use in textbooks of statistical thermodynamics. The surprising extrapolation result is:

> The mass spectrum grows in exactly the manner (exponential) as is required for the presence of an absolute maximum temperature.

With this the circle closes:

– The property of the new "elementary particles" is that each is simultaneously in ever-changing ways being created from all the others,
– with the tremendously (exponentially) increasing mass number distribution of different types of such "elementary particles",
– leading to the existence of a "boiling point" for primordial matter.

These three seemingly different things are actually different manifestations of a single underlying physics principle – provided that you take any one of these three as a general postulate valid beyond the currently experimentally studied range.

A theoretical model, such as this one, which is introduced as a postulate, where the behavior is extrapolated to infinity from the finite domain that is known, cannot be proved. Its consistency, its formal simplicity and the fact that its detailed quantitative predictions agree in the currently accessible experimental range, makes it interesting and credible until further notice. Should it be correct, then the old question of the ultimate constituents of matter disappears all by itself: This issue merges into the endless circle. Let's return to the analogy we developed with books: There is no "elementary book" from which all others are made. Yet when two books collide with each other violently enough, many new are produced – and each contains every other somehow in itself.

Before answering the last question: what does all this have to do with the "evolution of matter?" I offer a few remarks.

a) The situation described is typical of the physics of strong interactions, involving all nucleons and other particles responsible for the mediation of the nuclear forces. The electron is in this context irrelevant. The reason is that in such a short collision only the strong interactions can participate in formation of radiation equilibrium. There is no time for the electro-magnetic and weak forces to act; before they awake and can respond, everything is as if the collision had happened a few million years earlier.

b) The model described here relies on a speculation which posits what should happen for infinitely large particle masses by extrapolating what is observed at finite particle masses. There is another approach founded in similar yet very different more technical concept, namely the extrapolation towards stable "elementary particles", i.e. nucleon, mesons (stable under strong interactions). We attempt a description in which each such elementary particle emerges simultaneously from all the others: This is our so-called "Bootstrap-Theory," originating in the well-known "Baron Münchhausen" bootstraps. The gentleman is trying to pull himself out of the swamp by yanking on his own hair. Despite this analogy I think our particle bootstrap model is in principle correct – it's practically the same model as the one

I introduced above. However, it has, I believe, due to a technical defect, so far not functioned quite right: One has usually introduced only the few lowest mass particles in self-consistent bootstrap circles; the more stable particles one takes, the better the particle bootstrap should function, so all stable particles need to be included, after this is done there can no longer be an objection. On the lighter side, we recall that only when Münchhausen has yanked very strongly at his hair, was he able to move, and then not only himself, but taking with him the swamp, and the Earth – the whole world.

c) It is noteworthy that in the realm of today's particle physics (or High Energy Physics – we have seen that these two terms mean the same) no evidence is found that the existing principles of relativity and of quantum theory need to be corrected or extended in any way; even though we are in a new situation.

d) After my report, it might seem as if the end of elementary particle physics has come. However, what I have presented arises from speculative hypothesis. And even if everything were correct, we would not come to an end, but find ourselves at a new beginning: In all the above considerations only strong interactions were considered, and not in terms of particular form of forces, but only in terms of the ever-changing composition of the "elementary particles," and we have never spoken about their individual characteristics – therefore our conclusions were completely independent of all these additional known particle properties. Thus we have described the average behavior, the statistical behavior. But the main focus of high energy physics is precisely on all these more detailed individual properties of the new particles and the forces acting between them. And there is the question, why *these* forces? In this regard we stand at a new beginning.

e) Many physicists still believe in the possibility of exploring deeper and further to ever more elementary building blocks. One must follow this line experimentally and cannot be misled by intellectually satisfying speculation into believing that the scientific question is settled.

f) I have tried to describe everything in everyday language, in words, that we physicists use, when we talk about such things at tea. To you, the reader, everything must look very mysterious, especially the claim that each "elementary particle" in different ways has been created from all the others. Take it to be 'as-if-speech', as a blurry image of what can be formulated much more precisely with the help of mathematics or technical jargon.

With this report I also, as an aside, hope I have made you understand why we high energy physicists yearn so much for the next European 300-GeV accelerator, which will now probably be built.

Possible Consequences in the Large?

What does this all have to do with the creation of matter? At least a few theories about the beginning of the Universe assume a Big Bang, that is to say a creation explosion. Following previous ideas – based on traditional black body radiation –

the Universe began with infinite energy density, with energy density proportional to the pressure, and infinitely high temperature. Under such extreme conditions, traditional black body radiation no longer remains, but rather the conditions are found akin to the high-energy collisions of nucleons. And then when strongly interacting matter is present the temperature cannot be infinite, but only about 10^{12} K, and the pressure is not anymore proportional to the energy density but only proportional to its logarithm. This is a different scenario of the beginning of the Universe than was previously thought. A beginning is seen in experiments at CERN, where a proton melts with another for 10^{-23} sec into boiling primordial matter. Moreover, it cannot be excluded that even entire stars consist of boiling primordial matter.

We can wonder if this Big Bang, the origin of everything, including the beginning of time is an equally unsatisfactory assumption as is the existence of the very final building blocks of matter. Just as you can ask: And how did that building block come about?, so you can ask : And what was before Big Bang? How did it happen? We do not know. Maybe we will find one day that this question in a similar way is irrelevant as – possibly – the one about the final building blocks.

I close with an anecdote: On the bulletin board of a German university the following could once be read among lecture announcements: *Tuesdays 9-11 AM, free for all discussion session about the structure of the Universe – only for the advanced. signed X.* We will, alas, always be beginners (see Fig. 16.1).

Astron. & Astrophys. 5, 184—205 (1970)

Thermodynamics of Strong Interactions at High Energy and its Consequences for Astrophysics *

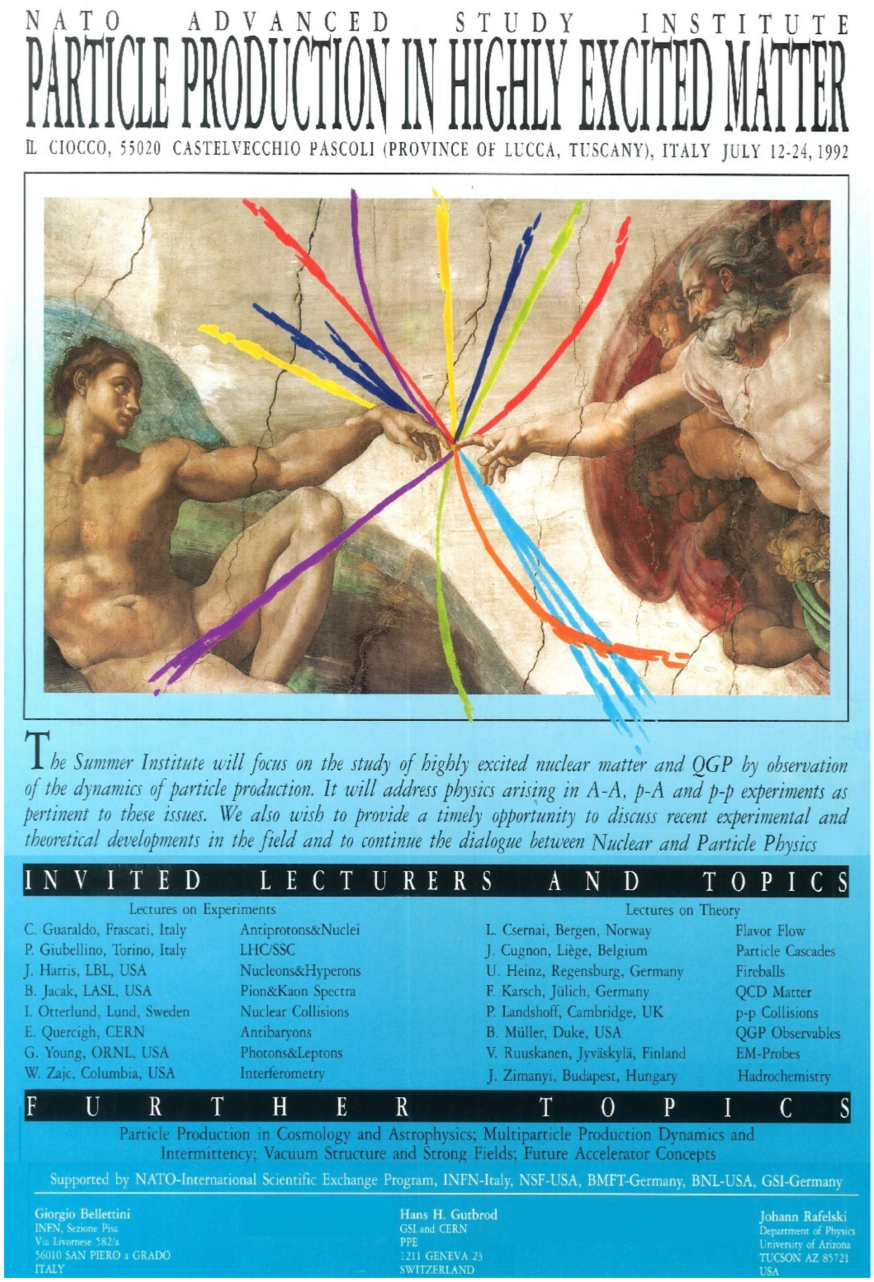
R. Hagedorn

European Organization for Nuclear Research, Geneva

Received October 14, 1969, revised December 1, 1969

Statistical thermodynamics in a particular form derived from high energy physics is used to describe the thermodynamical properties of what might have been our universe before its energy density became much lower than nuclear density. The main features are:
even if it started with infinite energy density, it never had a temperature greater than $T_0 = 160$ MeV (1.86×10^{12} °K), which is the universal highest temperature in this theory;
for very large energy density the pressure is not, as in usual theories, proportional to the energy density but only its logarithm;
inside each elementary volume V_0 ($\approx$ nucleon volume) the energy fluctuates by an amount of the order of the total energy contained in V_0. For infinite energy density this fluctuation does not vanish as in ordinary theories, but tends to $\Delta E/E \sim 0.4$. The conjecture is proposed that smaller but still substantial fluctuations of the baryonic quantum number may go along with the energy fluctuations.

Fig. 16.1 Within a year of this popular level lecture, Hagedorn presented a scientific account of his views as shown here (Astron. & Astrophys. **5** 184-205 (1970)). In doing this he contributed decisively to the establishment of the 'Hot Big Bang' as the standard cosmological model. The recognition of the phase boundary between boiling-quark and melting-hadron primordial universe arrived a decade later.



In 1992 a Summer School took place that united experts and students working on hadron production and quark-gluon plasma in laboratory and cosmology. The meeting was organized by G. Belletini, H.H. Gutbrod and J. Rafelski with the principal sponsor being the NATO Scientific Affairs Division. The above presents in abridged format the meeting poster.

Chapter 17
The Long Way to the Statistical Bootstrap Model – 1994

Rolf Hagedorn

Abstract I describe the long way from the first theoretical ideas about multiple particle production up to the situation in which constructing of a statistical model of strong interactions seemed natural. I begin in 1936, and argue that the statistical method came to be from a large network of observations and theoretical ideas. I shall pick up only a few primary lines, chosen for their common end point: the statistical bootstrap model of 1964/65.

> *'It is the nature of a hypothesis when once a man has conceived it, that it assimilates everything to itself, as proper nourishment; and, from the first moment of your begetting it, it generally grows the stronger by everything you see, hear, read or understand. This is of great use.'* [1]

17.1 Introduction

The Statistical Bootstrap Model (SBM) is a statistical model of strong interactions based on the observation that hadrons not only form bound and resonance states but also decay statistically into such states if they are heavy enough. This leads to the concept of a possibly unlimited sequence of heavier and heavier bound and resonance states, each being a possible constituent of a still heavier resonance, while at the same time being itself composed of lighter ones. We call these states clusters (in the older literature heavier clusters are called fireballs; the pion is the lightest 'one-particle-cluster') and label them by their masses. Let $\rho(m)\mathrm{d}m$ be the number of such states in the mass interval $\{m, \mathrm{d}m\}$; we call $\rho(m)$ the 'SBM mass spectrum'. Bound and resonance states *are due to* strong interactions; if introduced as new, independent particles in a statistical model, they also *simulate* the strong interactions to which they owe their existence. To simulate all *attractive* strong interactions we need all of them (including the not yet discovered ones), that is, we need the

---

Keynote talk at the "Hot Hadronic Matter: theory and experiment" workshop, Divonne 1994, Published in Proceedings, NATO-ASI **346**, pp13-46, (Plenum Press, New York 1995)

complete mass spectrum $\rho(m)$. To simulate *repulsive* forces we may use proper cluster volumes à la van der Waals. In order to obtain the full mass spectrum, we require that the above picture, namely that a cluster is composed of clusters, be self-consistent. This leads to the 'bootstrap condition and/or bootstrap equation' for the mass spectrum $\rho(m)$. The bootstrap equation (BE) is an integral equation embracing all hadrons of all masses. It can be solved analytically with the result that the mass spectrum $\rho(m)$ has to grow exponentially. Consequently, any thermodynamics employing this mass spectrum has a singular temperature T_0 generated by the asymptotic mass spectrum $\rho(m) \sim \exp(m/T_0)$. Today this singular temperature is interpreted as the temperature where (for baryon chemical potential zero) the phase transition hadron gas $\longleftrightarrow$ quark–gluon plasma occurs.

The main power of the SBM derives from the postulate that the strong interaction – as far as needed in statistical thermodynamical models – is completely simulated by the presence of clusters with an exponential mass spectrum and with mass-proportional proper volumes. This postulate implies that *in SBM the strongly interacting hadron gas is formally replaced by a non-interacting (i.e., ideal) infinite-component cluster gas with van der Waals volume corrections and exponential mass spectrum*, which can be handled analytically without recourse to perturbative methods.

The story of how this model was first conceived in the language of the grand canonical ensemble, reached maturity in the language of the microcanonical ensemble (i.e., phase space), and was finally equipped with finite particle volumes in order to become applicable to heavy-ion collisions and to the question of the phase transition is presented in Chapter 25 [2].

Here I describe the long way from the first theoretical ideas about multiple particle production up to the situation in which constructing SBM seemed natural. The story starts in 1936, and in my record I omit everything that did not lie on or near the way leading to SBM. What I wish to show is that SBM did not suddenly appear in 1965 as a *deus ex machina*, but was rather the logical consequence of a history of almost 30 years. Thus, from a large network of observations and theoretical ideas, I shall pick up only a few lines, chosen for their common end point: SBM. A complete and impartial picture of this history up to 1972 is presented by E.L. Feinberg in his exhaustive and instructive report [3], which is an indispensable complement to the present biased lecture.

I will try to be as non-technical as possible. Formulae are meant merely as illustrations (often oversimplified); for hard information the reader should consult the quoted literature. Units are $\hbar = c = k$ (Boltzmann) = 1; energy in MeV or GeV.

17.2 From 1936 to 1965

We list here experimental facts and theoretical concepts which were important and instrumental to the construction of SBM.

Fireballs

How did we come to believe that 'fireballs' – the things called 'clusters' in SBM – exist?

Multiple production: Heisenberg (1936)

Before Yukawa's paper postulating the pion [4], one tended to believe that the particles produced in cosmic ray events were electron–positron pairs. The only field theory then known, quantum electrodynamics (as yet without a consistent renormalization scheme), suggested that events with many secondaries should have vanishingly small cross-sections [proportional to $(e^2)^n$]. This led many theoreticians to the interpretation that such events must be the result of many interactions with different nucleons in the same heavy nucleus, each single interaction producing just one pair, a point of view [5–7] persisting even after the advent of meson theory and in spite of growing experimental evidence in favour of multiple production. Heisenberg – still unaware of Yukawa's paper – was the first to claim that, in a single elementary interaction, many secondaries might be produced [8], which at that time was a heretical idea – the pion was discovered 11 years later! Heisenberg followed this idea through many years (until $\sim$ 1955) and devised different theoretical approaches to it, all invoking strong non-linearities and/or diverging field theories. The final, irrevocable decision between his views and his opponents' only came with the first hydrogen bubble chamber pictures: Heisenberg's revolutionary idea had been right.[1] We summarize this line of thought in 'Lesson 1':

> **Lesson One (L1).** In a single elementary hadron–hadron collision, many secondaries can be produced.

Today this is so obvious that calling it a 'lesson' seems ridiculous, but seen in a historical perspective, it challenged a strong prejudice.

Dulles–Walker variables (1954)

Assume a source of particles (a 'fireball') moving with velocity β [Lorentz factor $\gamma = (1-\beta^2)^{-1/2}$] as seen from the lab and assume further that this source emits particles with velocities β_i' isotropically in its own rest frame. We put the z-axis in the direction of motion and call θ_i the polar angle under which particle i is emitted. Quantities in the fireball's rest frame are primed, those in the lab frame are not. In any book on relativistic kinematics, one finds the formula for the angle transformation:

$$\tan\theta_i = \frac{1}{\gamma}\frac{\sin\theta_i'}{\cos\theta_i' + \beta\beta_i'} \approx \frac{1}{\gamma}\tan\frac{\theta_i'}{2}\ , \tag{17.1}$$

[1] Although the various theoretical models he constructed, and which he himself considered as preliminary, did not give final answers to the why's and how's.

where the last approximation is true when β and β_i' are both near 1, which will be assumed from now on.

The fraction F of particles emitted inside a cone of polar angle θ' is, from elementary geometry, in the fireball's frame:

$$F = \frac{1}{2}(1 - \cos\theta') = \sin^2\frac{\theta'}{2}\ , \tag{17.2}$$

while in the lab the same particles – and the same fraction F – will be found inside the cone of angle [see Eq. (17.1)]

$$\tan\theta \cong \frac{1}{\gamma}\tan(\theta'/2) \tag{17.3}$$

so that in the assumed approximation

$$\gamma^2\tan^2\theta \cong \frac{F}{1-F}\ . \tag{17.4}$$

Hence

$$\log\frac{F}{1-F} \cong 2\log\gamma + 2\log\tan\theta\ . \tag{17.5}$$

We note in passing that, with $F = 1/2$, we find the angle $\theta_{1/2}$ of the cone into which half of the particles fall:

$$\tan\theta_{1/2} = \frac{1}{\gamma}\ . \tag{17.6}$$

Now define for each particle i the fraction F_i of particles falling inside the cone of polar angle θ_i (i.e., those having an angle smaller than or equal to that of particle i) and plot the points $y_i = \log F_i/(1-F_i)$ versus $x_i = \log\tan\theta_i$. These points will – *under the supposed conditions* $\beta_1\beta_i' \approx 1$ *and isotropy* – scatter about the straight line given by Eq. (17.5) with slope 2 and intercepts

$$x(y=0) = -\log\gamma\,, \qquad y(x=0) = 2\log\gamma\,, \tag{17.7}$$

as depicted in Fig. 17.1.

The discovery of these variables by Dulles and Walker [9] proved of great importance for the analysis of cosmic ray events: if the points are plotted according to the above rule, then if anything similar to a straight line emerged, an isotropically emitting centre had to be conjectured and its Lorentz factor γ could be read off. Although things were not that simple, the method revealed a lot of information, as we shall soon see.

'Constant' mean transverse momentum (1956)

The invariance of the transverse momenta (of the produced particles) under a Lorentz boost in the z-direction made them interesting from the beginning. The

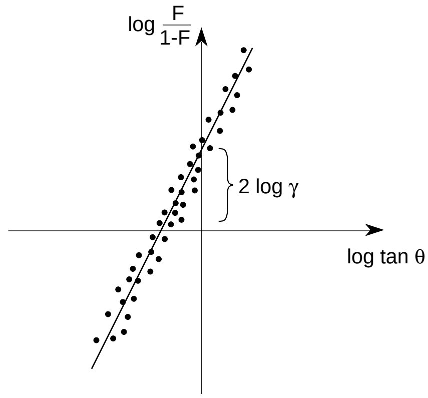


Fig. 17.1 Relativistic particles ($\beta_i' \approx 1$) emitted isotropically in the 'fireball' frame, which itself moves with $\beta \approx 1$ as seen from the lab, will scatter about a straight line with slope 2 and y intercept $2 \log \gamma$ when plotted with Dulles–Walker variables.

amazement was therefore great when it gradually turned out that their average $\langle p_\perp \rangle$ seemed to be practically independent of the primary energy of the collision from which they emerged. This was reported in so many papers over so many years that I cannot quote all of them. Probably J. Nishimura was the first to have pointed it out [10]. The result was by 1958 rather well confirmed [11] and remained so until the 'large transverse momenta' were discovered in 1973 [12], which – important as they were – corrected this result only slightly. We write down 'Lesson 2':

> **Lesson Two (L2).** Secondaries emerging from high-energy hadron collisions have mean transverse momenta of order $\langle p_\perp \rangle \approx 500$ MeV/c, rather independently of the collision energy.

The two-centre model (1958)

The most prominent qualitative feature of the particle tracks in emulsions and/or cloud chambers was that they were arranged in two cones: a wide one and, inside it, a narrow one. No measurements were needed to see this and to guess a simple mechanism that would produce it: two 'centres', one moving slowly and another moving fast[2] along the collision line, both emitting particles isotropically and with rather small, energy-independent momenta [Lesson 2] in their respective rest frames. I do not know whether one could say that a particular physicist had this idea first (it might have been Takagi [13], but I am not sure): it must have appeared obvious to anyone who saw pictures of these events. It was another thing to analyse such pictures quantitatively. The pioneers were the Cracow–Czech collaboration [14] and G. Cocconi [11]. They exploited the powers of the Dulles–Walker representation.

[2] In the lab; in the CM frame, one forward, one backward.

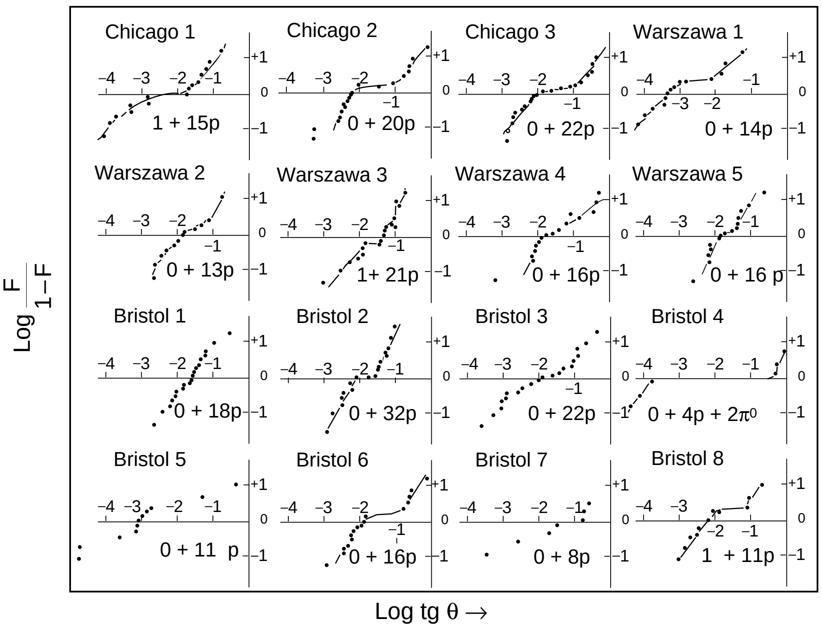


Fig. 17.2 Experimental data plotted in Dulles–Walker variables.

The story went like this: one applied the Dulles–Walker plot to the available events [11, 14] and, instead of finding the points representing the tracks scattering about a straight line – as expected for a single 'fireball' – one found something very different. The result is show in Figs. 17.2 and 17.3 which I copy from Cocconi's paper [11]. The spirit of Cocconi's paper is so well concentrated in a few original passages that I repeat them here. Cocconi says:

> It is evident from an examination of Fig. 2 [our Fig. 17.2] that in most cases the relativistic secondaries are separated into two groups as if they were emitted, in the CM centre of the collision, not by a single centre but by two bodies, as described in Section II(d)[3]. The evidence is so striking that we are going to analyse these events in a slightly different manner, more adjusted to the model.
>
> Instead of considering all the relativistic particles produced in the collision together, let us divide them into two groups: the forward group, b_1, and the backward group, b_2 (the narrow and wide cones).
>
> Let n_1 and n_2 be the number of particles falling in each group and let us analyse them in terms of $\log\tan\theta$ versus $\log[F_1/(1-F_1)]$ and versus $\log[F_2/(1-F_2)]$. The results are plotted in Fig. 3 [our Fig. 17.3].

Figures 17.2 and 17.3 and Cocconi's remarks need no further comment. It should be noted, however, that he is aware of the possibility of other interpretations, in which not individual 'fireballs', but a two-jet structure produces much the same effect.

[3] Cocconi proposes a two-centre model in Section II(d) of his paper (with two 'leading nucleons' not contained in the 'fireballs').

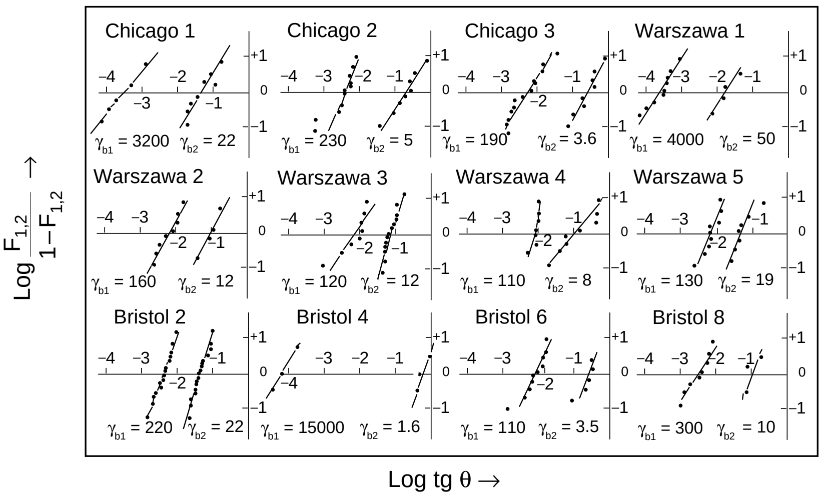


Fig. 17.3 Some of the events of Fig. 17.2. Forward and backward (CM) particles plotted separately.

The two-centre model was popular for a long time, as witnessed by the review paper written by Gierula [15] in 1963, five years later, and based on more than 100 events with $E_{\text{lab}} \gtrsim 10^3$ GeV. If I remember well from those years, the model did not always work – sometimes three or more fireballs had to be invoked – but on the whole it was rather successful. That it seems never to have been disproved came perhaps from the shift of interest to other questions arising from working with accelerators, where single events were analysed mostly in the hope of discovering new particles, but not to prove or disprove a two-centre model. The famous 'flat rapidity plateau' was, of course, no argument against a two- (or few-) centre model, as it arose from averaging over the impact parameter in many collisions contributing to the measured inclusive distributions. We thus draw 'Lesson 3':

Lesson Three (L3). Secondaries produced in elementary hadron collisions seem to be emitted from few (≈ 2) 'fireballs' rather isotropically with small momenta in the fireball's rest frame.

Conclusion: Fireballs with limited $\langle p \rangle$ exist

We conclude that 'fireballs', decaying with limited momenta, do exist. In other words, *lumps of highly excited hadronic matter keep together for a very short time before they decay in a very specific and – on this level – not yet understood way.*

Statistical and thermodynamical methods

Having collected, in the previous section, the arguments in favour of the existence of ‘fireballs’, we now turn to their description. The methods and the models used eventually for this description were developed long before the existence of their final objects was established. In fact, the story goes back to two early theoretical ideas:

- the compound nucleus of Bohr in 1936,
- the incorporation of interaction in statistical thermodynamics via scattering phase shifts by Beth and Uhlenbeck in 1937.

Bohr’s compound nucleus (1936)

Bohr [16] proposed the following picture for a certain class of nuclear reactions: if a heavy nucleus is hit by a nuclear particle, then the strong interaction among the constituents and with the projectile can often lead to a complete dissipation of the available energy, so that no single nucleon gets enough of it to escape at once. This excited ‘compound nucleus’ will then live a rather long time before it decays by emitting nucleons which accidentally obtained sufficient kinetic energy to overcome the binding force. Of course, this picture cries out for a statistical description.

The Weisskopf evaporation model (1937)

We did not have to wait long for it. Weisskopf [17] writes, in his famous paper on nuclear evaporation:

> Qualitative statistical conclusions about the energy exchange between the nuclear constituents in the compound state have led to simple explanations of many characteristic features of nuclear reactions. In particular the use of thermodynamical analogies has proved very convenient for describing the general trend of nuclear processes. The energy stored in the compound nucleus can in fact be compared with the heat energy of a solid body or a liquid, and, as first emphasized by Frenkel [18],[4] the subsequent expulsion of particles is analogous to an evaporation process.

Weisskopf is cautious. He does not claim right away that thermodynamics is applicable to nuclei; he rather derives first from elementary quantum mechanics a formula for the emission of a neutron by the excited nucleus A, leaving another excited nucleus B behind. For this he uses the principle of detailed balance by considering the inverse reaction $B+n \to A$, of which the cross-section is supposed to be known. From this, the emission probability can be calculated; it is a very simple expression containing the above cross-section, the level densities $\omega_A(E_A)$ and $\omega_B(E_B)$ of nuclei A and B, at their respective energies, and the phase space available for a neutron of given kinetic energy ε. Then he introduces quite formally an ‘entropy’, viz.,

$$S(E) = \ln \omega(E) , \tag{17.8}$$

[4] Our list of references.

and a 'temperature', viz.,

$$T(E) = (\mathrm{d}S/\mathrm{d}E)^{-1} . \tag{17.9}$$

In these variables, the derived formula for the emission probability assumes the usual form of an evaporation probability with a Boltzmann spectrum:

$$W(\varepsilon)\mathrm{d}\varepsilon \sim \varepsilon \exp(-\varepsilon/T)\mathrm{d}\varepsilon . \tag{17.10}$$

The rest of the paper discusses when the formula is valid and what corrections are necessary. What interests us here is that this is (to my knowledge) the first time that it was shown quantitatively that thermodynamics might be applied to such a tiny system as a nucleus. *The reason is the enormous level density* of heavy nuclei at high excitation energy. Note also that the formula was derived for the emission of a single neutron with only a few degrees of freedom (phase space). We conclude with 'Lesson 4':

> **Lesson Four (L4).** Thermodynamics and/or statistics might be (cautiously!) applied to very small systems, provided these have a very large level density (whatever that means).

When Weisskopf wrote his paper, not much was known about the level densities of nuclei, and he proposed to learn about them from the observed emission spectra, taking his formulae for granted.

Koppe's attempt and the Fermi statistical model (1948/1950)

Although traditionally all credit for the invention of a statistical model for particle production goes to Fermi (see below), it was actually H. Koppe who proposed, in fact two years earlier, the essence of such a model. He wrote [19]

> In a recent paper [20],[5] a simple method has been given for the calculation of the yield of mesons produced by the interaction of light nuclei. It was based on the assumption of strong interaction between mesons and nucleons which should make it possible to treat a nucleus as a 'black body' with regard to meson radiation and to calculate the probability for emission of a meson by statistical methods.

At that time, available energies were not high (Berkeley: α-particles of ~ 380 MeV) and consequently the temperatures remained small ($\sim$ 15 MeV), well below the pion rest mass. Yet the model did not work too badly. Note that (for him) the high level density justifying the treatment was located not in the meson field but in the interacting nuclei ('black body').

Fermi [21] then takes the important step of considering the pion field itself as the thermal (or better, statistical) system without the need for a background 'black body' à la Koppe. Thus he claims that, e.g., a proton–proton collision could be treated statistically. He writes [21]:

[5] Our reference; the paper is written in German.

> When two nucleons collide with very great energy in their CM system, this energy will be suddenly released in a small volume surrounding the two nucleons. We may think pictorially of the event as of a collision in which the nucleons with their surrounding retinue of pions hit against each other so that all the portion of space occupied by the nucleons and by their surrounding pion field will be suddenly loaded with a very great amount of energy. Since the interactions of the pion field are strong, we may expect that rapidly this energy will be distributed among the various degrees of freedom present in this volume according to statistical laws. One can then compute statistically the probability that in this volume a certain number of pions will be created with a given energy distribution. It is then assumed that the concentration of energy will rapidly dissolve and that the particles into which the energy has been converted will fly out in all directions.

After some further discussion he writes down his basic formula for the production of n pions (in modern notation):

$$S(n) = \frac{1}{n!}\left[\frac{V_0}{(2\pi)^3}\right]^n \int \delta\left(E - \sum_{i=1}^{n} E_i\right)\prod_{i=1}^{n} 4\pi p_i^2 \mathrm{d}p_i\,, \qquad E_i = \sqrt{p_i^2 + m^2}\,, \quad (17.11)$$

where V_0 is the 'interaction volume' [order $(4\pi/3)m_\pi^{-3}$ and Lorentz-contracted or not, according to taste], E the total centre-of-mass (CM) energy, and m the pion mass (or that of another species if considered).

The rest of his paper discusses applications at low, medium, and very high energies; in the latter case a thermodynamic formulation is proposed, where the temperature is proportional to $E^{1/4}$ – that is, a Stefan–Boltzmann gas of (massless) pions is assumed. The way to this is already prepared when he discusses medium energies, where a good number of pions are produced: in order to use the only existing analytical expressions for n-body momentum space (namely for $m = 0$ and/or for $m \to \infty$), he treats pions as massless and nucleons as non-relativistic. At this time (1950), these assumptions were reasonable. The discovery of 'limited transverse momenta' [10], which of course would invalidate them, was to come only six years later. He also mentions angular momentum conservation, but only to argue that it is unimportant; he soon comes back to this question in an attempt to explain the observed anisotropy in CM [22], which failed. We pass over these details.

What is important for us is that Fermi actually tries to describe the disintegration of what we called above 'fireballs' – eight years before they were discovered experimentally [11, 14].

While the model fails quantitatively (Heisenberg [23] quotes a measured event with an estimated primary energy of 40 GeV, where about 27 pions were actually produced, in contrast to 2.7 predicted by Fermi in the thermodynamic version), it is nevertheless the starting point for the development leading to the SBM.

Looking back we see a line of thought that leads from the Bohr compound nucleus directly to the *theoretical* concept of a hadronic fireball and its statistical (thermodynamical) description.

Beth–Uhlenbeck, Belenkij (1937/1956)

At the beginning of this section two main theoretical ideas were said to be essential for a statistical description of fireballs. One was the Bohr compound nucleus, leading directly to the Fermi model. The second is found in a paper by Beth and Uhlenbeck [24]. The authors incorporate interaction in statistical thermodynamics quantum mechanically via scattering phase shifts. We shall only sketch the idea. Details may be found in [25] and [26].

Suppose you have an ideal gas consisting of N non-interacting particles with masses $m_1, m_2, \ldots, m_N$ at total energy E enclosed in a volume V. Let the level density of this gas be $\sigma_N(E, V, m_1, \ldots, m_N)$. If a force acts between particles numbered 1 and 2, they may form a bound state m_{12}, and (if nothing else happens) the level density of this new system becomes $\sigma_{N-1}(E, V, m_{12}, m_3, \ldots, m_N)$. The interaction has changed the level density and the system with interaction would be described as a mixture of two ideal gases with and without bound states.

Beth and Uhlenbeck extend this argument to the case where the interaction leads not to a bound state but only to scattering. Single out from our gas two particles that are to scatter on each other, and take as normalization volume a sphere of radius R centred at the point of impact of the two particles. The density of states of this two-particle subsystem will be affected by the scattering process in that the ℓth partial wave of the common wave function of our two particles will be, asymptotically:

$$\psi_\ell(r, p) \sim \frac{1}{pr} \sin\left[pr - \frac{\ell\pi}{2} + \eta_\ell(p)\right] , \tag{17.12}$$

where p is their relative momentum, r their distance, and $\eta_\ell(p)$ the scattering phase shift. The wave function should vanish at $r = R$:

$$pR - \frac{\ell\pi}{2} + \eta_\ell(p) = n\pi , \quad n = 0, 1, 2, \ldots . \tag{17.13}$$

Thus n labels the allowed (discrete in R) two-body momentum states $\{p_0, p_1, \ldots\}$. For a fixed p', there are $n(p')$ states below p', the density of states near p' is

$$\frac{\mathrm{d}n}{\mathrm{d}p'} = \frac{R}{\pi} + \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}p'}\eta_\ell(p') . \tag{17.14}$$

Without interaction, $\eta_\ell(p) \equiv 0$. Hence, *the interaction changes the two-particle density of states by* $(1/\pi)\mathrm{d}\eta_\ell/\mathrm{d}p$. Of course this argument has to be repeated for all partial waves and all particle pairs with the final result that the sum over ℓ gives a contribution to the partition function containing the derivative of the scattering amplitude [25, 27]. The *formal* extension of this method to include all interactions is due to Bernstein, Dashen, and Ma [28].

For the following argument of Belenkij [29], the simple equation Eq. (17.14) is most illustrative. Let the two-body subsystem have a sharp resonance at relative momentum p^*. Then the phase shift rises there by π within a short interval, so that $(1/\pi)\mathrm{d}\eta_\ell(p)/\mathrm{d}p \approx \delta(p - p^*)$. Such a δ-function appearing in the density of states is equivalent to introducing an additional particle with mass $m^* = m(m_1, m_2, p^*)$ into the system, very much as a bound state would be introduced. The actual proof is somewhat complicated due to the switching between different sets of momenta. Belenkij does this in detail. We state 'Lesson 5':

> **Lesson Five (L5).** If in a statistical–thermodynamical system two-body bound and resonance states occur, then they should be treated as new, independent particles. Thereby a corresponding part of the interaction is taken into account.

Note that after doing so, the system is still formally an 'ideal gas', but now with some additional species of particles (simulating part of the interaction).

Belenkij's motivation for his work had been the known fact that Fermi's model gave wrong multiplicities: "This discrepancy may be as high as 20 times." He had hoped that his new remedy [expressed by (L5)] would cure the disease of the Fermi model; it did so only partially, for reasons to become clear soon.

When we adopted Belenkij's argument for including resonances, we did so because it was intuitively obvious that resonances should be included even when the formal derivation could not be directly invoked, as for instance in a process $A + B \to$ resonance $\to n$ particles, where a phase shift increasing by π is not defined. Incorporating resonances quite generally was later justified by Sertorio via the S-matrix approach to statistical bootstrap in an important paper [30].

The CERN statistical model (1958–1962)

When in 1957 the CERN PS was near completion, planning of secondary beam installations required estimates of particle production yields and momentum spectra. Bruno Ferretti, our division leader at that time, asked Frans Cerulus and me to do some calculations with the Fermi model ("just a fortnight of easy work ...", he said), not surmising that by that request he triggered a new development. In fact, we soon found out that the Fermi model, as it was, could not be used:

1. In the fireball rest frame, neither were pions ultrarelativistic nor nucleons nonrelativistic; indeed Lesson 2 (limited transverse momenta) excluded this, so there were no analytic formulae available to calculate momentum space integrals [Eq. (17.11)].
2. Interaction between the produced particles might be important; the ideal gas approximation could lead to large errors.

For the second problem Belenkij had already given the solution: include all known particles and resonances (Lesson 5). For the rest, we were confident: fireballs seemed to exist (Lesson 3 was known to us by hearsay) and their statistical description in principle possible (Lesson 4). We earnestly hoped that an improved Fermi model would do. Problem 2 being trivial (thanks to Belenkij), once problem 1 was overcome, we concentrated on the calculation of momentum space integrals. Cerulus had the idea to use the Monte Carlo method and we worked it out. At that time this was a new method, not familiar to physicists; moreover the first CERN computer was only to come in a year or so. So we had tried our new methods [31] with the help of the Institute of Applied Mathematics at Darmstadt, where an IBM computer was available (not very powerful in 1958!) and we had found that momentum space integrals with up to ~ 15 particles could be computed in reasonable time and reliably with *prescribable* error (5–10%).

I then took it over to write a program (my first!) for the expected CERN computer, a Ferranti Mercury. It was an adventure: I had to learn to program a non-existing machine, still under development, with no possibility of checking written parts of the program. Everything was to be expressed in machine code – a simple addition required four lines of code and all store addresses were absolute. One had to keep account of where each number (including intermediary results) was stored – and there were thousands of them (momentum spectra of some 15 species of particles).

After half a year I had finished the program (so I thought) and went to Saclay in France to test it on the first delivered Mercury machine. It failed beyond all expectation. Correcting was even more tricky than writing the program (which consisted of thousands of lines of numbers – no letters, no symbols; find the error!). It even required manual skill: input and output was via punched paper tape and one had to find the erroneous part (reading the tape by eye had to be learned, too), cut it out, and replace it carefully by the corrected piece, gluing everything together properly in case the teletape reader refused it or tore it to pieces.

In short, it was a mess, but finally it came to work, and in hundreds of hours we produced kilometres of tape with our precious results: the first accurate evaluations of the Fermi model including some interaction (all known particles plus some resonances) at several primary energies from 2 to 30 GeV (lab) and for *pp* and πp collisions. Cerulus [32] had used a very elegant group theoretical method to solve the problem of charge distribution, and then employed the same method to implement angular momentum conservation in phase space [33], in the hope of reproducing the known anisotropy. It failed (because it required more computing than was then possible and also) because the process was not so statistical as we had hoped: angular momentum conservation could not produce the pronounced anisotropy found in cosmic ray events and well accounted for by two-centre models [11, 14] – a fact strongly suggesting that phase relations between partial waves survived the statistical mixing assumed in the Fermi model. In principle we had the tools to build and correctly evaluate a two–centre model, but it would have required at least ten times more computing (summing over impact parameters with varying fireball energies), which was impossible (we had already spent several years to do all the computing for single fireballs). Thus the angular distribution could not be described adequately.

But we had a more important success [34]: from the calculated momentum spectra it followed that the mean kinetic energy of all particles was practically independent of the primary lab energy (6 to 30 GeV). Thus the model more or less correctly produced the empirical fact stated in Lesson 2 (limited transverse momenta). However, this was only a numerical result, due to the large number of species of particles entering our calculations: counting spin, isospin, and antiparticles of the included ones (π, ρ, ω, K, N, Δ, Λ, Σ, Ξ), we came to 83 different particle states, equivalent to 83 species. This proved important in the following development, where it was the key opening the door to the statistical bootstrap model, when we tried to understand this mechanism analytically.

We state ‘Lesson 6’:

> **Lesson Six (L6).** A properly evaluated Fermi model with some interaction (resonances; L5) produces, in a limited interval of primary energy, practically constant mean kinetic energies of secondaries and reasonable multiplicities.

A review of our work is given in [34]. See also [35].

The decisive turn of the screw: Large-angle elastic scattering

Early in 1964 evidence was growing that the elastic *pp* cross-section around 90° (CM) decreased exponentially with the total CM energy, at least in the then known region $10 \leq E \leq 30$ GeV of primary energies (lab). Many experiments contributed to this, and we cannot list them all here. The situation – theoretical and experimental – is well described in a paper by G. Cocconi [36], where references to the original experiments are given.

One can include angles a little below and above 90° by using transverse momentum $p_\perp = p \sin\theta$. Orear [37] obtained in this way an impressive fit to large-angle elastic scattering (in what follows, E is always the total CM energy and $\mathrm{d}\omega$ is the solid angle, frequently denoted $\mathrm{d}\Omega$):

$$E^2 \frac{\mathrm{d}\sigma}{\mathrm{d}\omega}\bigg|_{e\ell} = \text{const.} \times \exp\left(-\frac{p_\perp}{0.158}\right) , \tag{17.15}$$

which is shown in Fig. 17.4. The cross-section follows this empirical formula over nine orders of magnitude in the interval $1.7 \leq p_0 \leq 31.8$ GeV/c (primary lab momentum). Moreover, the reaction $p + p \to \pi + d$, as well as πp elastic scattering, showed the same behaviour. In particular the exponential decrease had the same slope as in *pp* elastic scattering [37].

It is tempting to interpret Eq. (17.15) as a thermal Boltzmann spectrum. In that case 0.158 GeV would be the ‘temperature’ at which the two nucleons were emitted.[6] *It thus seemed that there was something statistical*, a suspicion strangely corroborated by the observation that the same law (with slightly different ‘temperature’) was obeyed by secondaries in inelastic processes [36]. Almost two years before the Orear plot was published, L.W. Jones had already proposed a statistical interpretation: the two colliding particles would sometimes form a fireball – in analogy to the compound nucleus – which would decay into many channels, among them the two-body channel containing the original particles. This two-body decay could only be observed far outside the diffraction peak, that is, at large angles. He asked

[6] If E is used at 90° instead of $p_\perp \cong E/2$, the exponent becomes $-E/0.31$, which was sometimes misinterpreted as $T \approx 0.3$ GeV.

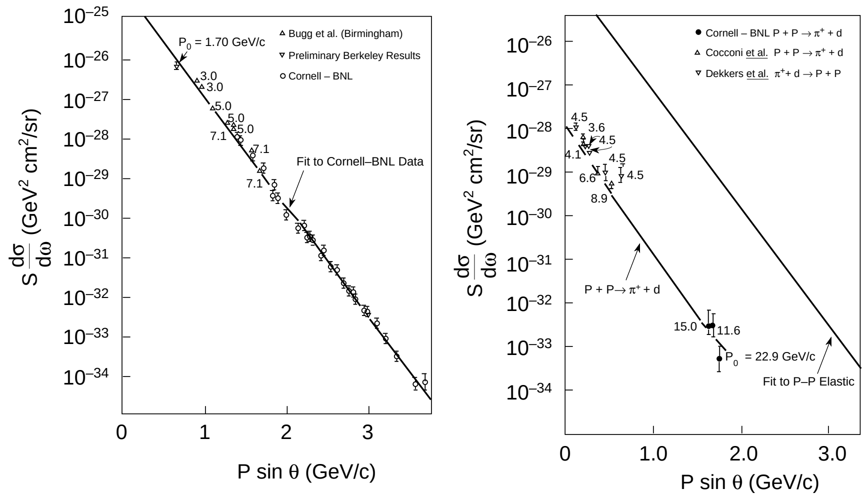


Fig. 17.4 The two original figures of Orear [37] for large-angle pp scattering and for $p+p \to \pi + d$, showing the exponential decrease with the same slope.

me whether such a picture could be described quantitatively by our statistical model.

Statistical model description of large-angle elastic scattering

For some obscure reasons, we had archived all results obtained since 1958,[7] and even included the two-body channel. It was simple to analyze them again and to find the amazing result

$$E^2 \frac{d\sigma_{e\ell}}{d\omega}\bigg|_{90^\circ} \sim E^2 \frac{P_0}{\sum_b P_b} \cong \text{const.} \times E \exp(-3.17E)\,, \tag{17.16}$$

where P_0 is the probability of the two-body channel and ΣP_b the sum over the probabilities of all channels. P_0 and all the P_b were numerical results from hundreds of phase-space calculations as described above. When we established Eq. (17.16), no free parameters were available, everything was in our archived data. This result [38] agreed reasonably well with early experimental data [36], but when the Orear fit [37] was published, the agreement became perfect: the number 3.17 in the exponent of Eq. (17.16) corresponds to a temperature $T = 0.158$ if $E = 2p_\perp$ (at 90°) is inserted; then Eq. (17.15) results (the factor E in front of the exponential is negligible).

Thus L.W. Jones' proposal was immensely successful. An independent confirmation by the observation of Ericson fluctuations [39] would have been desirable, but I do not know if it was ever tried. It was probably too difficult.

[7] Several people had contributed to the accumulation of statistical model results: J. von Behr, F. Cerulus, H. Faissner, G. Fast, K.H. Michel, J. Soln, and myself.

Thermal description

Our results were so convincing to me (unfortunately not to most others; among the few exceptions was G. Cocconi) that I firmly believed that in Eq. (17.16) we really had an exponential function and not something approximately exponential. This belief (which was directly leading to the statistical bootstrap model) had to be justified better than by Eq. (17.16), which was merely a numerical result established in a rather small range of energy $\sim 2.4 \leq E \leq 6.8$ GeV.

However, the belief was seriously challenged by A. Białas and V.F. Weisskopf [40] who had given a thermodynamical description based on assumptions that I considered unsuitable, but which nevertheless also gave a good fit to the data. What were these assumptions? Mainly these:

- The compound system is a hot gas.
- As constituents only pions are considered. K mesons and resonances are assumed to be unimportant.
- Pions are taken as massless.

These were exactly the assumptions that Fermi [21] had already made and that had led to wrong multiplicity estimates [23, 29].

From the above assumptions, it follows immediately that the gas is at the black-body temperature (Stefan–Boltzmann law)

$$T(E) = \text{const.} \times E^{1/4} \ . \tag{17.17}$$

Therefore a Boltzmann spectrum for elastic scattering at 90° would be of the form

$$\exp\left[-\frac{p_{\perp}}{T(E)}\right] = \exp(-\text{const.} \times E^{3/4}) \ , \tag{17.18}$$

instead of our result $\sim \exp(-\text{const.} \times E)$ as given in Eq. (17.16). For $(\mathrm{d}\sigma/\mathrm{d}\omega)_{90^\circ}$, the authors derive an expression that contains Eq. (17.18) as the essential part, the rest being algebraic factors.

The difference is in principle fundamental, but it is numerically insignificant in the range of energies then available. Apparently the Orear plot [37], which might have pleaded in favour of a pure exponential, was not yet available to the authors (as seen from the dates of reception of the two papers).

Exponential or not?

This question was so important that I wish to formulate it in two different ways:

1. Our result was [see Eq. (17.16)] that ΣP_b grows exponentially with E (the other factors being algebraic). Now, a given phase-space integral for b particles is

the density of states of the b-particle system at energy E; thus ΣP_b is the total density of states of the 'fireball' at energy E. *If our result Eq. (17.16) were true, it would mean that the density of states of hadronic fireballs would grow exponentially with their mass* $(= E)$ *up to at least m = 8 GeV.*

2. The second formulation is a consequence of the first. The entropy is the logarithm of the density of states, hence the entropy of a fireball would be

$$S(E) = \text{const.} \times E \tag{17.19}$$

and therefore its 'internal temperature' would be

$$T = (\mathrm{d}S/\mathrm{d}E)^{-1} = \text{const.} \tag{17.20}$$

In words, *if our result Eq. (17.16) were true, it would mean that the internal temperature of hadronic fireballs would be independent of their mass* $(M \equiv E)$.

This would also explain (in a thermodynamic language) why our phase-space calculations had given 'constant' mean kinetic energies [Lesson 6]: particles were emitted with a Boltzmann spectrum at an energy-independent temperature. We had suspected that this behaviour was due to our including interaction by admitting all relevant species of particles and resonances known to us, but that had remained a speculation up to then.

Cocconi had clearly seen what was going on. He writes [36]:

> If the dependence of S on E is of the form $S = aE^n$, it follows that $\mathrm{d}\sigma/\mathrm{d}\omega = const. \times \exp(-aE^n)$ and that the temperature of the compound system is $T = (naE^{n-1})^{-1}$. The value of n characterizes the 'gas' of the compound system [...]; $n = 1$ corresponds to the case of a system in which, *for E increasing, the number of possible kinds of particles increases so as to keep the energy per particle, and hence the temperature, constant.*[8]

Commenting on our phase-space results [34], he wrote:

> This model produces an essentially 'constant temperature' because, in the compound system, *beside the nucleons and mesons, also the known excited states are counted separately.*

All this can be conveniently summarized in Lesson 7:

> **Lesson Seven (L7).** The exponential decrease in the elastic *pp* cross-section at large angles up to a CM energy of about 8 GeV had empirically established the existence of 'fireballs' (clusters; compound states) up to at least $m = 8$ GeV. Moreover, their density of states had to grow exponentially as a function of their mass up to at least $m = 8$ GeV, which means that, if the level density is interpreted as a mass spectrum, there were an unexpectedly large number of resonance-like states above those few then explicitly known.

The question now was: *could a reasonable analytical model for fireballs be constructed, which would lead to an exponentially growing density of states and, consequently, to an energy-independent temperature?*

[8] The italics are mine.

Asympotics of momentum space

The question just formulated was in the mind of several people who therefore investigated the asymptotic behaviour of momentum space integrals for $E \to \infty$ [41–43]. They all consider essentially a pion gas and show that for $E \to \infty$ the masses become negligible and that the asymptotics can be evaluated there. All authors agree that (in general) the density of states for $E \to \infty$ then grows like $\exp(\text{const.} \times E^{3/4})$, just as for a gas of particles with $m = 0$. Vandermeulen as well as Auberson and Escoubès consider also the pathological case where the usual factor $1/n!$ in front of the phase-space integral is omitted. They discover the amazing fact that, *if the factor $1/n!$ in front of the phase-space integral is omitted, then the density of states for $E \to \infty$ grows like $\exp(const. \times E)$*. I give here a simple derivation of this result, taking all masses $m = 0$ from the outset and passing over subtleties such as the difference between $\langle E \rangle$ and E in thermodynamics.

For zero masses, the particle energies equal their momenta and the n-body phase-space integral (= n-body density of states at energy E) with spatial volume V becomes

$$\sigma_n(E,V) = \frac{1}{n!}\left[\frac{V}{(2\pi)^3}\right]^n \int \delta\left(E - \sum_{i=1}^{n} p_i\right) \prod_{i=1}^{n} 4\pi p_i^2 \mathrm{d}p_i \, . \tag{17.21}$$

The n-body partition function is then the Laplace transform of σ:

$$Z_n(T,V) = \int_0^\infty \sigma_n(E,V)\mathrm{e}^{-\beta E}\mathrm{d}E = \frac{1}{n!}\left(\frac{V}{8\pi^3}\right)^n \left(\int_0^\infty \mathrm{e}^{-\beta p} 4\pi p^2 \mathrm{d}p\right)^n \, , \tag{17.22}$$

where $\beta = 1/T$ (we use β and T for convenience). The last integral equals $8\pi T^3$, so that

$$Z_n(T,V) = \frac{1}{n!}\left(\frac{VT^3}{\pi^2}\right)^n = \frac{1}{n!} Z_1(T,V)^n \, . \tag{17.23}$$

Summing over n gives the partition function for our gas with particle number not fixed:

$$\begin{aligned} Z(T,V) &= \Sigma Z_n(T,V) = \exp\left[Z_1(T,V)\right] \, , \\ \ln Z(T,V) &= Z_1(T,V) = \frac{VT^3}{\pi^2} = -\frac{F}{T} = -\frac{1}{T}(E - TS) = S - \beta E \, , \end{aligned} \tag{17.24}$$

where F is the free energy and S the entropy. It follows that

$$E = -\frac{\mathrm{d}}{\mathrm{d}\beta} \ln Z = \frac{3VT^4}{\pi^2} \, , \tag{17.25}$$

which is the Stefan–Boltzmann law for Boltzmann statistics. Further

$$S = \beta E + \ln Z = \frac{4VT^3}{\pi^2} \, . \tag{17.26}$$

If S is expressed as a function of E (as it should be), we have

$$S = \left(\frac{256V}{27\pi^2} \right)^{1/4} E^{3/4} , \tag{17.27}$$

and the density of states becomes, as derived more rigorously in the above papers,

$$\sigma(E,V) = \mathrm{e}^S = \exp(\text{const.} \times E^{3/4}) . \tag{17.28}$$

But the situation changes drastically if the factor $1/n!$ *is omitted.* Go back to Eqs. (17.23) and (17.24) and drop $1/n!$ there. The sum now gives

$$Z(T,V) = \frac{1}{1-Z_1} = \frac{1}{1-VT^3/\pi^2} = \frac{T_0^3}{T_0^3 - T^3} , \qquad T_0 = (\pi^2/V)^{1/3} . \tag{17.29}$$

For $T \to T_0$ the partition function diverges. Hence T_0 is a singular temperature for this gas.

Now the miracle happens. Assume V to be the usual 'interaction volume' of strong interactions [21] (without Lorentz contraction):

$$V \approx \frac{4\pi}{3} m_\pi^{-3} \quad \text{gives } \; T_0 \cong \left(\frac{3\pi}{4} \right)^{1/3} m_\pi \cong 0.184 \text{ GeV} . \tag{17.30}$$

This is almost the mysterious 'constant' temperature so often encountered in this report. Following the standard procedure, we calculate the energy and entropy. Both become simple for $T \to T_0$:

$$E \approx \frac{3T_0^4}{T_0^3 - T^3} , \qquad S \approx \frac{E}{T_0} + \ln \frac{E}{3T_0} . \tag{17.31}$$

The energy diverges for $T \to T_0$ (therefore T_0 is the *maximum* temperature for this gas). For the level density, we obtain

$$\sigma(E,V) = \mathrm{e}^S \cong \frac{E}{3T_0} \exp \frac{E}{T_0} , \tag{17.32}$$

that is, omitting the factor $1/n!$ leads to a maximum temperature and to an exponentially growing density of states. Equation (17.31) implies that, for $E \geq 10T_0$, one always finds a temperature $0.9T_0 \leq T < T_0$, hence practically constant.

This result brought me – *me*, but nobody else – to a state of obsession. Did it not explain one of the most intriguing features of strong interaction processes? And was it not obviously wrong because of its unrealistic assumptions? Yet, there was an interpretation that opened the way to a better model.

Interpretation: distinguishable particles and Pomeranchuk's ansatz

The factor $1/n!$ in front of the phase-space integral Eq. (17.21) serves to compensate for 'double' counting: given a set of fixed momenta $\{\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_n\}$, all $n!$ permutations of this set occur during the integration over $p_1, p_2, \ldots, p_n$. If the particles are indistinguishable, one has therefore to divide by $n!$. If all n particles are different from each other, one should not divide. This was exactly the point: in our statistical model calculations we had used ~ 80 different particle states and had therefore to replace

$$\frac{1}{n!} \to \frac{1}{\Pi n_k!} , \quad k = 1, \ldots, 80 , \tag{17.33}$$

in front of the phase-space integrals. However, since the number of produced secondaries remained far below 80, the values n_k remained, *for all essentially contributing phase-space integrals*, either 0 or 1. Hence, practically all $n_k! = 1$, and $1/n!$ was *effectively* replaced by 1. If *this situation was to be simulated* analytically by a solvable model [namely all masses = 0 (for $E \to \infty$)], then in order to come near to reality, the factor $n!$ should be dropped, as if the particles were distinguishable.

This argument led Auberson and Escoubès to look at the case where $1/n!$ is dropped. They also considered a scenario corresponding to Eq. (17.33), namely where there are r different species of particles, while inside each species, particles are indistinguishable. They are cautious in the interpretation of their results [41]:

> If it is probable that the discernibility hypothesis is the most realistic at low energies, one cannot very well locate the energy at which this hypothesis must be abandoned (if at all).

And later:

> Clearly r could be larger than 3, to take into account the resonances at high energies.[9] (If, however, in reality the strongly interacting particles should have an infinity of excited states [...] we fall back essentially on discernible particles.)

They leave the question open.

In the present context, a paper by Pomeranchuk [44] must be mentioned. He proposes to improve the Fermi model by admitting that real pions are not pointlike. Therefore n pions would not find room in a volume V_0 $(\approx 4\pi m_\pi^{-3}/3)$, but need at least a volume nV_0. Thus the space volume factor in front of the integral in Eq. (17.21) would be $[nV_0/8\pi^3]^n$ instead of the one appearing in Eq. (17.21). However, for large n,

$$n! \approx \sqrt{2\pi n}\, n^n \mathrm{e}^{-n} , \tag{17.34}$$

the factor n^n arising from the corrected volume will essentially cancel the factor $1/n!$, and one thus arrives effectively at a model with 'distinguishable' particles in a volume eV_0. In this way, Pomeranchuk also obtains a maximal temperature of the order of m_π, that is, a practically constant temperature. His paper came about thirteen years too early – or at least five, since the constant mean transverse momenta

---

[9] r is the above number of different species of particles, roughly 80 in our old phase-space calculations.

became popular only after 1956 [10] (the decisive large-angle scattering took shape around 1964).

While the model of distinguishable particles was useful because it produced the surprise that motivated the investigations described in the following section, it was clear that all further efforts had to be made on the realistic basis of massive particles with Bose and/or Fermi statistics. The principal lesson to be kept in mind was that there should be many, many different species of particles.

17.3 The Statistical Bootstrap Model (SBM)

Up to here we have collected everything that helped to motivate the construction of SBM. We now describe this construction. For what follows, a few formulae need to be recalled; although everybody knows them, it is necessary to have them ready at hand in order not to interrupt the argument. We use Boltzmann statistics for simplicity (the first paper on SBM used correct statistics [45]).

A few well-known formulae

We go back to Eq. (17.21), rewrite it for relativistic massive particles, and follow the same derivations as there. The density of states of n particles of mass m enclosed in a volume V at energy E is then

$$\sigma_n(E,V,m) = \frac{1}{n!}\left[\frac{V}{(2\pi)^3}\right]^n \int \delta\left(E - \sum_{i=1}^{n} E_i\right) \prod_{i=1}^{n} 4\pi p_i^2 \mathrm{d}p_i \ , \quad E_i = \sqrt{p_i^2 + m^2} \ . \tag{17.35}$$

Its Laplace transform is the n-particle partition function:

$$Z_n(T,V,m) = \frac{1}{n!}\left[\frac{V}{(2\pi)^3}\right]^n \left(4\pi \int \mathrm{e}^{-\beta\sqrt{p^2+m^2}} p^2 \mathrm{d}p\right)^n \ , \tag{17.36}$$

and the integral is

$$\int \mathrm{e}^{-\beta\sqrt{p^2+m^2}} p^2 \mathrm{d}p = m^2 T K_2(m/T) \ , \tag{17.37}$$

where K_2 is the second modified Hankel function [which for $m \gg T$ goes as $(\pi T/2m)^{1/2} \exp(-m/T)$ and for $m \ll T$ as $(T/m)^2$].

We obtain

$$Z_n(T,V,m) = \frac{1}{n!}\left[\frac{V}{(2\pi)} m^2 K_2(m/T)\right]^n = Z_1(T,V,m)^n/n! \ , \tag{17.38}$$

by which the ‘one-particle partition function’ Z_1 is defined. Summing over n gives the (grand canonical) partition function for an unfixed particle number:

$$Z(T,V,m) = \sum_{n=0}^{\infty} \frac{1}{n!} Z_1^n = \exp\left[\frac{VT}{2\pi^2} m^2 K_2\left(\frac{m}{T}\right)\right] . \tag{17.39}$$

For a mixture of two gases with particles of masses m_1 and m_2, respectively, we have $Z(T,V,m_1,m_2) = Z(T,V,m_1)Z(T,V,m_2)$. We generalize this to a mixture of gases of many different sorts of particles by introducing the (as yet unknown) mass spectrum $\rho(m)$:

$$\rho(m)\mathrm{d}m = \text{number of different species of particles in } \{m, \mathrm{d}m\} , \tag{17.40}$$

and obtain

$$Z^{(\rho)}(T,V) = \exp\left[\frac{VT}{2\pi^2} \int_0^{\infty} m^2 K_2\left(\frac{m}{T}\right) \rho(m)\mathrm{d}m\right] . \tag{17.41}$$

On the other hand, any $Z(T,V)$ can be written as

$$Z(T,V) = \int_0^{\infty} \sigma(E,V) \exp\left(-\frac{E}{T}\right) \mathrm{d}E . \tag{17.42}$$

Given $Z(T,V)$, the energy spectrum (density of states) $\sigma(E,V)$ can be calculated, and vice versa.

Doing the same steps, i.e., summing over n and introducing a mass spectrum without, however, executing the Laplace transformations, yields the phase-space analogue of Eqs. (17.41) and (17.42):

$$\sigma^{(\rho)}(M,V) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{V}{(2\pi)^3}\right]^n \int \delta\left(M - \sum_{i=1}^{n} E_i\right) \prod_{i=1}^{n} 4\pi p_i^2 \rho(m_i) \mathrm{d}p_i \mathrm{d}m_i , \tag{17.43}$$

where $E_i = (p_i^2 + m_i^2)^{1/2}$. *Note the enormous difference between the two densities of states,* $\rho(m)$ *and* $\sigma(M,V)$. Suppose there is a single species with mass m_0; then $\rho(m) = \delta(m - m_0)$ is zero everywhere except at $m = m_0$, while $\sigma(M,V)$ grows (for $M \gg m_0$) as $\exp(\text{const.} \times M^{3/4})$ as shown in Eq. (17.28).

Introducing the statistical bootstrap hypothesis

From an article that does not otherwise concern our subject, R. Carreras [46] picked up a *bon mot* which may serve very well as a motto for this section:

> [...] all of these arguments can be questioned, even when they are based on facts that are not controversial.

Here are these arguments:

- If anything deserves the name ‘fireball’, then it is the lump of hadronic matter in the state just before it decays isotropically into a two-body final state, as observed in large-angle elastic [$p + p \to p + p$, $\pi + p(n) \to \pi + p(n)$] or two-body inelastic scattering [$p + p \to \pi^+ + d$].

- This fireball answers, within experimental accuracy, to the description by an improved Fermi statistical model, as witnessed by the agreement of our phase-space results with the Orear plot (Fig. 17.4).
- We therefore postulate that fireballs describable by statistical models do exist, provided that in such models interaction is taken into account by including known particles and resonances (Lessons 5, 6, and 7).
- While *practically* a limited number of sorts of particles and resonances was already sufficient to describe, within experimental accuracy, fireballs up to a mass of 8 GeV, we should *in principle* include all of them with the help of an as yet unknown mass spectrum $\rho(m)$.
- Recalling Eqs. (17.41), (17.42), and (17.43), we observe that there are two mass spectra appearing in the statistical description:

  1. $\sigma(M,V_0)\mathrm{d}M$ is the number of states (of species) of *fireballs* (volume V_0) in the mass interval $\{M,\mathrm{d}M\}$.
  2. $\rho(m)\mathrm{d}m$ is the number of species of possible *constituents* (of such fireballs) having a mass in $\{m,\mathrm{d}m\}$.

- A glance at the *Review of Particle Properties* [47] informs us, under the headings *Partial Decay Modes* that heavy resonances [to be counted in $\rho(m)$] have many decay channels, some of them containing resonances once again. Thus, heavy resonances 'consist' (statistically) of particles and lighter resonances – just as fireballs do.
- Therefore *there is no principal difference between resonances and fireballs*: the states counted in $\sigma(M,V_0)$ should also be admitted as possible constituents of fireballs of larger mass – that is, they should be counted in $\rho(m)$.
- We conclude that $\rho(m)$ and $\sigma(m,V_0)$ count essentially the same set of hadronic masses and that therefore they must be – up to details – the 'same' function.
- They cannot be exactly equal, because $\rho(m)$ starts with a number of δ-functions (π, K, $p,\ldots$) while $\sigma(m,V_0)$ is continuous above $2m_\pi$.
- Leaving the door open for such differences and others, we postulate only that the corresponding entropies should become asymptotically equal:

$$\frac{\log\sigma(m,V_0)}{\log\rho(m)} \underset{m\to\infty}{\longrightarrow} 1\,. \tag{17.44}$$

  We call this the 'bootstrap condition' [45], which is a very strong requirement in view of the great difference between ρ and σ in 'ordinary' thermodynamics (see the remark at the end of the last section).

The solution

The rest is mathematics (and the above motto no longer applies). It could be shown [45] that ρ and σ have to grow asymptotically like const. $\times\, m^{-\alpha}\exp(m/T_0)$, while possible solutions growing faster than exponentially are inadmissible in statistical thermodynamics. Nahm [48] proved that, by adding certain refinements, the condition Eq. (17.44) could be sharpened and that the power of the prefactor is then $\alpha = 3$. He also derived sum rules, which allowed him to estimate T_0 to lie in the region of 140 to 160 MeV, results which agreed with Frautschi's (and collaborators) numerical results [49]. Thus the question put after Lesson 7 had been answered in the affirmative: SBM was born.

It is a self-consistent scheme in which the 'particles' – i.e., clusters or resonances or fireballs, call them what you like – are at the same time:

- the object being described,
- the constituents of this object,
- the generators of the interaction which keeps the object together.

Thus it is a 'statistical bootstrap' [49] embracing all hadrons.

Further developments

Everything that happened to SBM after its birth is reported in some detail, and with all references known to me, in the review Chapter 25 [2], which I shall not try to sum up here.

Even so, a few more important steps must be mentioned:

- The thermodynamic description of fireballs is so simple that it can be combined with collective motions (as in two-centre models [11]) and summed over impact parameters. Leading particles and conservation laws are easily taken care of. In this way, Ranft and myself constructed the so-called 'thermodynamical model', which proved useful for predicting particle momentum spectra [50].
- Frautschi, in a most important paper [49], had written down and solved the first phase-space formulation of SBM. His work triggered an avalanche of further papers reviewed in Chapter 25 [2], see also Chapter 22, leading to a new development. His 'bootstrap equation' (BE) is much stronger and more elegant than our above 'bootstrap condition' Eq. (17.44). Later it was put in a manifestly Lorentz invariant form and analytically solved by Yellin [51]. This formulation has become standard. The Laplace-transformed BE is a functional equation for the Laplace transform of the mass spectrum. This equation was already known[10] in 1870 [52] and independently rediscovered by Yellin. All this was so important that I cannot resist illustrating it with the help of a simple toy model, in which clusters are composed of clusters with vanishing kinetic energy. In this limit the Frautschi–Yellin BE reads

$$\rho(m) = \delta(m - m_0) + \sum_{n=2}^{\infty} \frac{1}{n!} \int \delta\left(m - \sum_{i=1}^{n} m_i\right) \prod_{i=1}^{n} \rho(m_i) \mathrm{d}m_i \,. \qquad (17.45)$$

[10] In another context, as one might guess.

In words, the cluster with mass m is either the 'input particle' with mass m_0 or else it is composed of any number of clusters of any masses m_i such that $\Sigma m_i = m$. We Laplace-transform Eq. (17.45):

$$\int \rho(m)\exp(-\beta m)\mathrm{d}m = \exp(-\beta m_0) + \sum_{n=2}^{\infty}\frac{1}{n!}\prod_{i=1}^{n}\int \exp(-\beta m_i)\rho(m_i)\mathrm{d}m_i \,. \tag{17.46}$$

Define

$$z(\beta) := \exp(-\beta m_0)\,, \qquad G(z) := \int \exp(-\beta m)\rho(m)\mathrm{d}m\,. \tag{17.47}$$

Thus Eq. (17.46) becomes $G(z) = z + \exp[G(z)] - G(z) - 1$ or

$$z = 2G(z) - \exp[G(z)] + 1\,, \tag{17.48}$$

which is the above-mentioned functional equation for the function $G(z)$, the Laplace transform of the mass spectrum. This function proved most important in all further development. For instance, the coefficients of its power expansion in z are directly related to the multiplicity distribution of the final particles in the decay of a fireball [53]. It is most remarkable that the 'Laplace-transformed BE' Eq. (17.48) is 'universal' in the sense that it is not restricted to the above toy model, but turns out to be the same in all (non-cutoff) realistic SBM cases [49, 51]. Moreover, it is independent of:

- the number of space-time dimensions [54],
- the number of 'input particles' (z becomes a sum over modified Hankel functions of input masses),
- Abelian or non-Abelian symmetry constraints [55].

What is wanted is of course $G(z)$, given implicitly by Eq. (17.48). Solutions are reviewed in Chapter 25 [2]. Simplest is its graphic solution: we draw $z(G)$ according to Eq. (17.48) and exchange the axes (see Fig. 17.5). One sees immediately that (universally!)

$$z_{\max}(G) =: z_0 = \ln 4 - 1 = 0.3863\ldots\,, \qquad G_0 = G(z_0) = \ln 2\,.$$

The parabola-like maximum of $z(G)$ implies a square root singularity of $G(z)$ at z_0, first remarked by Nahm [48]. Upon inverse Laplace transformation, this leads to $\rho(m) \sim m^{-3}\exp(\beta_0 m)$ where (in our present case, not universally!):

$$\beta_0 = -\frac{1}{m_0}\ln z_0 = \frac{0.95}{m_0} \qquad \text{[see Eq. (17.47)]}\,. \tag{17.49}$$

Putting $m_0 = m_\pi$, we find a reasonable value for $T_0 = \beta_0^{-1}$:

$$T_0(\text{toy model}) = 0.145\ \text{GeV}\,. \tag{17.50}$$

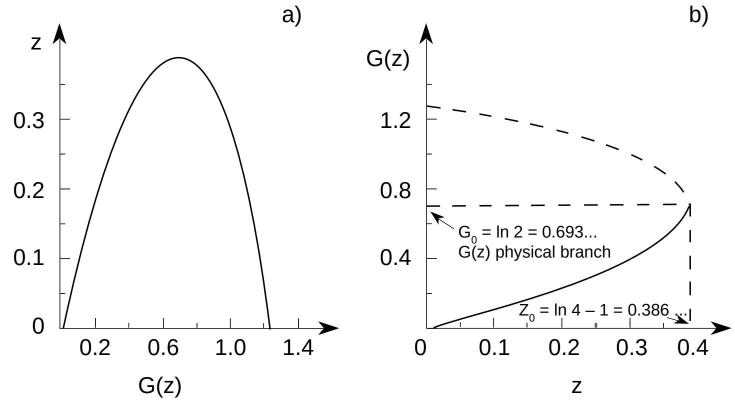


Fig. 17.5 (**a**) $z(G)$ according to Eq. (17.48). (**b**) $G(z)$, the graphical solution of Eq. (17.48).

Thermodynamics of a gas of the above clusters Eq. (17.45) has T_0 as a singular temperature. Thus, the simple toy model already yields all essential features of SBM. For a very short representation of the more realistic case of pion clustering in full relativistic momentum space, see [53, Sect. 2].

- Much work was done by groups in Bielefeld, Kiev, Leipzig, Paris, and Turin to clear up the relation of SBM to other trends in strong interaction physics (Regge, Veneziano, etc.) and to the theory of phase transitions; this is reviewed in Chapter 25 [2].
- In the mid-seventies J. Rafelski arrived at CERN and immediately began pushing me: SBM should be polished up to become applicable to heavy-ion collisions. There were two problems: baryon-number (and, eventually, strangeness) conservation and proper particle volumes – pointlike heavy ions would be nonsense. Thus we introduced baryon (strangeness) chemical potential and – less trivially – proper particle volumes, first in the BE [56], then in the ensuing thermodynamics [57]. We found that particle volumes had to be proportional to particle masses with a universal proportionality constant. The argument was the following. Let a cluster (fireball) of mass m and volume V be composed of constituents with masses m_i and volumes V_i. In contrast to standard assumptions in thermodynamics, the cluster is not confined to an externally imposed volume; rather it carries its volume with it (as already stressed by Nahm [48]), and so does each of its constituents. Let any one of them have four-momentum p_i^μ. Then its volume moves with four-velocity p_i^μ/m_i. With Touschek [58], we define a 'four-volume'

$$V_i^\mu = \frac{V_i}{m_i} p_i^\mu \ . \tag{17.51}$$

The constituents' volumes have to add up to the total cluster volume and their momenta to the total momentum:

$$\frac{V}{m}p^{\mu} = \sum_{i=1}^{n} \frac{V_i}{m_i}\, p_i^{\mu} \ , \qquad p^{\mu} = \sum_{i=1}^{n} p_i^{\mu} \ . \tag{17.52}$$

This is possible for arbitrary n and p_i^{μ} if and only if

$$\frac{V}{m} = \frac{V_i}{m_i} = \text{const.} = 4\mathscr{B} \ , \tag{17.53}$$

where the proportionality constant is written $4\mathscr{B}$ in order to emphasize the similarity to MIT bags [59], which have the same mass–volume relation. Moreover, as the energy spectrum of SBM clusters and MIT bags is the same even in the detail [60, 61], one is led to consider these two objects to be the same, at least in the sense that statistical thermodynamics of MIT bags is identical to that of SBM clusters. This 'identity' is interesting, because MIT bags 'consist of' quarks and gluons, SBM clusters of hadrons: it suggests a phase transition from one to the other.

- The thermodynamics of clusters with proper volumes still had a singularity at T_0, but a weaker one: while in the old (point-particle) version of SBM the energy density diverged at T_0 (thus making T_0 an 'ultimate' temperature), the energy density was now finite at T_0, making a phase transition (already anticipated by Cabibbo and Parisi [62]) to a quark–gluon plasma possible [57, 63, 64].
- Technical problems in handling the particle volumes explicitly could be elegantly solved [65–69] by using the 'pressure (or isobaric) partition function' invented by Guggenheim [70]. This technique allows one to treat the thermodynamics of bags with an exponential mass spectrum (pioneered by Baacke [71]) in a beautiful way: Letessier and Tounsi [49, 72, 74] succeeded, following the methods of the Kiev group [65–69], in describing with a *single* partition function the hadron gas, the quark–gluon plasma, and the phase transition between the two in a realistic case. This opens the way to solving a number of problems connected with proving (or disproving?) the actual presence of a quark–gluon phase in the first stage of relativistic heavy-ion collisions.

17.4 Some Further Remarks

The difficulty in killing an exponential spectrum

The most prominent feature of SBM is its exponentially increasing mass spectrum. Many objections to it were put forward: symmetry constraints would forbid a number of the states counted in it; correct Bose–Einstein and Fermi–Dirac statistics would also reduce the number of states; and in composing clusters of clusters and so on, one should take into account the Pauli principle, which again might eliminate many states. Furthermore, the original argument for including resonances was based on the Beth–Uhlenbeck method [see Eq. (17.14) and Lesson 5], which invokes phase shifts and their sudden rise by π when going through a resonance. But then the phase shifts should go to zero for infinite momentum and the Levinson theorem states that $\delta_\ell(0) - \delta_\ell(\infty) = N_\ell \pi$, with N_ℓ = number of bound states with angular momentum ℓ. Therefore phase shifts cannot go on increasing by π for each of our (exponentially growing number of) resonances – they must decrease again. That is, each and every one of the masses added somewhere to the mass spectrum must be (smoothly) subtracted later on. How then can an exponential mass spectrum survive?

All these objections turn out to leave the mass spectrum intact, because the exponential function is extremely resistant to manipulations: multiplication by a polynomial of any (positive or negative) order, squaring, differentiating, or integrating it will not do much harm (consider the leading term of its logarithm!) – at most change its exponential parameter $(T_0 \to T_0')$.

Therefore, once our self-consistency requirement – crude as it may be – has led to this particular mass spectrum, it is difficult to get rid of it. Incidentally, in the first paper on SBM [45], correct Bose–Einstein/Fermi–Dirac statistics was employed (easy in the grand canonical formulation, awful in phase space [75–79]) and the result was that the mass spectrum $\rho(m) = \rho_{\mathrm{Bose}}(m) + \rho_{\mathrm{Fermi}}(m)$ had to grow exponentially. The role of conservation laws has been dealt with in the literature (see references in Chapter 25 [2]). None of these explicit attacks killed the leading exponential part of the mass spectrum.

It remains, as an illustration of the resistance of $\rho(m)$, to assume that, for whatever reason, every mass once added to it, has to be eliminated again. (I do not know of any serious argument which would require this. The Levinson theorem derived in non-relativistic potential scattering [80] cannot be invoked in a situation where all kinds of reactions between constituents take place – but assume it had to be so.) Then, arriving at mass m, we subtract everything that had been added at $m - \Delta m$, whence

$$\rho(m) \longrightarrow \rho(m) - \rho(m - \Delta m) \approx \mathrm{e}^{\beta_0 m}\left[1 - \mathrm{e}^{-\beta_0 \Delta m}\right] ,$$

and the leading exponential remains untouched (a kind of differentiation).

What is the value of T_0?

The most fundamental constant of SBM, namely T_0, escapes precise determination. There are several ways to try to fix T_0:

1. Theoretically,
   a. inside SBM,
   b. from lattice QCD.
2. Empirically,
   a. from the mass spectrum,
   b. from the transverse momentum distribution,
   c. from production rates of heavy antiparticles ($\overline{\mathrm{He}^3}$, $\overline{d}$),

d. from the phase transition to the quark–gluon phase.

We obtain the following.

1a. T_0 from inside SBM

The crude model of Eq. (17.45) yields with pions only	$T_0 \approx 0.145$ GeV
If K and N were added to the input	$T_0 \approx 0.135$ GeV
The unrealistic model of distinguishable, massless particles as described by Eq. (17.31) gives	$T_0 \approx 0.184$ GeV
A more realistic model (pions + invariant phase space) yields [81]	$T_0 \approx 0.152$ GeV
Hamer and Frautschi [82] solve their BE by numerical iteration and read off	$T_0 \approx 0.140$ GeV
Nahm derives a sum rule from which he found, under different assumptions (which particles are admitted?) [48]:	$T_0 \approx 0.154$ GeV or 0.142 GeV

1b. T_0 from Lattice QCD. The determination of T_0 from lattice QCD is still hampered with difficulties. First estimates using pure gauge gave rather high T_0, while the introduction of quarks pose their own problems. Nevertheless compromises have been devised which circumvent these problems and provide a way of dealing with quarks (but paying a price depending on what one is after). Table 17.1 is taken from a review article by F. Karsch, where the methods are described and references to original work are given [83, Table 2] (see also [84]).

Table 17.1 T_0 from lattice QCD. $T_0 \approx 0.145$ GeV or 0.130 GeV. $\sqrt{\sigma} = 0.42 \pm 0.02$ GeV is the string tension and n_f the number of light flavours, $n_\mathrm{f} = 2$ or 4 being the most physical choice. Taken from [83], for contemporary results see Chapters 7, 10, 14, and 21

n_f	T_0 from $\sqrt{\sigma}$	T_0 from m_ρ	T_c from m_N
0	239 ± 13	239 ± 23	225 ± 30
2	–	145 ± 7	113 ± 9
4	160 ± 22	130 ± 7	105 ± 9

It is believed that the value 'T_0 from m_ρ' is the most realistic. It agrees rather well with the above-listed estimates of T_0 from inside SBM [except the one for distinguishable massless particles which – accidentally? – lies nearer to the pure gauge ($n_\mathrm{f} = 0$) value].

2a. From the Mass Spectrum. Here the difficulty is that approximate completeness of the empirical mass spectrum ends somewhere around 1.5 GeV, because the density of mass states increases and the production rate decreases (both exponentially, as predicted by SBM), and the identification of all masses rapidly becomes impossible. On the other hand, we know only the asymptotic form of the mass spectrum $\sim m^{-3}\exp(m/T_0)$ and have to guess an extrapolation towards lower masses, which does not diverge for $m \to 0$. Various attempts (after 1970):

Hagedorn and Ranft [85] obtain, with large uncertainties	$T_0 \cong 0.148$ GeV
Letessier and Tounsi [86] find	$T_0 \cong 0.155$ GeV
See also [87]	$T_0 \cong 0.158$ GeV

2b. From the $p_\perp$ Spectrum

Large-angle elastic scattering. Orear [37] finds an apparent temperature $T = 0.158$ GeV, which should lie near to T_0. Hence	$T_0 \gtrsim 0.158$ GeV
Folklore has it that the $p_\perp$ distribution in the soft region is $\exp(-6p_\perp)$. The exact formula for the distribution is [88] quite different from $\exp(-p_\perp/T)$, but for $p_\perp \gg m_\pi$ one might generously accept $\exp(-6p_\perp)$. Then	$T \approx T_0 \cong 0.167$ GeV
A serious attempt to fix T_0 from the $p_\perp$ distribution is reported in [89]. In a region where $p_\parallel$ is very small (no integration over $p_\parallel$), the authors fit the $p_\perp$ distribution to a Bose–Einstein formula and obtain the surprisingly low result	$T_0 > T \approx 0.117$ GeV

They give reasons why they identify this with T_0, although the primary momentum is only 28.5 GeV (Brookhaven), so that T_0 could still lie somewhat higher (as I believe).

Remark. The determination of T from $p_\perp$ suffers from a number of perturbing effects, which have been discussed in detail in [88]: resonance $(\rho, \Delta, \ldots)$ decay, leakage of 'large $p_\perp$' down to the soft region, etc. It seems that none of these effects influence the two-body large-angle scattering, so that the value found by Orear [37] might be more trustworthy than the values obtained by fitting the soft $p_\perp$ distribution by various formulae (partly not well justified).

2c. From Production Ratios of Heavy Antiparticles. Production rates of anti-^{3}He, antitritium ($\bar{\mathrm{t}}$), antideuteron ($\bar{d}$), and of many other particles have been measured [90].[11] By taking ratios we avoid (at least in part) problems coming from the pion production rate, not well known theoretically, and from (not very) different momenta and target materials. From the quoted paper [90], we take ratios $\overline{^3\mathrm{He}}/\bar{d}$ and $\bar{\mathrm{t}}/\bar{d}$ and average the values [all around (0.8 to 2) $\times 10^{-4}$]. We find

[11] I am grateful to P. Sonderegger (CERN) for making me aware of this work.

$$\left\langle \frac{\overline{^3\mathrm{He}} \text{ or } \bar{\mathrm{t}}}{\overline{d}} \right\rangle = (1.4 \pm 0.7) \times 10^{-4} \,. \tag{17.54}$$

From SBM – taking into account the fact that, for each produced antibaryon, another baryon must be produced along with it – one easily works out (spin, etc., factors included)

$$\left\langle \frac{\overline{^3\mathrm{He}} \text{ or } \bar{\mathrm{t}}}{\overline{d}} \right\rangle = V \left(\frac{3}{2}\right)^{3/2} \left(\frac{m_{\mathrm{N}} T}{2\pi}\right)^{3/2} \frac{8}{3} \exp(-2m_{\mathrm{N}}/T) \,. \tag{17.55}$$

The exponential is easily understood: $\overline{^3\mathrm{He}}$ or $\bar{\mathrm{t}}$ require production of six nucleon masses, while $\overline{d}$ requires 4, and in the ratio, four of the six cancel out.

Putting in numbers, one finds that the experimental values 17.54 are obtained with a temperature between 0.15 and 0.16 GeV when for V we assume the usual $4\pi/3m_\pi^3$. Hence (at a primary momentum of ~ 200 GeV/c) $T_0 \gtrsim 0.155$ GeV

2d. From the Phase Transition to the Quark–Gluon Phase. As the existence of the quark–gluon phase is still hypothetical, no direct measurement is available.

A theoretical estimate was proposed by Letessier and Tounsi [91]: they require that the curves $P = 0$ for an SBM hadron gas and for a quark–gluon phase coincide “as well as can be achieved”. They find $T_0 \approx 0.170$ GeV

Remark 1. The collective motions expected in the expansion and decay of the ‘fireballs’ produced in relativistic heavy-ion collisions will ‘Doppler-shift’ the temperatures read off from transverse momentum distributions. Too high temperatures will result if the collective transverse motion is not corrected for. Thus in our early work on heavy-ion collisions [92], we (erroneously?) assumed a value $T_0 \approx 0.19$ GeV, which does not seem, in view of all the other estimates, to be realistic.

Remark 2. While no precise value can yet be assigned to T_0, it is satisfying that so many different methods yield values which differ typically by less than 20%. An average over all values listed above gives:[12] $T_0 = 0.150 \pm 0.011$ GeV

Where is Landau, where are the Californian bootstrappers?

History as told above makes it evident that Landau’s model [93] is orthogonal to our approach and did not influence its development. There was, however, one moment after the formulation of SBM, namely when we tried to combine it with collective motions to obtain momentum spectra of produced particles, when we considered a

[12] The two extreme values 0.117 and 0.184 were omitted; the error quoted is the mean standard error arising from the listed values (without taking their individual errors into account).

combination of Landau's hydrodynamical approach with SBM – only to discard it almost immediately. Landau dealt with 'prematter' expanding after a central collision, while we needed the evolution of hadron matter after collisions averaged over impact parameter. We had to take into account various sorts of final particles (π, K, N, hyperons, and antiparticles) obeying conservation laws (baryon number and strangeness). We had to care for leading particles, etc. All that forced us to pursue the semi-empirical 'thermodynamical model' [50] whose aim was not theoretical understanding, but practical predictions for use in the laboratory.

Moreover there was a psychological obstacle which I never overcame: namely the Lorentz-contracted volume from which everything was supposed to start. True, when two nucleons hit head on, then just before the impact they are Lorentz-contracted (seen from the CM); then they collide, heat up, and come to rest. When they start to expand, they are at rest and hence no longer Lorentz-contracted. On the other hand, one can conceive that the mechanical shock has indeed compressed them. But into what state would be another complicated hydro-thermodynamical problem in which their viscosity, compressibility, specific heat, and what-not would enter.[13] Why should this state of compression, which constitutes the initial condition for the following expansion, be exactly equal to the Lorentz 'compression' before the shock? Since nobody among the people working on this model shared my uneasiness, I guessed it was my fault – but it somehow prevented me from ever becoming enthusiastic about the Landau model.

This is the place to mention another Russian physicist whose work would have inspired us, had we been aware of it. In 1960, five years before SBM, Yu.B. Rumer [94] wrote an article with the title *Negative and Limiting Temperatures*, in which he states the necessary and sufficient conditions for the existence of a limiting temperature – namely an exponential spectrum – and gives an example: an ideal gas in an external logarithmic potential. Unfortunately, he remained essentially on the formal side of the problem and did not connect it with particle production in strong interactions. Otherwise, who knows?

Now for the Californian bootstrappers. Even a most modest account of what has been done in the heroic effort of a great number of theoreticians on the program of 'Hadron Bootstrap' or 'Analytic *S*-Matrix' would fill a whole book. For me the question is: did it in any way help the conception of SBM in 1964? And the answer is negative. To realize that, one only has to remember the above-reported history up to 1964 and hold it against the best non-technical expositions of the basic ideas and the general philosophy of 'Hadron Bootstrap', namely, the two articles by G.F. Chew: *'Bootstrap': A scientific idea?* [95] and *Hadron bootstrap: Triumph or frustration?* [96]. After 1964, however, the influence was enormous, although not technically. But it was of great value for all those who worked on SBM to see their philosophical basis shared with others. Most influential, of course, was that S. Frautschi, one of the leading Californian bootstrappers, joined our efforts, not only lending his prestige, but indeed giving SBM a turn that upgraded it (he also

[13] These are problems now occupying theoreticians working on relativistic heavy-ion collisions – could these problems be trivial? See, however, the Post Scriptum at the end of this paper.

coined its name ‘statistical bootstrap’) and made it acceptable to particle physicists: the phase-space formulation and its numerous consequences.

The Californian bootstrappers credo was the analytic *S*-matrix: Poincaré invariant, unitary, maximally analytic (with crossing, pole–particle correspondence), which was believed – or hoped – to be uniquely fixed by these requirements. Another aspect was that it should generate the whole hadron spectrum, where each hadron “plays three different roles: it may be a ‘constituent’, it may be ‘exchanged’ between constituents and thereby constitute part of the force holding the structure together, and it may *be* itself the entire composite” [95].

I know of only two realizations of this aspect: the π–ρ system [97,98] and SBM – both remaining infinitely far behind the ambitious bootstrap program. In spite of a large number of other achievements obtained in and around the program (dispersion relations, Regge poles, Veneziano model, even string theories), one sees today a strong resurrection of Lagrangian field theories, which have now taken the lead in the race toward a ‘theory of everything’. I believe that the bootstrap philosophy and the Lagrangian fundamentalism must complement each other. Each one alone will never obtain complete success.

17.5 Conclusion

Had I been asked to speak only five minutes, my lecture might have been much better. Here it is. On the long way to SBM, we stopped at a few milestones:

- The realization that in a *single* hadron–hadron collision, many secondaries can be produced (1936).
- The discovery of limited $\langle p_\perp \rangle$ (1956).
- The discovery that fireballs exist and that a typical collision seems to produce just two of them (1954–1958).
- The concept of the compound nucleus and its thermal behaviour (1936–1937).
- The construction of simple statistical/thermodynamical models for particle production in analogy to compound nuclei (1948–1950).
- The introduction of interaction into such models via phase shifts at resonance (1937, 1956).
- The discovery that large-angle elastic cross-sections decrease exponentially with CM energy (1963).
- The discovery that a parameter-free and numerically correct description of this exponential decrease existed already, buried in archived Monte Carlo phase-space results (1963).

The birth of SBM in 1964 was but the logical consequence of all this. Between 1971 and 1973 the child SBM became a promising youngster, when it was reformulated and solved in phase space. It became adult in 1978–1980, when it acquired finite particle volumes. Today, another fourteen years later, it shows signs of age and is ready for retirement: in not too long a time, all of its detailed results will have been derived from QCD, maybe from 'statistical QCD'.

So then, was that all? I believe that something remains: SBM has opened a (one-sided but) intuitive view of strong interactions, revealing:

- their 'thermal behaviour' and thus their accessibility to statistical thermodynamical descriptions,[14]
- the existence of clusters (fireballs),
- the production rates of particles and their typical multiplicity distributions,
- large-angle elastic scattering,
- the 'universal' soft-$p_\perp$ distribution (plotted against $\sqrt{p_\perp^2 + m^2}$, please!), and
- the existence of a singular temperature T_0 where a phase transition takes place.

In fact, all these are nothing else than obvious (and calculable) manifestations of one single, fundamental property of strong interactions, namely the fact that they possess an exponential mass spectrum $\rho(m) \propto \exp(m/T_0)$.

And most remarkably, the constant T_0 is roughly equal to the *lowest* hadron mass, m_π. I believe the simple interpretation to be that strong interactions are as strong as they can possibly be, before becoming *too* strong. Too strong, that is, if they caused the mass spectrum to grow only a little faster than exponentially,[15] say, $\rho(m) \sim \exp\left[(m/m_0)^{1+\alpha}\right]$, $\alpha > 0$, the 'entropy' of its clusters would be $S_\mathrm{c} \sim \ln\rho \sim (m/m_0)^{1+\alpha}$. Then clusters would swallow each other up (if in reach) to become giant superclusters – a sort of hadronic black holes – which could not live in thermal equilibrium with each other while they remained cold inside:

$$T_\mathrm{interior}(m) = \left(\frac{\mathrm{d}S}{\mathrm{d}m}\right)^{-1} = \mathrm{const.} \times \left(\frac{m_0}{m}\right)^{\alpha} \underset{m\to\infty}{\longrightarrow} 0\,.$$

Maybe, even, the combined forces of gravitation and superstrong interactions might have stopped the expansion of the Universe at some early state. Anyway, we can save the effort of working out all the consequences of superstrong interactions: the state of the world seems not to favour such a hypothesis.

Don't ask me why strong interactions are actually as strong as permissible – this will have to be answered by some future unified theory (maybe the only possible one?) not yet known (to me). In the meantime, I would like to know the reaction of a physicist who, in 50 years, comes accidentally upon this review and takes the trouble to read it. He might quote [99]

How finely we argue upon mistaken facts!

[14] Many-body systems can often be described statistically, with final states emerging from single two-body collisions only in strong interactions.

[15] In view of the 'stability' of the exponential function (Sect. 17.4), this might require a superstrong interaction which could be *much* stronger than the actual one.

Acknowledgements

J. Rafelski has initiated and largely organized this workshop (Fig. 17.6). I wish to thank him for this opportunity to deliver a paper which otherwise would not have been written, and I wish to thank Marie-Noëlle Fontaine (Fig. 1.4) for the beautiful typescript and her infinite patience with it (and with me).

Fig. 17.6 Poster of the Divonne 1994 meeting where this lecture was presented.

Post scriptum

As kindly pointed out to me by E.L. Feinberg, my uneasiness about Lorentz contracted interaction volumes mentioned under the leading *Where Is Landau and ...* is indeed due to my fault, or rather to my ignorance about some fundamental ingredients of Landau's model. The relevant points are put forward in E.L. Feinberg's beautiful paper *Can the relativistic change in the scales of length and time be considered the result of action of certain forces?* [100]. I wish to thank him for bringing it to my attention.

References

1. L. Sterne: *The Life and Opinions of Tristram Shandy, Gentleman* (Dodsley, London, 1760), Book II, Chapter 19
2. R. Hagedorn: In *Quark Matter '84*, Helsinki Symposium, ed. by K. Kajantie, published in Lecture Notes in Physics **221** (Springer, Heidelberg, 1985); see How We Got to QCD Matter from the Hadron side: 1984, chapter 25 in this volume
3. E.L. Feinberg, Phys. Rev. TS **5**, 237 (1972)
4. H. Yukawa: Proc. Phys. Math. Soc. Japan **17**, 48 (1935)
5. W. Heitler: Proc. Cambr. Phil. Soc. **37**, 291 (1941)
6. W. Heitler and H.W. Peng: Proc. Cambr. Phil. Soc. **38**, 296 (1942)
7. L. Janossy: Phys. Rev. **64**, 345 (1943)
8. W. Heisenberg: Z. Phys. **101**, 533 (1936)
9. N.M. Dulles and W.D. Walker: Phys. Rev. **93**, 215 (1954)
10. J. Nishimura: Soryushiron Kenkyu **12**, 24 (1956)
11. G. Cocconi: Phys. Rev. **111**, 1699 (1958)
12. F.W. Büsser et al. (CERN–Columbia–Rockefeller College): Phys. Lett. B **46**, 471 (1973), and a wealth of further papers in conference reports and review articles.
13. S. Takagi: Progr. Theor. Phys. **7**, 123 (1952)
14. P. Ciok, T. Coghen, J. Gierula, R. Holyński, A. Jurak, M. Miesowicz, T. Saniewska, O. Stanisz and J. Pernegr: Nuovo Cimento **8**, 166 (1958);
    P. Ciok, T. Coghen, J. Gierula, R. Holyński, A. Jurak, M. Miesowicz, T. Saniewska, and J. Pernegr: Nuovo Cimento **10**, 741 (1958)
15. J. Gierula: Fortschr. der Physik **11**, 109 (1963)
16. N. Bohr: Nature **137**, 344 (1936)
17. V.F. Weisskopf: Phys. Rev. **52**, 295 (1937)
18. I. Frenkel: Soviet Phys. **9**, 533 (1936)
19. H. Koppe: Phys. Rev. **76**, 688 (1949)
20. H. Koppe: Zs. f. Naturforschung **3a**, 251 (1948)
21. E. Fermi: Progr. Theor. Phys. **5**, 570 (1950)
22. E. Fermi: Phys. Rev. **81**, 683 (1951)
23. W. Heisenberg: Naturwissenschaften **39**, 69 (1952)
24. E. Beth and C.E. Uhlenbeck: Physica **4**, 915 (1937)
25. L. Landau and E. Lifshitz: *Statistical Physics* (MIR, Moscow, 1967)
26. R. Hagedorn: In Cargèse Lectures, **6**, 643 (1973), ed. by E. Schatzmann (Gordon& Breach, New York)

27. K. Huang: *Statistical Mechanics* (John Wiley and Sons, New York, 1963)
28. J. Bernstein, R. Dashen, and S. Ma: Phys. Rev. **187**, 1 (1969)
29. S.Z. Belenkij: Nucl. Phys. **2**, 259 (1956)
30. L. Sertorio: Rivista del Nuovo Cimento **2**, No. 2 pp. 1-130 (1979)
31. F. Cerulus and R. Hagedorn: Suppl. Nuovo Cimento **9**, 646, 659 (1958)
32. F. Cerulus: Nuovo Cimento **19**, 528 (1961)
33. F. Cerulus: Nuovo Cimento **22**, 958 (1961)
34. R. Hagedorn: Nuovo Cimento **15**, 434 (1960)
35. S.Z. Belenkij, V.M. Maksimenko, A.I. Nikisov, and I.L. Rozental: Usp. Fiz. Nauk **62**, 1 (1957), in Russian; ibid. Fortschr. d. Physik **6**, 524 (1958), same in German
36. G. Cocconi: Nuovo Cimento **33**, 643 (1964)
37. J. Orear: Phys. Lett. **13**, 190 (1964)
38. G. Fast, R. Hagedorn, and L.W. Jones: Nuovo Cimento **27**, 856 (1963); G. Fast and R. Hagedorn: Nuovo Cimento **27**, 208 (1963)
39. T.E.O. Ericson: Phys. Lett. **4**, 258 (1963)
40. A. Białas and V.F. Weisskopf: Nuovo Cimento **35**, 1211 (1965)
41. G. Auberson and B. Escoubès: Nuovo Cimento **36**, 628 (1965)
42. H. Satz: Nuovo Cimento **37**, 1407 (1965)
43. J. Vandermeulen: Bull. Soc. Roy. Sci. Liège (Belgium) **34**, 455 (1965)
44. I. Pomeranchuk: Dokl. Akad. Nauk SSSR **78**, 889 (1951)
45. R. Hagedorn: Suppl. Nuovo Cimento **3**, 147 (1965)
46. R. Carreras: Picked up for you this week, No 26 (CERN, Geneva, 1993)
47. Particle Data Group: Phys. Rev. D **50**, No 3, Part I (1994)
48. W. Nahm: Nucl. Phys. B **45**, 525 (1972)
49. S. Frautschi: Phys. Rev. D **3**, 2821 (1971)
50. R. Hagedorn and J. Ranft: Suppl. Nuovo Cimento **6**, 169 (1968); H. Grote, R. Hagedorn, and J. Ranft: *Particle Spectra* (CERN, Geneva, 1970)
51. J. Yellin: Nucl. Phys. B **52**, 583 (1973)
52. E. Schröder: Z. Math. Phys. **15**, 361 (1870)
53. G.J.H. Burgers, C. Fuglesang, R. Hagedorn, and V. Kuvshinov: Z. Phys. C **46**, 465 (1990)
54. H. Satz: Phys. Rev. D **20**, 582 (1979)
55. K. Redlich and L. Turko: Z. Phys. C **5**, 201 (1980)
56. R. Hagedorn, I. Montvay, and J. Rafelski: *Proc. Workshop on Hadronic Matter at Extreme Energy Density*, Erice, 1978, p. 49, ed. by N. Cabibbo and L. Sertorio (Plenum Press, New York, 1980); and J. Rafelski and R. Hagedorn, Chapter 23 in this volume
57. R. Hagedorn and J. Rafelski: Phys. Lett. B **97**, 136 (1980)
58. B. Touschek: Nuovo Cimento B **58**, 295 (1968)
59. A. Chodos, R.L. Jaffe, K. Johnson, C.B. Thorn, and V.F. Weisskopf: Phys. Rev. D **19**, 3471 (1974)
60. J.I. Kapusta: Nucl. Phys. B **196**, 1 (1982)
61. K. Redlich: Z. Phys. C **21**, 69 (1983)
62. N. Cabibbo and G. Parisi: Phys. Lett. B **59**, 67 (1975)
63. R. Hagedorn and J. Rafelski: Bielefeld Symposium on *Statistical Mechanics of Quarks and Hadrons*, p. 237, ed. by H. Satz (North Holland, Amsterdam, 1981)
64. J. Rafelski and R. Hagedorn: as in [63], p. 253
65. M.I. Gorenstein: Yad. Fiz. **31**, 1630 (1980)
66. M.I. Gorenstein, V.K. Petrov, and G.M. Zinovjev: Phys. Lett. B **106**, 327 (1981)
67. M.I. Gorenstein, G.M. Zinovjev, V.K. Petrov, and V.P. Shelest: Teor. Mat. Fiz. **52**, 346 (1982)

68. R. Hagedorn: Z. Phys. C **17**, 265 (1983)
69. M.I. Gorenstein, S.I. Lipskikh, and G.M. Zinovjev: Z. Phys. C **22**, 189 (1984)
70. E.A. Guggenheim: J. Chem. Phys. **7**, 103 (1939)
71. J. Baacke: Acta Phys. Pol. B **8**, 625 (1977)
72. J. Letessier and A. Tounsi: Nuovo Cimento A **99**, 521 (1988)
73. J. Letessier and A. Tounsi: Phys. Rev. D **40**, 2914 (1989)
74. J. Letessier and A. Tounsi: Preprint PAR/LPTHE 90-17, Univ. Paris VII (1990) presented at IInd Internat. Bodrum Physics School, 1989, University of Istanbul, Turkey
75. K. Fabricius and U. Wambach: Nucl. Phys. B **62**, 212 (1973)
76. M. Chaichian, R. Hagedorn, and M. Hayashi: Nucl. Phys. B **92**, 445 (1975)
77. J. Engels, K. Fabricius, and K. Schilling: Phys. Lett. B **59**, 477 (1975)
78. J. Kripfganz: Nucl. Phys. B **100**, 302 (1975)
79. M. Chaichian and M. Hayashi: Phys. Rev. D **15**, 402 (1977)
80. M. Levinson: Kgl. Danske Vid. Selsk. Mat. Fys. Med. **25**, No. 9, pp. 25-30 (1949)
81. R. Hagedorn and I. Montvay: Nucl. Phys. B **59**, 45 (1973)
82. G.J. Hamer and S. Frautschi: Phys. Rev. D **4**, 2125 (1971)
83. F. Karsch: In *QCD 20 Years Later*, Aachen Symposium, 1992, Vol. 2, p. 717, ed. by P.M. Zerwas and H.A. Kastrup (World Scientific, Singapore, 1993)
84. J. Fingberg, U. Heller, and F. Karsch: Nucl. Phys. B **392**, 493 (1993)
85. R. Hagedorn and J. Ranft: Nucl. Phys. B **48**, 157 (1972)
86. J. Letessier and A. Tounsi: Nuovo Cimento A **13**, 557 (1973)
87. A. Tounsi, J. Letessier and J. Rafelski, in "Hot Hadronic Matter: theory and experiment" NATO-ASI **346**, pp. 105-116 (Plenum Press, New York, 1995) edts. J. Letessier, H.H. Gutbrod, and J. Rafelski
88. R. Hagedorn: Rivista del Nuovo Cimento **6**, No. 10, pp. 1-50 (1983)
89. A.T. Laasanen, C. Ezell, L.J. Gutay, W.N. Schreiner, P. Schúblin, L. von Lindern, and F. Turkot: Phys. Rev. L **38**, 1 (1977)
90. A. Bussière et al.: Nucl. Phys. B **174**, 1 (1980)
91. J. Letessier and A. Tounsi: Nuovo Cimento A **99**, 521 (1988)
92. R. Hagedorn and J. Rafelski: Phys. Lett. B **97**, 136 (1980)
93. L.D. Landau: *Collected Papers*, ed. by D. Ter Haar (Gordon and Breach, New York, 1965)
94. Yu.B. Rumer: Soviet Phys. (JETP) **11**, 1365 (1960); translation from original JETP **38**, 1899 (1960)
95. G.F. Chew: Science **161**, 762 (1968)
96. G.F. Chew: Physics Today **23**, 23 (1970)
97. G.F. Chew and S. Mandelstam: Nuovo Cimento **19**, 752 (1961)
98. F. Zachariasen: Phys. Rev. L **7**, 112 (1961)
99. L. Sterne: *The Life and Opinions of Tristram Shandy, Gentleman* (Dodsley, London, 1761), Book IV, Ch. 27
100. E.L. Feinberg: Sov. Phys. Usp **18**, 624 (1976); translation from original Usp. Fiz. Nauk. **116**, 709 (1975)

Chapter 18
About 'Distinguishable Particles'

Johann Rafelski

Abstract I present the context in which Hagedorn withdrew his first work on limiting temperature, show his withdrawal note, and discuss what was lost from view for 50 years while the manuscript shown in Chapter 19 lingered in the archives. I close describing the contemporary meaning of Hagedorn temperature.

18.1 Withdrawn Manuscript

In early 1978 Rolf Hagedorn shared with me a copy of his unpublished manuscript 'Thermodynamics of Distinguishable Particles: A Key to High-Energy Strong Interactions?', a preprint CERN-TH-483 dated 12 October 1964. He said there were two copies; I was looking at one; another was in the CERN archives. A quick glance sufficed to reveal that this was, actually, the work proposing a limiting temperature and the exponential mass spectrum. Hagedorn explained that upon discussions of the contents of his paper with Léon Van Hove, he evaluated in greater detail the requirements for the hadron mass spectrum and recognized a needed fine-tuning. Hagedorn concluded that his result was therefore too model-dependent to publish, and in the CERN archives one finds Hagedorn commenting on this shortcoming of the paper. These comments are printed below, and can be appreciated in full after reading Chapter 19.

However, Hagedorn's 'Distinguishable Particles' is a clear stepping stone on the road to a better understanding of strong interactions and particle production. The insights gained in this work allowed Hagedorn to rapidly invent the Statistical Bootstrap Model (SBM). The SBM paper 'Statistical Thermodynamics of Strong Interactions at High Energies', preprint CERN-TH-520 dated 25 January 1965, was published in Nuovo Cim. Suppl. **3**, pp. 147–186 (1965).

In the SBM model, a hadron exponential mass spectrum with the required properties is a natural outcome. However, it took time for the SBM model to be recog-

Department of Physics, The University of Arizona, Tucson, AZ 85721, USA

nized. Arguably, this was so because the stepping-stone ‘Distinguishable Particles’ manuscript had not seen the light of day. The need for the ‘right’ mass spectrum was not evident to the reader of the SBM paper. SBM is presented in this book both conceptually and with mathematical detail as relating to hot nuclear matter in Chapter 23, and the relation to phase transition to quark matter is further developed in Chapter 27. A historical SBM perspective that discusses the role of distinguishable particles is offered in Chapter 17. The unpublished ‘Distinguishable Particles’ paper, motivating SBM, is published here for the first time in the following Chapter 19.

18.2 Note by Rolf Hagedorn of 27 October 1964

The logical difficulty mentioned on p. 210 has been removed as follows, and the result is disappointing. Let α and β label possible momenta (in a two-dimensional box of volume $V_0^{2/3}$) and kinds of particles, respectively. Then

$$\log Z = \sum_{\alpha,\beta} \log\left(1 - \mathrm{e}^{-\sqrt{p_\alpha^2+m_\beta^2}/T}\right)^{-1} .$$

We replace

$$\sum_\alpha \longrightarrow \frac{V_0^{2/3}}{2\pi}\int_0^\infty p\mathrm{d}p\ldots , \qquad \sum_\beta \longrightarrow \int_0^\infty \rho(m)\mathrm{d}m\ldots ,$$

and expand the logarithm:

$$\log Z = \frac{V_0^{2/3}}{2\pi} T^2 \sum_{n=1}^\infty \frac{1}{n^3}\int_0^\infty \rho(m)\mathrm{d}m\left(1+\frac{nm}{T}\right)\mathrm{e}^{-nm/T} .$$

Now everything depends on the asymptotic behaviour of the mass spectrum $\rho(m)$:

1. If $\rho(m)$ grows faster than exponentially, $\log Z$ diverges for all $T > 0$. No thermodynamics is possible.
2. If $\rho(m)$ grows $\sim \mathrm{e}^{m/T_0}$, then $\log Z$ diverges for $T > T_0$. Whether it diverges for $T = T_0$ and if so, how it diverges for $T = T_0$ depends on the detailed behaviour of $\rho(m)$. In any case, an upper bound T_0 exists.
3. If $\rho(m)$ grows less than exponentially, no upper bound T_0 exists; if $\rho(m)$ grows exponentially up to some large M and afterwards grows less than exponentially, then the system would – over some energy range which depends on M – behave as if a highest temperatures existed.
4. In the ‘distinguishable particles’ model, Z diverges as $T_0/(T_0 - T)$ for $T \to T_0$ (two-dimensional case). Such behaviour can be obtained from a particular choice of $\rho(m)$, e.g.,
$$\rho(m) \longrightarrow \frac{1}{1+m/T_0}\frac{2}{m}\sinh\frac{m}{T_0}$$

and with $V_0 = (2\pi)^{3/2}/T_0^3$. It is seen, therefore that every type of exponential growth leads to an upper bound T_0, but only a very particular type leads to a behaviour which, for $T \to T_0$, coincides with that of our model. The model describes therefore a highly singular case[1] and should be abandoned. *Its main point, the highest temperature T_0 has still a chance to survive*, maybe only as an apparent 'highest temperature' in a large energy range (this, at least, is suggested by the experiments).

5. As the upper bound T_0 does not depend on the volume any more the simple description by means of a longitudinal and a transverse temperature must be abandoned.

18.3 From Distinguishable Hadrons to SBM

The beginning of a new idea in physics often seems to hang on a very fine thread: was anything lost when 'Thermodynamics of Distinguishable Particles' remained unbpublished? And what would Hagedorn do after withdrawing his first limiting-temperature paper? My discussion of the matter with Hagedorn suggests that his vision at the time of how limiting temperature could be justified evolved very rapidly. Presenting his final insight was what interested Hagedorn and motivated his work. Therefore, he opted to work on the more complete theoretical model.

While the withdrawal of the old, and the preparation of an entirely new paper seemed to be the right path to properly represent the evolving scientific understanding, today's perspective is different. In particular the insight that the appearance of a large number of different hadronic states allows to effectively side-step the quantum physics nature of particles within statistical physics became essentially invisible in the ensuing work. Few scientistd realize that this is a key property in th SBM, and the fundamental cause allowing the energy content to increase without an increase in temperature, as Hagedorn explains in the withdrawal note, see also the end of Sections 17.2 and 19.1.

The loss of relevance of quantum physics in hot hadronic matter is the scientific fact that we lost sight of after 'Distinguishable Particles' was withdrawn. To the best of my knowledge the dense, strongly interacting hadronic gas is the only physical system where this happens. Normally, the greater the density of particles, the greater the role of quantum physics. After surfacing briefly in Hagedorn's withdrawn 'Thermodynamics of Distinguishable Particles' paper, this finding faded from view. This indeed was a new idea in physics hanging on a very fine thread which ripped.

On the other hand, the Hagedorn limiting temperature lived on. Within a span of only 90 days between the withdrawal of his manuscript, and the date of his new CERN-TH preprint, Hagedorn formulated the Statistical Bootstrap Model. Its salient feature is that the exponential mass spectrum arises from the principle that hadrons are clusters comprising lighter (already clustered) hadrons. The key point

[1] Editor's note: today called 'fine-tuned'

of this second paper is a theoretical model based on the very novel idea of hadrons made of other hadrons. Such a model bypasses the need to identify constituent content of all these particles.

Models of this type are employed in other areas of physics. The simplest one is the statement that while atomic nuclei are made of individual nucleons, a great improvement in the understanding of nuclear structure is achieved if we cluster 4 nucleons (two protons and two neutrons) into an α-particle introducing the α-substructure, and so on. Hagedorn simply made all hadrons be clusters of lightest mesons, pions. The difference to the nuclear α-model is that the number of pions and more generally of all strongly interacting particles is not conserved. That turns out to have a big consequence and is the origin of the limiting temperature behavior.

Clustering pions into new hadrons and then combining these new hadrons with pions, and with already preformed clusters, and so on, turned out to be a challenging but soluble mathematical exercise. The outcome in this new Statistical Bootstrap Model (SBM) was that the number of states of a given mass was growing exponentially. Thus, in SBM, the exponential mass spectrum required for the limiting temperature arose naturally ab-initio. Furthermore the model established a relation between the limiting temperature, the exponential mass spectrum slope, and the pion mass, which provides the scale of energy in the model.

18.4 Hagedorn Temperature as a General Physics Concept

The presentation of the original limiting temperature article Chapter 19 is in part motivated by the reccent developments which adopt the concept in domains of physics that are entirely unrelated; it is the physics principle that leads thee way. Examples taken from search of the name ‘Hagedorn’ in title of a publication unrelated to the domain of physics in which Hagedorn worked are for example:

- T. Biswas, T. Koivisto and A. Mazumdar, “Atick-Witten **Hagedorn** Conjecture, near scale-invariant matter and blue-tilted gravity power spectrum,” JHEP **1408**, 116 (2014) [arXiv:1403.7163].
- A. Arslanargin and A. Kaya, “Open Strings on D-Branes and **Hagedorn** Regime in String Gas Cosmology,” Phys. Rev. D **79**, 066013 (2009) [arXiv:0901.4608].
- R. H. Brandenberger, A. Nayeri, S. P. Patil and C. Vafa, “Tensor Modes from a Primordial **Hagedorn** Phase of String Cosmology,” Phys. Rev. Lett. **98**, 231302 (2007).

Chapter 19
Thermodynamics of Distinguishable Particles: A Key to High-Energy Strong Interactions?

Rolf Hagedorn

Abstract A new kind of thermodynamical model for strong interactions at high energies is proposed. We start from the fact that strong interactions produce so many possible particle states (from π over its resonances to nucleons, strange particles and their resonances, up to highly excited 'fireballs') that in an actual process each of these states practically never occurs more than once. We use this in order to treat the very first instant of a high-energy collision by statistical thermodynamics of a system of an illimited number of distinguishable particles. The model shows surprising properties: there exists a universal highest possible temperature T_0 of the order of 150–200 MeV (corresponding to $\approx 10^{12}$K) which governs all high-energy processes of strongly interacting particles, independently of the actual energy and independently of the particle number, from cosmic ray jets down to elastic scattering. If a Lorentz contracted volume is introduced, the transverse momentum distribution in jets as well as in elastic scattering is described in agreement with experimental results. Paradoxically, this distribution is independent of whether or not 'thermal equilibrium' is reached. If it is not reached – in the majority of cases it is not reached – then the jet structure for production processes is the consequence. If the model turns out to be as good as present experiments indicated, then the existence of a highest temperature is very likely; it implies that, from higher and higher energy experiments, not much new can be learnt about the structure of strong interactions, since the mode of excitation (which depends on the dynamical details we would like to know) has no influence on what is finally observed. The situation would then be similar to that in ordinary thermodynamics, where no experiment could possibly reveal how a certain system was brought into its thermodynamical state. In astrophysics, the method of thermodynamics of distinguishable particles may have important consequences for the treatment of the highly compressed inte-

---

Preprint CERN-TH-483, October 1964, http://cds.cern.ch/record/929730?ln=en
archived in electronic format in year 2006/2010

(deceased) CERN-TH, 1211 Geneve 23, Switzerland

rior of heavy stars ('neutron stars') where Fermi statistics would have to be replaced by the one used here.

19.1 Introduction

In the past 10–15 years, it has become a more and more established (and more and more accepted) habit to publish conjectures, models, speculations – no matter whether:

- the foundations are safe,
- or all features have been worked out and compared with experiments,
- or even the correct physical interpretation of the resulting formulas is understood.

The model presented here suffers from all these deficiencies – and so far it is in good company. This is the first excuse to publish it. The second is that it shows some very remarkable features agreeing with some experimental facts and that there is a hope that it can be brought into a state where it becomes a theory. Whether it will survive this development is an open question. The few striking features to be presented below seem to make it interesting enough to publish it and initiate a discussion.

There are two roots of the present model:

1. Root one is a well-known observation already made by Fermi [1] in his first paper on the statistical model: the statistical model of particle production starts from expressions for channel probabilities (final channel f_j with N_j particles):

$$\begin{aligned} P_j &= \int \left|\langle f_j|T|i\rangle\right|^2 \prod_{k=1}^{N_j} \delta(p_k^2 - m_k^2)\mathrm{d}^4 p_k \\ &= \left(\prod_{k=1}^{N_j} C_k\right) F_{\mathrm{stat}} \int \delta^4\left(p_i - \sum_i^{N_j} p_k\right) \prod_1^{N_j} \frac{\mathrm{d}^3 p_k}{2E_k} \\ &= \left(\prod_{k=1}^{N_j} \Omega_k\right) F_{\mathrm{stat}} \int \delta^4\left(p_i - \sum_i^{N_j} p_k\right) \prod_1^{N_j} \mathrm{d}^3 p_k \,, \end{aligned} \tag{19.1}$$

   where $\prod_{k=1}^{N_j} C_k$ and $\prod_{k=1}^{N_j} \Omega_k$ stand for the mean values of the squared matrix element with respect to the invariant or non-invariant phase space, respectively, and F_{stat} takes into account spin and isospin weight and contains $1/n_i!$ when n_i particles of type i are present. It was then observed by Fermi that a statistical model starting from Eq. (19.1) fits smoothly into a thermodynamical model once the energy E and thus the particle number become large enough. Thus, in discussing high-energy limits, one will use conveniently the methods of statistical thermodynamics which are far easier to handle than the expressions in Eq. (19.1).

2. The second root of the present model is the observation that a very particular kind of statistical thermodynamics is necessary to fit the actual behaviour of the statistical model of type (19.1): it was found by evaluating numerical calculations made at CERN (1958–1962) at various c.m. energies (using the non-invariant phase space) that the ratio of the probability P_0 for the elastic channel to the sum over all probabilities behaves like an exponential function of the c.m. energy over a fairly large range [2] ($3 \leq E \leq 7.6$):

$$\left.\frac{p_0}{\sum p_j}\right|_{\mathrm{pp}} = \mathrm{e}^{-3.10(E-2)}\,, \tag{19.2}$$

with units such that $\hbar = c = k = M_p = 1$, k is Boltzmann's constant. This, and a similar result for πp collisions, was used to predict large angle elastic [2, 3] and exchange scattering, i.e., $p + p \to \mathrm{A} + \mathrm{B}$, etc. [4]. The predictions for *pp* elastic scattering at about 90° fit the experiments qualitatively over a range where the cross-section changes by 5 powers of ten.

It was natural to ask the question whether the exponential behaviour could be understood analytically since Eq. (19.2), being the result of hundreds of hours of computer time, is practically an empirical result. This question was attacked by several authors [5–8]. Bialas and Weisskopf [6] obtained[1]

$$\frac{p_0}{\sum p_j} \sim \mathrm{e}^{-\alpha(E-2)^{3/4}}\,, \qquad \alpha \text{ a constant}\,, \tag{19.3}$$

starting directly from statistical thermodynamics. Satz [8] considered the asymptotic behaviour for $E \to \infty$ of $\sum p_j$ with p_j in the form of Eq. (19.1) and found

$$\frac{p_0}{\sum p_j} \sim \mathrm{e}^{-aE^{2/3}}\,. \tag{19.4}$$

Vandermeulen [7] considered a special case of Eq. (19.1), namely the one where all masses are neglected (which can be justified), and found

$$\frac{p_0}{\sum p_j} \sim \mathrm{e}^{-bE}\,. \tag{19.5}$$

All the authors mentioned so far started from definite assumptions and obtained definite results which do not all agree with Eq. (19.2), although in one case it could be shown that, in the limited range where Eq. (19.2) was computed, expressions Eqs. (19.2) and (19.3) deviate little numerically from each other (see Fig. 4 of Bialas and Weisskopf [6]).

For our present discussion, the most interesting paper is that by Auberson and Escoubès [5], because these authors discuss various different assumptions and find different forms, namely,

---

[1] Here and in the following quotations, we consistently neglect algebraic expressions in E as compared to the exponential.

$$\frac{p_0}{\sum p_j} \sim \mathrm{e}^{-cE^\alpha}\ , \qquad \alpha = \frac{1}{2}\,,\ \frac{2}{3}\,,\ \frac{3}{4}\,,\ 1\ , \tag{19.6}$$

according to what the assumptions are. In order to be close to the calculations leading to Eq. (19.2), they work with the non-invariant phase space. An overall result is, whatever the particular assumptions are, that the masses of the particles produced can be neglected when $E \to \infty$. (Except if the main contributions come from ever new 'particles' with higher and higher mass values as E increases, such that $E/\overline{m}$ remains constant, $\overline{m}$ being the mean value of the masses produced. This is not at all likely in view of the purely geometrical fact that the phase space for small or negligible masses is so much bigger than for masses such that $E/\overline{m}$ is constant; only some very peculiar dynamical properties, which so far we have no reason to expect, would be able to counteract this tendency of phase space.) A particular result is that

$$\sum p_j = \sum_{n=2}^{\infty} \left[\frac{\Omega^n}{n!} \int \delta \left(E - \sum_{i=1}^{n} \varepsilon_i \right) \delta^3 \left(\sum_{i=1}^{n} \mathbf{p}_i \right) \mathrm{d}^3 p_1 \dots \mathrm{d}^3 p_n \right] \tag{19.7}$$

tends asymptotically to e^{cE} only if the factor of $1/n!$ is omitted, in other words, if the particles are considered to be distinguishable.[2]

This is not as unreasonable as it first sounds. And since it is the main point of the model which will be presented below, we must explain this more carefully. Auberson and Escoubès seek an asymptotic formula for the statistical model which fits the detailed numerical calculations at moderately high energies. In these detailed calculations, many different particles were considered (Ξ^-, Ξ^0, Σ^+, Σ^0, Σ^-, Λ, p, N, $\mathrm{N}^*_{3/2}$, K^0, K^+, K^-, π^+, π^0, π^-, and in some calculations ρ, ω, and η). It turned out that the calculated average particle numbers hardly exceeded the value one, and even for the pions they remained below two (per charge state). Had we included all the presently known resonances, then all average occupation numbers of the various states of the particles would have remained below one. Now in those calculations the factor $1/n!$ actually takes the form $1/n_1!n_2!\dots$, where n_1, n_2, etc., are the numbers of particles of type 1, type 2, etc.[3] Since then $\langle n_i \rangle$ actually turns out to be $\lesssim 1$, it follows that mainly those channels contribute to $\sum p_j$, where the n_i are either 0 or 1, in other words, where the whole factor $1/n_1!n_2!\dots$ equals one.

Since we presently know many more states of the fundamental particles, and since it seems likely that higher and higher excited states may be found, we expect that feeding them into the statistical model will have the effect that, even at very high energies, the main contributions will come from channels in which the n_i are zero or one. If therefore a simplified model of the type given in Eq. (19.7) with particles of equal masses is used to find out the asymptotic behaviour, then one should omit the factor $1/n!$ in order to come as near as possible to reality. It is reassuring that just in

---

[2] In fact, Ω must also be kept independent of E, contrary to what was done in the numerical calculations leading to Eq. (19.2). But there the masses were not neglected, and one sees easily that at moderate energies this will have the effect of increasing the power of E in the exponential (the details are difficult).

[3] Here, π^+, π^0, and π^-, etc., are of course considered to be different particles.

doing that one finds the exponential behaviour which is indicated by our numerical result (19.2) and which fits the experiment [9].

As one should expect, statistical thermodynamics of massless particles – all distinguishable from each other! – leads then to the same behaviour. The main formulas for such a statistics were worked out by Escoubès and the present author and included in the paper by Auberson and Escoubès [5].

In the following, we shall therefore discuss the model 'statistical thermodynamics of distinguishable particles' in some detail and try to understand its physical meaning. It should be clear from our considerations that the mechanism, which we imagine to take place, is the following. In the first instant of the collision, a certain number of particles – ranging from pions over kaons, nucleons, hyperons and their resonances to highly excited 'fireballs' – is produced according to the statistics of distinguishable particles. Then resonances decay according to their mode and 'fireballs' decay again according to the statistics of distinguishable particles, each one forming such a system. At the end of this chain of decays we arrive at pions, kaons, nucleons, and hyperons, where now the number of pions may be much larger than one without invalidating our treating the particles as distinguishable.

In Sect. 19.2, we present the simplest possible model of this kind. In Sect. 19.3, we speculate on its physical interpretation which, sometimes, is rather obscure. In Sect. 19.4, we discuss its weak points and possible improvements, and in Sect. 19.5, we sum up and draw a few general conclusions.

19.2 Statistical Thermodynamics of Distinguishable Particles

We write down the assumptions:

1. We consider a system of particles enclosed in a volume V in a temperature bath T. [We put the Boltzmann constant $k = 1$, the temperature T is then measured in nucleon masses: $T = 1$ ($= 939$ MeV) in those units corresponds to 1.1×10^{13}K.]
2. The number of particles is not limited.
3. All particles are distinguishable, i.e., can be labelled.
4. All particles have mass zero (we shall later consider massive particles) and no internal (mechanical) degrees of freedom (the $2s+1$ possible orientations of a particle with spin s are considered as $2s+1$ different particles).

Let ε_1, ε_2, ..., ε_α, ..., be the possible energy levels of one particle in the volume V. If we give a set of numbers $(n) = (n_1, n_2, \ldots, n_\alpha, \ldots)$ indicating by n_α how many particles of energy ε_α are present, then, in the usual case of indistinguishable particles, (n) would completely specify a quantum state of our gas. But as we consider the particles to be distinguishable, (n) stands for

$$\frac{N!}{n_1!n_2!\ldots n_\alpha!\ldots}$$

different states of the same energy $E = \sum_\alpha n_\alpha \varepsilon_\alpha$, with $N = \sum_\alpha n_\alpha$. For N particles, the partition function will then be

$$Z_N = \sum_{(n)} \frac{N!}{n_1!n_2!\dots} \exp\left(-\frac{1}{T}\sum n_\alpha \varepsilon_\alpha\right) \qquad \left(N = \sum n_\alpha\right)$$
$$= \left(\sum_\alpha \mathrm{e}^{-\varepsilon_\alpha/T}\right)^N \qquad (\text{with } N = \textstyle\sum_\alpha n_\alpha \text{ fixed}) . \tag{19.8}$$

We shall use the shorthand notations

$$x_\alpha \equiv \mathrm{e}^{-\varepsilon_\alpha/T} , \qquad z \equiv \sum_\alpha \mathrm{e}^{-\varepsilon_\alpha/T} = \sum_\alpha x_\alpha . \tag{19.9}$$

We calculate z for a massless particle ($p = \varepsilon$) in the usual way:

$$g(p)\mathrm{d}p = g(\varepsilon)\mathrm{d}\varepsilon = \frac{4\pi p^2 \mathrm{d}pV}{h^3}$$

as density of states gives in our units

$$z = \int g(\varepsilon)\mathrm{e}^{-\varepsilon/T}\mathrm{d}\varepsilon = \frac{VT^3}{\pi^2} . \tag{19.10}$$

We now drop the assumption that N is fixed and find

$$Z = \sum_{N=0}^{\infty} Z_N = \sum_{N=0}^{\infty} z^N = \frac{1}{1-z} = \frac{1}{1-\sum x_\alpha} = \frac{1}{1-VT^3/\pi^2} \tag{19.11}$$

for the partition function of our system. We observe here the striking feature, which will be of fundamental importance and indeed the very heart of our model, namely, that the partition function of our gas exists only if the temperature

$$T < T_0 = \left(\frac{\pi^2}{V}\right)^{1/3} . \tag{19.12}$$

We now calculate the expectation values of the energy E and particle number N of our system (as the system is in thermal contact with a temperature bath T, its energy is not fixed: our system is a member of a canonical, not microcanonical ensemble). We find from Eq. (19.11) that

$$\overline{E} = T^2 \frac{\partial}{\partial T} \log Z = \frac{3VT^4}{\pi^2} \frac{1}{1-VT^3/\pi^2} = Z\frac{3VT^4}{\pi^2} . \tag{19.13}$$

(Note that ordinary Bose statistics would give

$$\frac{1}{2}\left(\frac{V\pi^2}{15}T^4\right)$$

per internal degree of freedom, i.e., the Stefan–Boltzmann law. Our gas would, for $T \to 0$ thus behave like a light quantum gas with a slightly changed radiation constant.) We thus see that $\overline{E}$ diverges when $T \to T_0$.

The average occupation number of the energy level ε_α becomes with Eq. (19.11)

$$\overline{n}_\alpha = x_\alpha \frac{\partial}{\partial x_\alpha} \log Z = \frac{\mathrm{e}^{-\varepsilon_\alpha/T}}{1 - VT^3/\pi^2} = Z\mathrm{e}^{-\varepsilon_\alpha/T} \ . \tag{19.14}$$

This also determines the energy (= momentum) spectrum (see below). The expectation value of the particle number is

$$\overline{N} = \sum_\alpha \overline{n}_\alpha = zZ = \frac{VT^3}{\pi^2(1 - VT^3/\pi^2)} = Z\frac{VT^3}{\pi^2} \tag{19.15}$$

and the average energy (= momentum) of a particle is

$$\overline{\varepsilon} = \frac{\overline{E}}{\overline{N}} = 3T \ . \tag{19.16}$$

We observe that, in Eqs. (19.13)–(19.15), the relevant quantities all contain Z and therefore diverge when $T \to T_0$. Since we will be concerned throughout this paper with large energies, we consider the behaviour near $T = T_0$. Then in the slowly varying factors, T may be replaced by T_0 and we obtain the simple expressions

$$\overline{E} \ \to \ 3T_0 \frac{T_0^3}{T_0^3 - T^3}\, , \quad \overline{N} \ \to \ Z = \frac{T_0^3}{T_0^3 - T^3}\, , \quad \overline{n}_\alpha \ \to \ \mathrm{e}^{-\varepsilon_\alpha/T_0} \frac{T_0^3}{T_0^3 - T^3} \tag{19.17}$$

Let us draw $\overline{E}$ (or Z) as a function of T, or better, T as a function of $\overline{E}$. We obtain the behaviour shown in Fig. 19.1, where we omit the unphysical temperatures $T > T_0$ which lead to negative $\overline{E}$. This figure has obviously to be intepreted as follows: as soon as T comes very near T_0, we may achieve any (large) values of $\overline{E}$, $\overline{N}$, Z by only infinitesimal changes in the temperature. Consequently, as the mean values $\overline{E}$ and $\overline{N}$ suffer enormous changes for infinitesimal changes in T, we no longer expect the distributions of E and N over our canonical ensemble to have the sharp peaks we are used to in statistical thermodynamics. On the contrary, these distributions become flat and tend to a constant when $T \to T_0$. Indeed, as follows from the definition of Z [see Eqs. (19.8) and (19.11)], we have

$$T^2 \frac{\mathrm{d}\overline{E}}{\mathrm{d}T} = \overline{E^2} - \overline{E}^2 \ , \qquad z\frac{\mathrm{d}\overline{N}}{\mathrm{d}z} = \overline{N^2} - \overline{N}^2 \ , \tag{19.18}$$

which gives, when evaluated for $T \to T_0$ and $z \to 1$,

$$\frac{\overline{E^2} - \overline{E}^2}{\overline{E}^2} \ \longrightarrow \ 1 \ , \qquad \frac{\overline{N^2} - \overline{N}^2}{\overline{N}^2} \ \longrightarrow \ 1 \ . \tag{19.19}$$

It is easily seen that the distribution of N becomes a constant. Writing

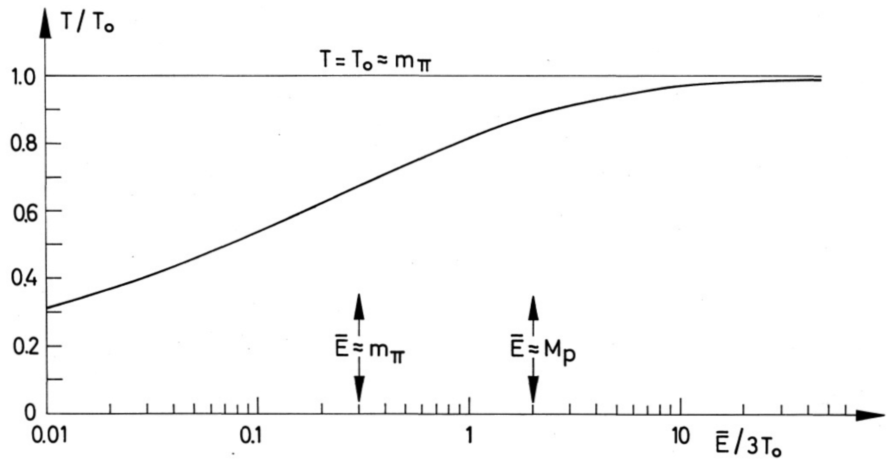


Fig. 19.1 Temperature T/T_0 as a function of the energy $\overline{E}/3T_0$ for particles of zero mass.

$$\overline{N} = \frac{\sum N z^N}{\sum z^N}$$

gives at once the probability of finding N particles as

$$W(N) = \frac{z^N}{\sum z^N} = \frac{z^N}{Z} \,. \qquad (19.20)$$

Since $z \to 1$ and $Z \to \infty$ when $T \to T_0$, we obtain $W(N) \to 0$. In other words, when $T \to T_0$, the energy and particle number of our system become undetermined.[4] Thus, from the point of view we have adopted so far, namely to consider our system as a member of a canonical ensemble (or as being in a temperature bath), we have failed to achieve anything and would be obliged to stop here. We shall see, however, in the next section, that just those circumstances which make it impossible to follow further the lines of the usual interpretation will serve to allow a new interpretation, if only by brute force.

19.3 The Interpretation of the Model

If, in statistical thermodynamics, we encountered a case where the mean energy of a system in a temperature bath T were undetermined, we would say that the system considered were not suited for a statistical treatment. A similarly unfavourable situation would, for instance, arise if a system contained, say, only two atoms: their total energy would be rather badly determined – or, if we fixed its energy in an ad

[4] The relative fluctuations of the occupation probabilities $\overline{W}_\alpha = \overline{(n_\alpha/N)}$ vanish, however. Hence, $\overline{W}_\alpha$ becomes 'sharp' when $T \to T_0$ (see Appendix A).

hoc manner and considered it as a member of a microcanonical ensemble, we could hardly speak of its temperature.

In our present model, however, the situation is different: the very fact that for $T \to T_0$ the system can have any energy and that $\overline{E}$ becomes very large can be reinterpreted as follows. Whenever a system is given – whose energy E is fixed and sufficiently large, namely $E \gg T_0$ – then we may think of it as a former member of (and now isolated from) the canonical ensemble embedded in a temperature bath $T \to T_0$. We may then forget about the canonical ensemble and simply postulate that we can ascribe to any system of sufficiently large energy the temperature T_0, even if the system chooses to have a small number of particles. By inverting Eq. (19.17), we have T as a function of $\overline{E}$:

$$T = T_0 \left(1 - \frac{T_0}{\overline{E}} + \cdots \right) , \tag{19.21}$$

so that $T \approx T_0$ whenever $\overline{E} \gg T_0$. Although this holds for the dependence of T on $\overline{E}$ and although we have seen that for $T \to T_0$ we might expect any, even small, energies, we play safe when we say that we ascribe to the system the temperature T_0 if it has an actual energy $E \gg T_0$; if the energy is small, it still could have belonged to the ensemble of temperature T_0, but it could as well have belonged to a lower temperature.

Next we observe that only $\overline{E}$ and $\overline{N}$ are (via T) coupled to each other, but not the actual energy E and particle number N of a system. Thus, once we have fixed $E \gg T_0$ and ascribed the temperature T_0 to the system, we may still expect any number N of particles; indeed, all N values become equally likely for $T \to T_0$.

By the reinterpretation of our model, we have of course abandoned the point of view of the canonical ensemble of temperature T and have obtained the description of a single system of given energy E. This system must no longer be thought of as being in contact with its surroundings – which would be hard to imagine for a high-energy collision – but, thanks to the peculiar behaviour of $\overline{E}(T)$, it has a temperature $T \to T_0$ in its own right.

As for the value of $T_0 = (\pi^2/V)^{1/3}$, it must be chosen such that, at least, the system itself conserves the main features of a thermodynamical system: the particles must be able to interact with each other. As we wish to describe high-energy collisions of strongly interacting particles, where the particles produced will escape radially from the region of interaction, they will cease to interact once their mutual distances become much larger than the range of forces, i.e., the Compton wavelength of the pion. Thus the volume V is to be taken as

$$V = \frac{4\pi}{3} \left(\frac{a}{m_\pi} \right)^3 , \tag{19.22}$$

where $a \approx 1$ is an adjustable parameter. With Eq. (19.12), we obtain

$$T_0 = \frac{m_\pi}{a} \left(\frac{3\pi}{4} \right)^{1/3} = m_\pi \frac{1.35}{a} \equiv b m_\pi , \tag{19.23}$$

where $b = 1.35/a$ is again of order one.

We now formulate the new interpretation in the following postulate.

Postulate. To every high-energy collision of strongly interacting particles (hadrons) of total centre-of-mass (kinetic) energy $E \gg m_\pi$, and also to every highly excited hadron ('fireball') with excitation $\Delta E \gg m_\pi$, we ascribe the temperature $T \to T_0 = bm_\pi$ with $b \approx 1$. (19.24)

We can consider this temperature $T \approx 140$ MeV (corresponding roughly to 10^{12}K) as the 'highest possible temperature' which, as a fundamental constant, governs all high-energy processes of strongly interacting particles. [Of course, this goes as far as the present simplified model is valid – introduction of masses and of a particular shape of the volume of interaction and other refinements may change this conclusion somewhat (see below).] Apart from such, we hope, minor changes, we would predict on this basis that T_0 will come into play whenever at least one strongly interacting particle takes part in a collision. Therefore high-energy reactions with initial states like $\mathrm{e} + p$, $\gamma + p$, $\nu + p$, and those in which p is replaced by any other hadron, will show certain features similar to $p + p$ collisions at high energies. The reason is that, whenever a sufficient amount of excitation energy ($\Delta E \gg T_0$) is transferred to a hadron, the excited hadron falls under the above postulate. This holds even for such reactions as $\mathrm{e} + \mathrm{e} \to \mathrm{e} + \mathrm{e} +$ hadrons. For two reasons, this does not apply to the weakly or electromagnetically interacting partners of the reaction:

- They have no reason to feel the temperature T_0 which has its origin in the range of strong interactions.
- They do not interact strongly enough to produce the many resonances required for a statistics of distinguishable particles. For indistinguishable particles, no T_0 exists.

It is interesting to compare these immediate consequences of our model (and of the above postulate) with recent speculations by T.T. Wu and C.N. Yang [10]. They assume that the sharp decrease with energy of differential cross-sections at large angles is due to a mechanism independent of the method of excitation and discuss the consequences of such a possibility. Our present model provides a natural basis for their assumption and leads to the same consequences (which were also partly drawn by the present author [4]).

We shall now draw some quantitative conclusions of our postulate and try to see how our system will behave experimentally. First we remark that, since we now consider the energy E to be given, and work in the centre-of-mass frame, we have to impose energy–momentum conservation, at least when the number of particles is small. In particular, we should extend the summation $Z = \sum z^N$ from $N = 1$ to ∞ and not – as we did above – from $N = 0$ to ∞. We shall keep this point in mind. It does, however, not change our conclusions about the gross features of the system. Since we work at $T = T_0$, the partition function diverges and it does not matter, in general, whether or not the first term is included.

Secondly, we remark that, putting all masses equal to zero is an oversimplification in some cases. For instance, the total energy E has then often to be interpreted as kinetic energy. Our assumption that the particles be distinguishable is based on the experimental fact that so many particle states (resonances, charge, hypercharge) are known and that the list of them grows steadily. Of course, the newly added particles have a tendency to have higher and higher mass values – it may well be that the 'fireballs' of cosmic ray events are the asymptotic form of them where the widths of the resonances are larger than their spacing. We shall consider here everything from a pion over kaon, nucleon, hyperon, and resonances up to the fireballs, as possible 'particles' appearing in our system, and this would force us to take the mass of such a particle into account. But presently, we shall simply put $m = 0$ and $T = T_0$ (if necessary, $\lim_{T\to T_0}$ is understood).

We shall consider:

- the number of particles produced (first generation),[5]
- the momentum spectrum of the particles (first generation).

These two are related to each other and treated together. From Eq. (19.14), we have for the expectation value of the number of particles with energy ε_α,

$$\overline{n}_\alpha = Z\mathrm{e}^{-\varepsilon_\alpha/T_0} ,$$

and from this

$$\overline{N} = \sum \overline{n}_\alpha .$$

Now, however, we must insist on energy conservation, that is, these formulas are subject to the condition

$$\sum \varepsilon_\alpha \overline{n}_\alpha = E . \tag{19.25}$$

The usual definition of $\overline{E}$, which we used in Eq. (19.13), namely

$$\overline{E} = T^2 \frac{\partial}{\partial T} \log Z ,$$

can be derived from the requirement $\sum \varepsilon_\alpha \overline{n}_\alpha = \overline{E}$ with $\overline{n}_\alpha$ defined by Eq. (19.14). We are thus forced, if we insist on energy conservation which leads to Eq. (19.25), to identify the expectation value $\overline{E}$ of the old interpretation (canonical ensemble) with the given sharp value E of the energy in our new interpretation. Consequently, wherever Z and $\overline{E}$ appear explicitly, we shall replace them by

$$\overline{E} \longrightarrow E , \qquad Z \longrightarrow E \frac{\pi^2}{3VT_0^4} = \frac{E}{3T_0} , \tag{19.26}$$

as suggested by Eqs. (19.10), (19.12), (19.13), and (19.25). Then with (19.21), the temperature T becomes

$$T = T_0 \left(1 - \frac{T_0}{E} + \cdots \right) \approx T_0 , \tag{19.27}$$

[5] 'First generation' means the distinguishable particles (produced in the first instant) which later, by a chain of further 'generations', decay into the observed pions, nucleons, etc.

and the number of particles with energy ε_α is

$$\overline{n}_\alpha = \frac{E}{3T_0}\mathrm{e}^{-\varepsilon_\alpha/T_0}\,, \qquad \overline{N} = \sum \overline{n}_\alpha = \frac{E}{3T_0}\,. \tag{19.28}$$

Our conclusion that the distribution of N values becomes constant remains true. We can even see how it tends to the constant $z = VT^3/\pi^2$ and T from Eq. (19.27) gives $z(Z) \simeq 1 - 3T_0/E$ ($E \gg T_0$). Hence, for the probability $W(N)$ of finding just N particles, Eq. (19.20) yields

$$W(N) \cong \frac{3T_0}{E}\left(1 - \frac{3T_0}{E}\right)^N \approx \frac{3T_0}{E}\exp\left(-N\frac{3T_0}{E}\right) \equiv \frac{1}{N}\mathrm{e}^{-N/\overline{N}} \longrightarrow 0\,. \tag{19.29}$$

Now this seems to be in spectacular disagreement with experiments: neither – as we know from cosmic ray evidence – does the number $\overline{N}$ of particles produced increase proportionally to E, nor is the distribution of the observed particle number constant. The way out of this apparent disagreement is again provided by the model itself: the N particles found in a particular case are by no means the final particles observed in photographic emulsions (mainly pions) – speaking of distinguishable particles, we have to consider them to be anything between a pion and a 'fireball'. Equation (19.29) gives us the probability of just finding N such not further specified objects. It states that all values of N are (for $N \sim \overline{N}$) practically equally likely. The question how probable is it to find N specified particles is quite another one, as we shall see in a moment when we discuss large angle elastic scattering. Presently, we note that, interpreting the N particles as 'fireballs' of unspecified excitation energy, Eq. (19.29) tells us that a 'two-fireball model' would never work,[6] as there will be contributions of almost the same weight from 3, 4, 5, ..., fireballs, a situation similar to that in the multiperipheral model of Amati, Fubine, and Stanghellini [12] and in considerations by K.G. Wilson [13].

But if this is so, then the number of pions and other final particles observed in experiments should even be larger than $\overline{N} = E/3T_0$, since these particles are produced in a chain of decays starting from the first N 'fireballs' and going into smaller and smaller ones. Here the answer is that introducing the masses and a contracted volume will bring that in order: we shall come back to this problem in Sect. 19.4, where it will be shown that $\overline{N}$, the number of 'fireballs', tends to ≈ 5 and becomes energy independent for $E \to \infty$.

Let us now consider the energy spectrum of our particles. First we treat the case where the question of how many particles we expect and the question of what their energies might be are intimately connected: large angle elastic and exchange scattering. In that case, we have two definite final particles, each with energy $E/2$, and from Eq. (19.28), we conclude that the probability of finding a particle in the energy level $\varepsilon_\alpha = E/2$ is given by

$$w_\alpha = \frac{\overline{n}_\alpha}{\overline{N}} = \mathrm{e}^{-\varepsilon_\alpha/T_0} = \mathrm{e}^{-E/2T_0}\,. \tag{19.30}$$

[6] This is in fact the experimental situation [11].

This is then also the probability of finding two specified particles. They may be the initial (elastic scattering) or some definite other ones, e.g., $p+p \to \mathrm{A}+\mathrm{B}$. Between Eq. (19.30) and the differential cross-section come, of course, some further considerations (flux factors, centrality condition, influence of the actual masses of A and B), which have been treated in another paper [4]. The main point is that, if we compare Eq. (19.30) with the numerical result of Eq. (19.2) which fits the observed large angle scattering well [4], we find that T_0 should have a value such that $1/T_0 = 6.2$. Thus in this case,

$$T_0 = 1.1 m_\pi = 151 \text{ MeV} . \tag{19.31}$$

This agrees well with our postulate Eq. (19.24). One could argue that this is a lucky accident, but then it will be hard to explain how this accident leads to a formula which fits the experiment well in a region where the observed cross-section varies over five orders of magnitude [4]. There are in fact not many formulas of physics which cover such a range.

The probability of finding two specified particles [see Eq. (19.30)] is indeed very different from that of finding any two particles: $W(2) = (3T_0/E)\mathrm{e}^{-6T_0/E}$. The obvious interpretation is that there is a large number of two-body final states, each with a probability of order $\mathrm{e}^{-2T_0/E}$, ranging from elastic scattering to two heavy fireballs, all contributing to $W(2)$. One can even estimate the number of two-body final states. It is of the order

$$n(2) \approx \mathrm{e}^{E/2T_0} W(2) = \frac{3T_0}{E} \mathrm{e}^{E/2T_0 - 6T_0/E} . \tag{19.32}$$

Putting $E = 5.6$, i.e., the c.m. energy minus 2 in a 25 GeV pp collision, one finds $n(2) \approx 3 \times 10^6$. This would mean that between the pion[7] and the heaviest (here possible) fireball ($\Delta E \approx 5.4$) lie some 3×10^6 different states. (The question of the mass spectrum of fireballs is treated below see page p. 197.) Similar considerations will apply to other few-particle channels, and seen from this angle, it no longer seems surprising that $W(2) \approx W(3) \approx W(4) \approx \ldots$.

We now turn to the energy spectrum in general. The density of states in the volume V was, in our units,

$$g(\varepsilon) = \frac{V\varepsilon^2}{2\pi^2} = \frac{\varepsilon^2}{2T_0^2} ,$$

and the number of particles to be expected in the level ε_α was given in Eq. (19.28). The number of particles between ε and $\varepsilon + \mathrm{d}\varepsilon$ then becomes

$$w(\varepsilon)\mathrm{d}\varepsilon = \frac{E}{6T_0^4} \varepsilon^2 \mathrm{e}^{-\varepsilon/T_0} \mathrm{d}\varepsilon .$$

We do not believe that the normalization factor $E/6T_0^4$, which makes $\int w(\varepsilon)\mathrm{d}\varepsilon = \overline{N} = E/3T_0$, is very meaningful because $\overline{N}$ itself is to be rather different in a more realistic model. We simply write

$$w(\varepsilon)\mathrm{d}\varepsilon \sim \varepsilon^2 \mathrm{e}^{-\varepsilon/T_0} \mathrm{d}\varepsilon . \tag{19.33}$$

[7] Since $p+p \to \text{fireball} + \pi$ leads to the smallest mass (m_π) of one of the final particles.

Remembering that we put $m = 0$, we may as well replace ε by p, the momentum. Then Eq. (19.33) reads

$$w(p)\mathrm{d}p \sim p^2 \mathrm{e}^{-p/T_0} \mathrm{d}p \,, \tag{19.34}$$

or, if we assume an isotropic distribution (as, so far, we are obliged to), we obtain

$$w(\mathbf{p})\mathrm{d}^3 p \sim \mathrm{e}^{-|\mathbf{p}|/T_0} \mathrm{d}^3 p \,. \tag{19.35}$$

It is a remarkable fact that a formula of this type applies apparently to all high-energy processes, if only we replace $|\mathbf{p}|$ by the transverse momentum $p_\perp = |\mathbf{p}| \sin\theta$. Let us simply do that and leave the question of how to justify it and get rid of the isotropy to later speculations. Then, if we write

$$w(p_\perp) \sim \mathrm{e}^{-p_\perp/T_0} \,, \tag{19.36}$$

conservation of total energy could be left to the longitudinal component. Thus Eq. (19.36) would hold true whatever the actual number of particles is. We would expect such a law to govern, not only processes like $p + p \rightarrow \mathrm{A} + \mathrm{B}$, but also the transverse momentum distribution of the many particles produced in cosmic ray jets. Our model would thus explain the hitherto obscure fact that experimentally the transverse momentum distribution in high-energy events is independent not only of the primary energy, but even of the number of particles involved. Since T_0 depends only on the range of interaction, cosmic ray jets and large angle scattering must show the same behaviour There might, of course, be some slowly varying factors (powers of p and/or E) in front of the exponential which differ from case to case, but the asymptotic behaviour should be dominated by Eq. (19.36). Orear [14] points out that Eq. (19.36) is a rather good fit to many processes. He quotes experimental results on *pp* elastic scattering, $p + p \rightarrow \pi + d$, $\pi + p \rightarrow \pi + p$, and finds that they are all well fitted [the $\pi + p$ data are rather meagre and can only be said not to disagree with Eq. (19.36)] by our Eq. (19.36) if one takes $T_0 = 158$ MeV (*pp* elastic), 160 MeV ($p + p \rightarrow \pi + d$).

Figure 19.2 may illustrate how good the fit actually is. Orear (from whose paper [14] the figure is taken) plots $E^2 \mathrm{d}\sigma_{\mathrm{el}}/\mathrm{d}\omega\big|_{\mathrm{pp}}$ as a function of $p_\perp = p \sin\theta$. The fit

$$E^2 \frac{\mathrm{d}\sigma}{\mathrm{d}\omega}\bigg|_{\mathrm{pp}} = \mathrm{const.} \times \mathrm{e}^{-p_\perp/T_0} \,, \qquad T_0 = 158 \text{ MeV} \,,$$

is really excellent if one keeps in mind that it covers a range of the primary (lab) momentum between $p_0 = 1.7$ GeV/c and $p_0 = 30.7$ GeV/c and a range of $\mathrm{d}\sigma/\mathrm{d}\omega$ of eight powers of 10. As an aside, we mention that the factor E^2, which seems necessary to make the fit so good, is predicted from a simple 'centrality condition' and is contained in the formula for large angle elastic and exchange scattering recently proposed by the present author [4].

The above values of T_0 agree well with our postulate $T \approx m_\pi$. Furthermore, Cocconi, Koester, and Perkins [15] and Fowler and Perkins [11] find from high energy nucleon–nucleon collisions that the transverse momentum distribution of

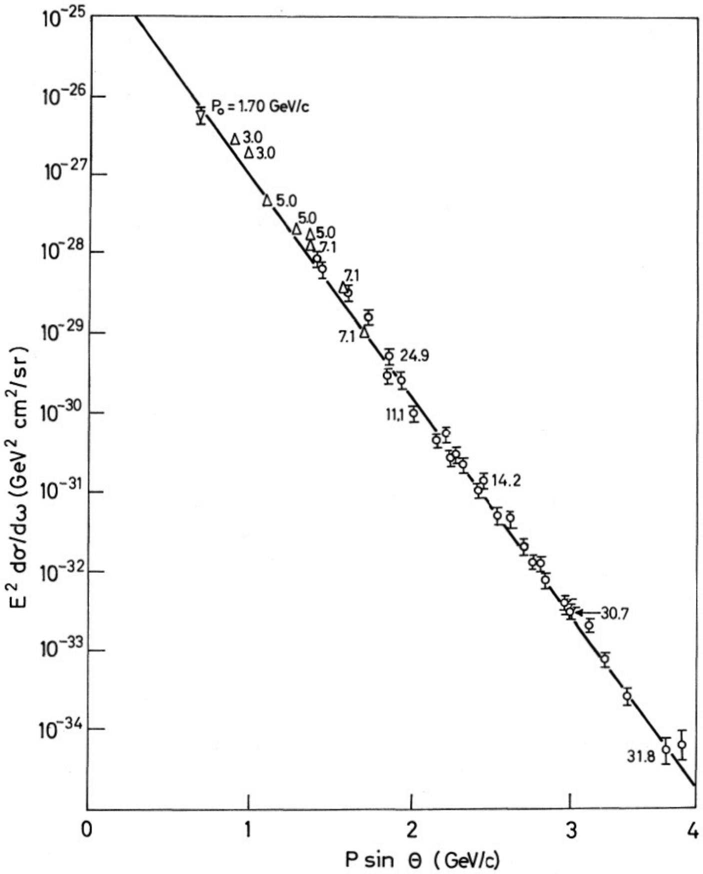


Fig. 19.2 $E^2 \mathrm{d}\sigma_{\mathrm{el}}/\mathrm{d}\omega\big|_{\mathrm{pp}}$ as a function of the transverse momentum. Taken from Ref. [14].

pions is given by Eq. (19.36) with an apparent value of $T_0 \approx 170$ MeV, a somewhat broader distribution than the one with $T_0 \approx 150$ MeV. This broadening is to be expected if we remember that the pions observed experimentally are *not* the particles produced in the first instant: they are the end-products of a chain of decays, each of which is governed by a law like Eq. (19.36), and the broadening is simply a kinematical effect. If one wishes to calculate this effect quantitatively, then one must no longer put the masses equal to zero. A very simple example is carried through in Appendix B. We assume that a fireball of mass $m^* \gg T_0$ emits a particle of mass m_ℓ (not $\gg T_0$) and we consider only one-dimensional (transverse) motion: the momentum distribution of m^* in the c.m. system is $w(p^*) = \exp(-\varepsilon^*/T_0)$, while that of m_ℓ in the rest frame of the fireball is $w(p_\ell) = \exp(-\varepsilon_\ell/T_0)$. Then the momentum distribution of the emitted lighter particle in the c.m. frame becomes $W(p) \approx \mathrm{const.} \times (\varepsilon + m^*)^{-1/2} \exp(-\varepsilon/T_{\mathrm{eff}})$, where T_{eff} increases monotonically and slowly with ε: for $\varepsilon = m_\ell$, it equals T_0 and for $\varepsilon \to \infty$, it reaches $2T_0$. For a fireball of mass $m^* = 1$ and for $\varepsilon \approx p = 1$ GeV, one finds $T_{\text{eff}} \approx 4T_0/3$, and a fireball with $m^* = 2$ would, for the same ε, lead to $T_{\text{eff}} \approx 6T_0/5$.

We stress once more (see Appendix A) that, although the relative fluctuation $(\overline{N^2} - \overline{N}^2)/\overline{N}^2$ tends to one, this is *not* so for $(\overline{w^2} - \overline{w}^2)/\overline{w}^2$. In other words, notwithstanding the flatness of the distribution of multiplicities, we should find a sharp distribution about the predicted momentum spectrum. It is instructive to look at the experimental Figs. 19.3 and 19.4. They show the sharp distribution about the predicted spectrum and the flat distribution of multiplicities.

We conclude this section by admitting that our reinterpretation of the original model was not always very convincing. In particular, the step from the energy distribution $\overline{n} = \mathrm{e}^{-\varepsilon_\alpha/T_0}$ to the transverse momentum distribution $w(p_\perp) \sim \mathrm{e}^{-p_\perp/T_0}$ is mainly suggested by the experimental evidence. But even with these inconsistencies in mind, we believe that the model contains some truth. We shall now try to strengthen this optimism by some speculations about possible improvements.

19.4 Speculations on a More Realistic Model

Our model of massless distinguishable particles in an energy independent volume V was worked out in such detail because of its simplicity. We shall now try to improve the model. Convincing improvements have not yet been carried through. Only an essay will be given. We discuss in turn:

- Angular distribution and multiplicity, see p. 197

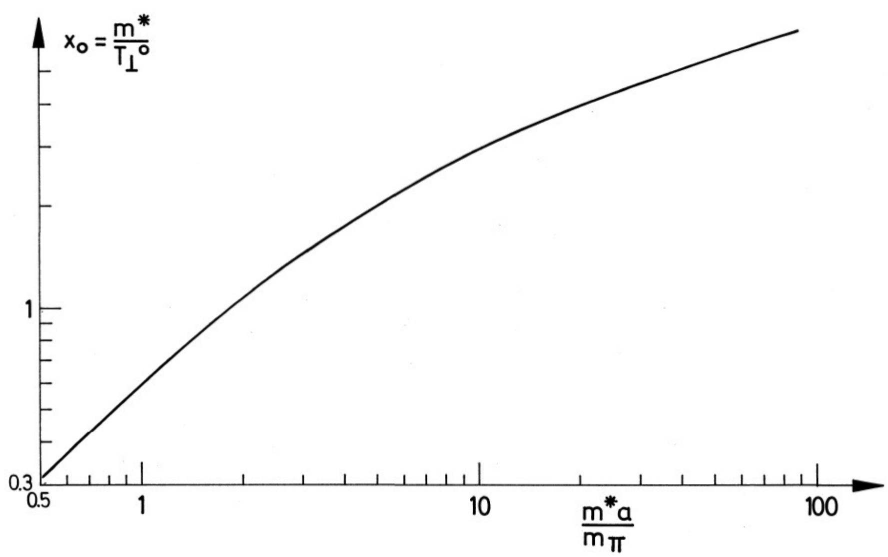


Fig. 19.3 Relation between the mass m^* and the temperature $T_\perp^0$.

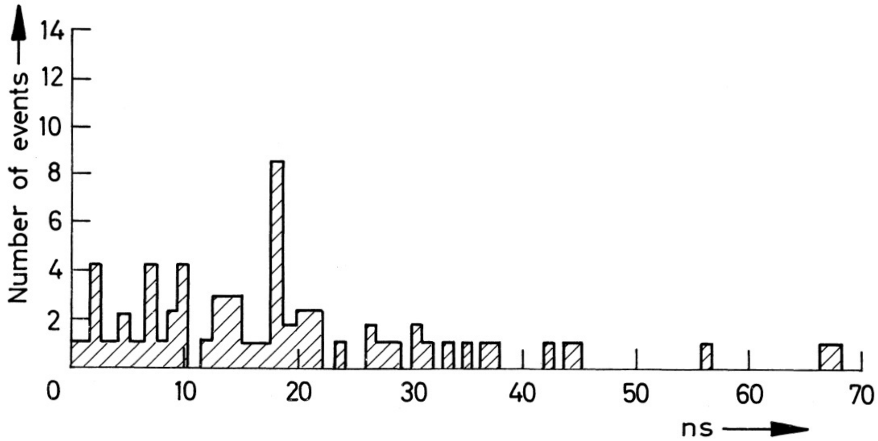


Fig. 19.4 Experimental distribution of multiplicities in 64 events at primary energies $6 \times 10^3 \leq E_p \leq 4 \times 10^4$ GeV. Taken from Ref. [11].

- The case of nonzero masses, see p. 203
- A speculation on the mass spectrum of fireballs, see p. 206
- Elastic (and exchange) scattering, see p. 207
- A logical difficulty of the model, see p. 210

Angular distribution and multiplicity

Fermi, in his paper initiating the statistical model [1], consider the possibility of a Lorentz contracted interaction volume. Assuming such a volume, we would already obtain from the uncertainty relation a suggestion of the character of the momentum distribution: the spherical volume $V = (4\pi/3)(1/m_\pi)^3$ would become a flat rotational ellipsoid with a transverse half-axis of length $1/m_\pi$ and a longitudinal (with respect to the collision axis) half-axis of length $1/\gamma m_\pi$, where $\gamma = (1-\beta_{\text{cm}}^2)^{-1/2}$ $(= E/2$ for pp collisions). The uncertainty relation requires a particle which has been kept in such a volume and which is suddenly set free to have momenta of the order of $p_\perp \approx m_\pi/2$, $p_\perp \approx \gamma m_\pi/2$. We would expect a similar effect for our model. However, one sees immediately that merely Lorentz contracting this interaction volume will have no other result than to replace V in all our formulas by V/γ and consequently T_0 by $T_0\gamma^{1/3}$. In such a contracted volume, the energy levels will be of the type

$$\varepsilon_\alpha = c\sqrt{\alpha_2^2+\alpha_1^2+\gamma^2\alpha_3^2}\,, \qquad \alpha_{1,2,3} = 0,\ \pm 1,\ \pm 2, \dots\,, \tag{19.37}$$

where $c = m_\pi(6\pi^2)^{1/3}$ for a rectangular box of volume

$$l_1 l_2 \frac{l_3}{\gamma} = \frac{1}{\gamma}\frac{4\pi}{3}\left(\frac{1}{m_\pi}\right)^3 ,$$

and our $z = \sum \mathrm{e}^{-\varepsilon_\alpha/T}$ becomes as usual

$$z(\gamma,T) = \int \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \exp\left(-\frac{C}{T}\sqrt{\alpha_2^2+\alpha_1^2+\gamma^2\alpha_3^2}\right) = \frac{V_0}{\gamma\pi^2}T^3 , \tag{19.38}$$

where $V_0 = 4\pi/(3m_\pi^3)$. From here, all the machinery of Sect. 19.2 runs as before, only V_0/γ replaces V everywhere. Therefore, not only is nothing gained from the angular distribution, but we even have to accept the unpleasant fact (in view of the experimental findings) that T_0 would no longer be constant ≈ 150 MeV, but instead would increase with energy:

$$T_0 = \left(\frac{E}{2}\right)^{1/3}\left(\frac{\pi^2}{V_0}\right)^{1/3} \approx \left(\frac{E}{2}\right)^{1/3} \times 150\ \mathrm{MeV} .$$

In order to overcome this difficulty, we now make the drastic assumption that the longitudinal and transverse motion can be treated independently (we shall try afterwards to give it a shade of justification). In that case, we again consider a rectangular box with volume V_0/γ, where $V_0 \approx (4\pi/3)(1/m_\pi)^3$ is our old V. We choose the sides to be

$$l_1 = l_2 = V_0^{1/3} , \qquad l_3 = \frac{1}{\gamma}V_0^{1/3} . \tag{19.39}$$

Then we have two independent problems of statistical thermodynamics, viz.,

- in a one-dimensional volume $V_\parallel = V_0^{1/3}/\gamma$ (longitudinal),
- in a two-dimensional volume $V_\perp = V_0^{2/3}$ (transverse).

The densities of the energy levels become in our units

$$\begin{aligned} g_\parallel(p_\parallel)\mathrm{d}p_\parallel &= \frac{V_\parallel \mathrm{d}p_\parallel}{h} = \frac{V_0^{1/3}}{2\pi\gamma}\mathrm{d}p_\parallel , \\ g_\perp(p_\perp)\mathrm{d}p_\perp &= \frac{V_\perp 2\pi p_\perp \mathrm{d}p_\perp}{h^2} = \frac{V_0^{2/3}}{2\pi}p_\perp \mathrm{d}p_\perp , \end{aligned} \tag{19.40}$$

and with

$$z = \sum_\alpha \mathrm{e}^{-\varepsilon_\alpha/T} \longrightarrow \int g(p)\mathrm{e}^{-p/T}\mathrm{d}p ,$$

we obtain

$$z_\parallel = \frac{V_0^{1/3} T_\parallel}{2\pi\gamma}\,, \qquad Z_\parallel = \frac{1}{1-z_\parallel} \text{ diverges for } T_\parallel^0 = \frac{2\pi\gamma}{V_0^{1/3}} = 2\gamma\pi^{1/3} T_0\,,$$
$$z_\perp = \frac{V_0^{2/3} T_\perp^2}{2\pi}\,, \qquad Z_\perp = \frac{1}{1-z_\perp} \text{ diverges for } T_\perp^0 = \frac{\sqrt{2\pi}}{V_0^{1/3}} = \sqrt{2\pi^{-1/3}}T_0\,. \tag{19.41}$$

As we decided to treat the longitudinal and the transverse motion independently, there is no reason to insist that the two temperatures associated with these motions should be equal. Since our old T_0 equals $(\pi^2/V_0)^{1/3}$, we see that $T_\perp^0 \approx T_0$ and $T_\parallel \approx \gamma 3 T_0$. To every high energy collision in which both energies $E_\parallel$ and $E_\perp$ are large (i.e., where total c.m. energy *and* momentum transfer are $\gg m_\pi$), we would again ascribe a temperature $\approx T_0$. We would, however, associate this temperature with the transverse motion only and provide another temperature $\approx 3\gamma T_0$ for the longitudinal one.

Is this picture justified? It is well known [17] that a consistent description of scattering should employ wave packets – aimed at each other – rather than plane wave states. A plane wave state would be non-localized and T would be zero (it would be zero for every completely non-localized state, whether or not $\mathbf{p}$ is sharp), but a scattering experiment is equivalent to a position measurement of the colliding particles with the high precision of the linear dimensions of the order of $1/m_\pi$. In that case we could ascribe a temperature $T \lesssim T_0$ to the localized wave packets before the collision. Let us then make the rather unconventional speculation and imagine that we could – on the basis of our considerations in Sect. 19.3 – ascribe to the (localized) incoming particle (in its rest system) a temperature $T \lesssim T_0$, which depends, as we saw, neither on the number of particles in a volume nor on the energy, but only on the volume. The (localized) hadron would then have a 'temperature' $T \lesssim T_0$ and only the conservation laws forbid it to radiate off mesons and nucleon–antinucleon pairs, etc. We may think of $T \lesssim T_0$ being the temperature of the cloud of virtual particles. The conservation laws would play the role of a box with rather rigid walls in which the virtual particles are enclosed and which they cannot leave. In the rest system of the nucleon, the energy spectrum of these virtual particles is isotropic; for the momentum distribution in the forward direction of the incoming nucleon, one has

$$w(p_\parallel^*) \sim \mathrm{e}^{-|p_\parallel^*|/T_0}\,,$$

where the star indicates the rest system of the nucleon. [This distribution is very different from that in a Newton–Wigner localized state [18]. This is not surprising as we are dealing with the localized state of a *physical* particle, whereas the Newton–Wigner state describes a *bare* localized (Klein–Gordon) particle. It may be possible to relate our philosophy to the 'bootstrap' model of hadrons.] Consider those virtual particles which go in the forward half-space. The $p_\parallel^*$ will appear in the c.m. system with momentum $p_\parallel = \gamma(p_\parallel^* + \beta E^*)$, where β is the velocity of the nucleon seen from the c.m. system of the collision. As $\beta \approx 1$ and $E^* \gtrsim p_\parallel^*$ (still neglecting masses), we have $p_\parallel \approx 2\gamma p_\parallel^*$. Seen from the c.m. system, the momentum distribution of the virtual particles around the incoming nucleons with then be roughly

$$\begin{aligned} &w_{\|}(p_{\|}) \sim \mathrm{e}^{-p_{\|}/2\gamma T_0} + \text{contribution from backward going particles}\,, \\ &w_{\perp}(p_{\perp}) \sim \mathrm{e}^{-p_{\perp}/T_0} \quad (\text{unchanged})\,. \end{aligned} \tag{19.42}$$

The collision, which now takes place, will 'carry out the position measurement' and loosen the constraints imposed by the conservation laws. In other words, the collision breaks the rigid walls of the volume V and the virtual particles can become real. They would escape with momentum spectra of the type (19.42), where $w_{\|}(p_{\|})$ contributes $\mathrm{e}^{-p_{\|}/2\gamma T_0}$ to the forward and backward directions – each nucleon to one of these directions. [The contributions from the particles which were emitted backward in the nucleon rest frame have low average momenta and do not follow the distribution $w_{\|}(p_{\|})$.]

We see that such a picture would rather naturally lead to an effective longitudinal temperature $T_{\|}$ being about γ times larger than $T_{\perp}$ (and indeed the two Lorentz contracted incoming particles will, when they get into touch with each other, look like *one* Lorentz contracted volume at rest in the c.m. system). We only have to assume that, in most cases, the collision time is so short that a thermal equilibrium ($T_{\|} \approx T_{\perp}$) cannot be reached. The simple model described in [4] shows that the fraction of collisions in which thermal equilibrium may be reached, is about $1/\gamma^2$ of all inelastic ones.[8] That means that in roughly $1 - 1/\gamma^2 = \beta^2 \approx 1$, i.e., in nearly all collisions, the thermal equilibrium is *not* reached and the longitudinal and transverse temperatures can be different and the volume of interaction looks Lorentz contracted. Then our above treatment would be justified. If, however, in the very few remaining collisions ($\sigma \approx \sigma_{\mathrm{inel}}/\gamma^2$) thermal equilibrium is (more or less) reached, then we would come back to our model of Sect. 19.2 and we would obtain the unique temperature $T_{\|} \approx T_{\perp} \approx T_0$ in a volume which is no longer Lorentz contracted. The emission of particles would have a tendency to become isotropic. The interesting point is, however, that then again the transverse (and, incidentally, the longitudinal) momentum distribution would be described by $\sim \mathrm{e}^{-p_{\perp}/T_0}$, since T_0 does not depend on the amount of energy transferred to the transverse degree of freedom. We thus come to the following conclusion:

Conclusion. Whenever in a collision – no matter whether central or not – total energy and momentum transfer are both much larger than m_{π}, we expect the transverse momentum distribution to be independent of the total energy, of the momentum transfer, and of the particle number, and to be approximately of the form $w(p_{\perp}) \sim \exp(p_{\perp}/T_0)$. The longitudinal momentum distribution will be roughly of the form $w(p_{\|}) \sim \exp(p_{\|}/\alpha\gamma T_0)$, $\alpha \approx 1$, except for very central collisions, where it becomes similar to the transverse distribution.

Having succeeded or failed (as the reader may decide) to justify the independent treatment of longitudinal and transverse motion, we go back to Eq. (19.41) and draw a few further conclusions.

---

[8] There is of course a continuous range of intermediate situations between 'central' and 'peripheral'.

First of all, the distributions of particles with $p_\parallel$ and $p_\perp$ become

$$
\begin{aligned}
\overline{n}_\parallel(p_\parallel) &= \frac{\overline{N}}{T_\parallel^0}\mathrm{e}^{-p_\parallel/T_\parallel^0}\,, \qquad T_\parallel^0 \approx 3\gamma T_0\,, \\
\overline{n}_\perp(p_\perp) &= \frac{\overline{N}}{{T_\perp^0}^2}p_\perp\mathrm{e}^{-p_\perp/T_\perp^0}\,, \qquad T_\perp^0 \approx T_0 \approx m_\pi\,,
\end{aligned}
\tag{19.43}
$$

which is just what we said in the above conclusion. Of course, we cannot expect these two independent distributions to be exact in cases of low multiplicity, where energy–momentum conservation imposes severe restrictions. In particular, in large angle elastic and exchange scattering, the transverse momentum distribution uniquely fixes the longitudinal one. It remains to explain why, when at most one of the two given distributions can hold, nature apparently chooses the transverse one (for evidence, see Fig. 19.2). In cases of large multiplicity, the two distributions describe a 'jet'. It is most interesting to learn from experiments [11, 15] that our formula actually fits[9] the transverse momentum distribution in jets up to at least $p_\perp \approx 1.2$ GeV/c and for primary energies between 25 GeV and 10^6 GeV with the one constant T_0 value of ≈ 160 MeV.

Although we decoupled the two directions of motion, we should require that the mean particle number be the same in both systems since we wish after all to describe actual events (in which of course the particles having longitudinal momentum components are just the same as those which have transverse ones). Since from Eq. (19.15) we have $\overline{N} = z/(1-z)$, it follows that

$$
\overline{N} = \frac{z_\parallel}{1-z_\parallel} = \frac{z_\perp}{1-z_\perp}\,, \qquad z_\parallel = z_\perp\,,
\tag{19.44}
$$

and consequently, the relation between the temperatures is

$$
T_\parallel = \gamma V_0^{1/3} T_\perp^2\,,
\tag{19.45}
$$

which (only) for $T \to T_0$ can be written

$$
T_\parallel^0 = \gamma\sqrt{2\pi}T_\perp^0\,.
\tag{19.46}
$$

Next let us consider energy conservation. We can write down the energies contained in the longitudinal and transverse motion, respectively (remembering that $m = 0$):

$$
\begin{aligned}
\overline{E}_\parallel &= \int p_\parallel \overline{n}_\parallel(p_\parallel)\mathrm{d}p_\parallel = \overline{N}T_\parallel^0 \approx 3\overline{N}\gamma m_\pi\,, \\
\overline{E}_\perp &= \int p_\perp \overline{n}_\perp(p_\perp)\mathrm{d}p_\perp = 2\overline{N}T_\perp^0 \approx 2\overline{N}m_\pi\,.
\end{aligned}
\tag{19.47}
$$

---

[9] An apparent slow increase in T_0 with primary energy can be understood as a kinematical effect (see Appendix B).

But is the total energy E the sum of these two? We might say so if the two belonged to two really independent systems – but it is just energy–momentum conservation which makes them not completely independent. We could think of defining the total energy by

$$\frac{1}{\overline{N}}\int\sqrt{p_\perp^2+p_\parallel^2}\,\overline{n}_\perp(p_\perp)\overline{n}_\parallel(p_\parallel)\mathrm{d}p_\perp\mathrm{d}p_\parallel\ , \tag{19.48}$$

where the correct energy for a particle with $p=(p_\perp,p_\parallel)$ is now summed up. But again, multiplying the two, supposedly independent, distributions does not yield the correct distribution $\overline{n}(p_\perp,p_\parallel)$ of the particles: it may only tend to the correct one for large multiplicities. We may leave the details and use the fact that, for large E (in pp collisions $\gamma=E/2$), Eq. (19.42) shows that almost all energy is contained in the longitudinal component, which contains γ times more than the transverse one. We thus identify $\overline{E}_\parallel$ with E for E very large ($\gamma\gg 1$ means $E\gg 1$, that is, much more than our usual condition $E\gg m_\pi$). We obtain

$$E\approx 3\overline{N}\gamma m_\pi\ ,$$

and therefore,

$$\overline{N}\approx\frac{E}{3\gamma m_\pi}\quad\left(=\frac{2}{3m_\pi}\approx 5\text{ for }pp\text{ collision}\right)\,. \tag{19.49}$$

This low and energy independent multiplicity does not of course concern the final one of pions, etc. In fact, $\overline{N}$ is the average number of more or less excited particles formed in the first instant (resonances and/or fireballs) which afterwards decay. The actual increase in multiplicities, which is experimentally observed, must be interpreted as an increase in the excitation of the fireballs. The energy independence of $\overline{N}$ and $T_\perp^0$ implies immediately [by Eq. (19.47)] that the average kinetic energy stored in the transverse motion is itself independent of the primary energy [although strongly fluctuating from event to event, see Eq. (19.19)].

Since the probability of finding just N particles is

$$W(N)=\frac{z^N}{\sum z^N}\ ,\qquad z=z_\parallel=z_\perp\ , \tag{19.50}$$

and since $z_\parallel\to 1$ for $T_\parallel\to T_\parallel^0$, it follows that here also all N values *become equally probable in the limit* (in such a way, however, that $\overline{N}\to\approx 5$). Although the average number of fireballs is ≈ 5, the actual number can therefore hardly be predicted. The introduction of masses will, of course, suppress very large N values. Nevertheless, even then we have to expect enormous fluctuations in the multiplicities of final particles produced in collisions at (fixed!) high energy.[10]

[10] This is actually found [11] at cosmic ray energies: the r.m.s. fluctuation in multiplicity is $\approx\overline{n}_\mathrm{s}$ (where $\overline{n}_\mathrm{s}$ is the charged multiplicity) (see Fig. 19.4). From the experimental angular distribution follow large fluctuations in the number of fireballs.

Although the present treatment of the problem of the angular distribution is certainly not yet fully correct, it may indicate the direction in which one has to go. Nevertheless, we find quite satisfying results:

- The transverse momentum distribution is independent of:
  - the total energy,
  - the number of particles,
  - the centrality of the collision,

  and always has the form $\sim \exp(-p_\perp/T_0)$, $T_0 \approx m_\pi$ (vanishing fluctuations). This holds whenever the total energy and momentum transfer are both $\gg m_\pi$. In that case, thermal equilibrium may or may not be attained. It is irrelevant.
- Almost all the energy is contained as *kinetic energy* in the longitudinal component (jet) and the (strongly fluctuating) transverse energy is on average independent of the primary energy.
- The multiplicity of the first generation of particles (fireballs) fluctuates strongly but is not large. The average value is of order 5 and independent of the primary energy. The strongly fluctuating multiplicity of the last generation (final pions, nucleons, hyperons) will on average increase very slowly, as most of the total energy is contained in the kinetic energy of the fireballs and only a little in their excitation.

These rather – for a statistical model – unusual features explain also in a most natural way *why* the conventional (Fermi) statistical model of particle production works so much better than one could (in view of the rareness of central collisions) reasonably expect: the mere existence of the 'highest temperature' T_0 guarantees, so to speak, an everlasting pre-established thermal equilibrium inside the incoming (localized) particles – the collision itself has only to break off the volumes (= conservation laws) in which the clouds were enclosed. Even in non-central collisions, this pre-established thermal distribution reveals itself, namely in the transverse momenta. Central collisions only help to transfer longitudinal energy into the transverse motion, without effect for the latter, except for an increase in multiplicity. Only in most central collisions can a thermal equilibrium be obtained in the usual sense, and then $T_\parallel^0 \approx T_\perp^0$.

The case of nonzero mass

We return to Eq. (19.41) and remark that with $\gamma = 1$, i.e., for a cubic box, we obtain $T_\parallel^0 = \pi^{1/3} 2T_0$ and $T_\perp^0 = \sqrt{2/\pi^{1/3}}\, T_0$, where T_0 is the value for the three-dimensional problem in the same volume. Since $\sqrt{2/\pi^{1/3}} = 1.17$, we see that $T_\perp^0 \approx T_0$. We also expect the two temperatures to be nearly equal in the case $m \neq 0$. We shall therefore discuss mainly $T_\perp^0$.

We first consider briefly $T_\parallel^0$ and then in more detail $T_\perp^0$, assuming now that all particles have the same mass m^* (to be thought of as the average mass):

1. For the longitudinal component, we have

$$z(m^*,T) = \sum_\alpha \mathrm{e}^{-\varepsilon_\alpha/T} \;\longrightarrow\; \int g(p)\mathrm{e}^{-\sqrt{p^2+m^{*2}}/T}\mathrm{d}p\ .$$

Then $g_\parallel(p_\parallel)$ taken from Eq. (19.40) gives

$$\begin{aligned} z_\parallel(m^*,T) &= \frac{V_0^{1/3}}{2\pi\gamma}\int_0^\infty \mathrm{e}^{-\sqrt{p^2+m^{*2}}/T}\mathrm{d}p \\ &= \frac{V_0^{1/3}m^*}{2\pi\gamma}\int_0^\infty \mathrm{e}^{-m^*\sqrt{1+x^2}/T}\mathrm{d}x \qquad (\text{put } x=\sinh y) \qquad (19.51)\\ &= \frac{V_0^{1/3}m^*}{2\pi\gamma}\left[-\frac{\mathrm{d}K_0(\tau)}{\mathrm{d}\tau}\right]_{\tau=m^*/T}\ . \end{aligned}$$

The condition that $Z_\parallel$ diverges is that $z_\parallel = \uparrow$. Hence

$$\frac{2\pi}{m^*V_0^{1/2}}\gamma = -\left.\frac{\mathrm{d}K_0(\tau)}{\mathrm{d}\tau}\right|_{\tau=m^*/T_\parallel^0} = K_1(m^*/T_\parallel^0)\ . \qquad (19.52)$$

Now the Bessel function $K_1(x)$ is a smooth, steadily decreasing function of x, with the asymptotic behaviour

$$K_1(x) \;\longrightarrow\; \begin{cases} \sqrt{\dfrac{2}{\pi x}}\mathrm{e}^{-x}\ , & x\to\infty\ ,\\ \dfrac{1}{x}\ , & x\to 0\ . \end{cases} \qquad (19.53)$$

Since the left-hand side of Eq. (19.52) goes to ∞ when $E\to\infty$, we conclude that $x\to 0$. Then we use the second line of Eq. (19.53) to obtain, for $E\to\infty$,

$$\frac{2\pi}{m^*V_0^{1/3}}\gamma = \frac{T_\parallel^0}{m^*}\ , \qquad T_\parallel^0 = 2\gamma\pi^{1/3}T_0\ . \qquad (19.54)$$

This agrees with the value found for $m^*=0$, as expected. In the longitudinal component, the mass of a fireball is practically always negligible compared to its momentum.

2. The transverse component is treated similarly. We arrive at

$$
\begin{aligned}
z_\perp(m^*,T) &= \frac{V_0^{2/3}}{2\pi}\int_0^\infty p\mathrm{e}^{-\sqrt{p^2+m^{*2}}/T}\,\mathrm{d}p \\
&= \frac{V_0^{2/3}m^{*2}}{2\pi}\int_0^\infty x\mathrm{e}^{-m^*\sqrt{1+x^2}/T}\,\mathrm{d}x \qquad (19.55)\\
&= \frac{V_0^{2/3}T^2}{2\pi}\left(1+\frac{m^*}{T}\right)\mathrm{e}^{-m^*/T}\ ,
\end{aligned}
$$

and the highest temperature $T_\perp^0$ is defined implicitly by

$$
1 = \frac{V_0^{2/3}T_\perp^{0\ell}}{2\pi}\left(1+\frac{m^*}{T_\perp^0}\right)\mathrm{e}^{-m^*/T_\perp^0}\ . \qquad (19.56)
$$

The numerical value of $T_\perp^0$ is not very relevant, because we are largely uncertain about the value of m^* [= average of masses produced at the given energy = very slowly varying function of the energy (?)] to be put in. Indeed, we should not use a single value for m^* but rather a mass spectrum (see below). *What is relevant is that the $m^* \neq 0$ case does not change the basic fact that a highest temperature exists. Therefore, all our conclusions drawn so far remain at least qualitatively valid.*

Let us nevertheless make a little numerical analysis of Eq. (19.56). For the average mass of 'fireballs', we may expect a value of perhaps the nucleon mass. This is in accordance with our observation that most of all energy in jets must be contained in the kinetic energy of the longitudinal component and only a little in the excitation (= mass) of fireballs. We rewrite Eq. (19.56), putting $x_0 = m^*/T_\perp^0$ and $V_0 = (4\pi/3)(a/m_\pi)^3$:

$$
\frac{1}{2\pi}\left(\frac{4\pi}{3}\right)^{2/3}\left(\frac{m^*a}{m_\pi}\right)^2 = \frac{x_0^2}{1+x_0}\mathrm{e}^{x_0}\ . \qquad (19.57)
$$

In Fig. 19.3 we plot

$$
\frac{m^*a}{m_\pi} = \sqrt{2\pi}\left(\frac{3}{4\pi}\right)^{1/3}\frac{x_0\mathrm{e}^{x_0/2}}{\sqrt{1+x_0}} \qquad (19.58)
$$

as a function of x_0.

Put $m^*=1$ and $T_\perp^0 = 170$ MeV so that $x_0 \approx 6$. We read off

$$
\frac{m^*a}{m_\pi} = \frac{a}{m_\pi} \approx 60\ ,
$$

or $a \approx 9$. This is certainly disappointing. It would mean that the volume in which interaction still takes place would have linear dimensions of the order of 9 times the pion Compton wavelength.

It seems, however, that there is a way out, maybe even two ways. The value $T_\perp^0$ which we inserted is taken from the fit of $\mathrm{e}^{-p_\perp/T_\perp^0}$ to experiments. Taking the masses to be nonzero, one should not expect $p_\perp$ in the Boltzmann law, but rather $\sqrt{p_\perp^2+m^2}$. Let $p_\perp$ be of the order m_π and consider the pion transverse momentum ($m=m_\pi$ in the Boltzmann law). Then roughly $\sqrt{p_\perp^2+m_\pi^2}\approx\sqrt{2}p_\perp$. If one fitted the experiments with $\exp(-\sqrt{p_\perp^2+m_\pi^2}/T)$ instead of with $\mathrm{e}^{-p_\perp/T_\perp^0}$, one would expect a T value which is roughly $\sqrt{2}$ times larger than the old one. Thus, for consistency, we should put $T_\perp^0\approx 240$ MeV ≈ 0.25. Then $x_0\approx 4$ and $am^*/m_\pi\approx 20$, $a\approx 3$ for $m^*=1$. Actually, if m^* were somewhat smaller than the nucleon mass, say ≈ 0.75, we would arrive at $x_0\approx 3$ and $am^*/m_\pi=10$, $a\approx 2$. This is already quite reasonable.

The other way out of the difficulty may lie in introducing a mass spectrum of excited states.

A speculation on the mass spectrum of 'fireballs'

If $\rho(m^*,T)\mathrm{d}m^*$ is the mass spectrum, the true $z(T)$ would be given by [see Eq. (19.55)]

$$z_\perp(T)=\int_0^\infty\rho(m^*,T)z_\perp(m^*,T)\mathrm{d}m^*=\frac{V_0^{2/3}T^2}{2\pi}\int_0^\infty\rho(m^*,T)\left(1+\frac{m^*}{T}\right)\mathrm{e}^{-m^*/T}\mathrm{d}m^*\,. \tag{19.59}$$

Since under no circumstances can $z_\perp(T)$ become large than one, the integral must converge. This puts a limit on the asymptotic behaviour of $\rho(m^*,T)$:

> The mass spectrum of highly excited hadrons (fireballs) must grow less than $\mathrm{e}^{m^*/T}$, where T is of the order of m_π.

That it will indeed grow almost that fast is seen when we consider that $\rho(m^*,T)\mathrm{d}m^*$ is the total number of states between m^* and $m^*+\mathrm{d}m^*$ of a 'fireball'. But such a fireball itself is again described by our model – an unspecified number of distinguishable particles in a volume V_c with a total energy $E=m^*$. The density of states of such a system is roughly e^S, where $S(E,V)$ is the entropy. Since the temperature of the system (at sufficiently high energy) becomes $T_0=$ const., it follows that asymptotically [namely when $(\log m^*)/m^*\to 0$],

$$S(E,V_0)\;\longrightarrow\;\frac{E}{T_0}\,,\qquad \rho(m^*,T)\;\longrightarrow\;\mathrm{e}^{m^*/T_0}\,. \tag{19.60}$$

We may then put

$$\rho(m^*,T)\equiv\frac{1}{T}f(m^*/T)\mathrm{e}^{m^*/T}\,, \tag{19.61}$$

and obtain for $z_\perp=1$,

$$\frac{2\pi}{V_0^{2/3}}=T_\perp^{0\,2}\int_0^\infty f(x)(1+x)\mathrm{d}x\,. \tag{19.62}$$

As experimental evidence shows that $T_\perp^0$ is of the order of m_π, it is required that

$$\int_0^\infty f(m^*/T_0)\left(1+\frac{m^*}{T_0}\right)\mathrm{d}(m^*/T_0)\approx 1\,,\quad f(m^*/T_0)\equiv T_0\rho(m^*,T_0)\mathrm{e}^{-m^*/T_0}\,. \tag{19.63}$$

It is not clear whether Eq. (19.63), which puts a condition on the mass spectrum, is compatible with the fact that the mass spectrum should follow from the theory itself. It could be that this leads to an interesting self-consistency problem with further consequences (a kind of 'bootstrap' at high temperature).

In any case, we see that the introduction of a mass spectrum $\rho(m^*,T)$ may resolve the apparent difficulty in reconciling the numerical value of $T_\perp^0$, as found experimentally, with the requirement $m^*\neq 0$ (and not too small). We may presently at least hope that the value of the integral in Eq. (19.63) is near to one and consequently neither $T_\perp^0$ nor V_0 have to have unreasonable values.

Elastic and exchange scattering

Whatever the actual value of T_0 may turn out to be, we know that it exists. Let us then assume that it is indeed of the order of m_π. Taking the masses seriously and still treating the transverse and longitudinal motion as independent, we expect the transverse momentum distribution to have the form [see Eq. (19.43)]

$$w(p_\perp)\sim p_\perp\mathrm{e}^{-\sqrt{p_\perp^2+m^{*2}}/T_0}\,, \tag{19.64}$$

where strictly speaking m^* is the mass of that type of fireball whose distribution we wish to describe. Since actually mostly pions are observed, and these come from a chain of decays, the observed distribution will be somewhat different, and in fact broader (see Appendix B). But, experimentally [11, 14, 15], it definitely is of the form given in Eq. (19.64) with $T_0\approx 160$ MeV. Experimentally, the T_0 used to fit the distribution by $p_\perp\mathrm{e}^{-p_\perp/T_0}$, seems to increase (very slowly) with the primary energy [11]. This can be due to the slowly increasing mean excitation of the fireballs which then, on average, decay in a number of steps into the final particles. This number of steps will increase with the mass m^* of the fireball. Each of these steps will broaden the spectrum resulting from the preceding decay. If one tries nevertheless to fit with $p_\perp\mathrm{e}^{-p_\perp/T_0}$, then T_0 must obviously increase somewhat (see Appendix B).

The situation is different if we apply this formula to elastic scattering of nucleons, for instance. There m^* really means the nucleon mass and then the differential elastic cross-section should obey

$$\frac{\mathrm{d}\sigma_{\mathrm{el}}}{\mathrm{d}w}\sim\mathrm{e}^{-\sqrt{p_\perp^2+m^2}/T_0}\,. \tag{19.65}$$

There will be kinematical factors in front of this expression, which are not too easily worked out, because conceptually $w(p_\perp)$ and $\mathrm{d}\sigma/\mathrm{d}w$ are somewhat different:

$\mathrm{d}\sigma/\mathrm{d}w$ applies to a system with fixed total energy, whereas $w(p_\perp)$ applies to a system where the energy of a single particle is not fixed. Indeed, $w(p_\perp)\mathrm{d}p_\perp$ shows how many there might be in the interval given by $\mathrm{d}p_\perp$. If we applied this $w(p_\perp)$ literally to elastic scattering, we would then have

$$\mathrm{d}p_\perp = \mathrm{d}(p\sin\theta) = \sin\theta\,\mathrm{d}p + p\cos\theta\,\mathrm{d}\theta = p\cos\theta\,\mathrm{d}\theta\ , \tag{19.66}$$

because $\mathrm{d}p = 0$ for fixed total energy. We see then that, for geometrical reasons, $w(p_\perp)\mathrm{d}p_\perp = 0$ for $\theta = \pi/2$ and hence that at 90°,

$$\frac{\mathrm{d}\sigma_{\mathrm{el}}}{\mathrm{d}w} 2\pi\sin\theta\,\mathrm{d}\theta \neq w(p_\perp)\mathrm{d}p_\perp\ , \tag{19.67}$$

since the left-hand side is nonzero. However, for smaller angles such a formula looks most natural. We may then tentatively simply replace $\mathrm{d}p_\perp$ by $p\mathrm{d}\theta$, which somehow compensates for our disregarding the strong correlations between longitudinal and transverse distributions in a two-body case. Then, using

$$w(p_\perp) = \text{const.} \times p_\perp \mathrm{e}^{-\sqrt{p_\perp^2+m^2}/T_0}\ ,$$

the result is[11]

$$\frac{\mathrm{d}\sigma_{\mathrm{el}}}{\mathrm{d}w} = \text{const.} \times p^2 \mathrm{e}^{-\sqrt{p_\perp^2+m^2}/T_0} \qquad (\text{angles not near } \pi/2)\ , \tag{19.68}$$

whereas for angles near 90°, some other factor should replace p^2.

This other factor may involve an extra condition: centrality. The point is this: the distribution $w(p_\perp)$ is, as we saw, independent of the extent to which thermal equilibrium (between longitudinal and transverse motion) is reached. This holds certainly only for systems of many particles where a large value of $p_\perp$ does *not* then require the collision to have been central. In the two-body case a large value $p_\perp \to p$ implies a central collision and consequently we expect this further condition to modify $w(p_\perp)$. As shown in [4], the centrality condition can be taken into account by multiplying the relevant total inelastic cross-section by $1/\gamma^2 = 4/E^2$. Then, for larger angles, we should expect

$$\frac{\mathrm{d}\sigma_{\mathrm{el}}}{\mathrm{d}w} \approx \text{const.} \times \mathrm{e}^{-\sqrt{p_\perp^2+m^2}/T_0} \underset{\text{near } 90^\circ}{\longrightarrow} \text{const.} \times \mathrm{e}^{-p_\perp/T_0}\ . \tag{19.69}$$

Perhaps the 'constant' still varies with E. In fact, the best fit is obtained by putting it equal to $1/E^2$ (see Fig. 19.2).

The most interesting property of our new formula is that, for $p_\perp \to 0$ (forward direction), it gives

---

[11] Note that $\mathrm{d}\sigma/\mathrm{d}w$ is a function of the two *independent* variables E and θ. Our formulas (19.68) and (19.69) claim to describe the differential elastic cross-section as a function of both of these variables.

$$\frac{d\sigma_{\text{el}}}{dw} \approx \text{const.} \times p^2 \exp\left(-\frac{1}{T_0}\frac{p_\perp^2}{2m}\right) \underset{\text{near } 0^\circ}{\longrightarrow} \text{const.} \times p^2 , \tag{19.70}$$

which is required by the optical theorem together with the empirical facts that the scattering amplitude tends to become purely imaginary and $\sigma_{\text{tot}} \to \text{const}$. Even the functional form coincides with that of the observed diffraction peak – for small angles, $p^2 \to -t$ (invariant square of the momentum transfer) and Eq. (19.70) reads

$$\left.\frac{d\sigma}{dt}\right|_{t\to 0} = \text{const.} \times e^{t/2mT_0} . \tag{19.71}$$

Experimentally, it is of the form e^{+bt} with $b \approx 10\ \text{GeV}^{-2}$. However, if T_0 were of the order of $m_\pi \approx m/6.8$, this would give $b \approx 3.4/m^2 = 3.9\ \text{GeV}^{-2}$. This value is much too small. Conversely, in our present model, it would mean that, as $p_\perp \to 0$, a temperature of

$$T_0^* \approx 0.4 T_0 \tag{19.72}$$

would be needed to fit the data. Now our model by no means excludes a temperature lower than T_0 and indeed we cannot even justly require that T_0 should be reached when the momentum transfer is extremely small. [Remember the conclusion on p. 200.] This may be interpreted as saying that the temperature of that cloud of virtual particles, which constitutes what we call a localized nucleon, is here about one half of T_0, because at this small momentum transfer, it was not so well localized. The temperature reaches T_0 only if a sufficient amount of energy is transferred to the transverse motion. (It would be interesting to see what our model has to say about peripheral collisions $p + p \to N^* + N \to N + N + \pi$, etc.)

Whatever the reason, as a matter of fact [16],

$$\frac{d\sigma_{\text{el}}}{dw} \sim e^{-\sqrt{p_\perp^2 + m^2}/T_0}$$

seems to fit the experimental data with $T_0 \approx 50$ MeV in the diffraction region and with $T_0 \approx 150$ MeV outside the diffraction region. The temperature required for the fit changes rather rapidly from one value to the other. If one puts

$$\Delta E_\perp = \sqrt{p_\perp^2 + m^2} - m ,$$

then $\Delta E_\perp$ is the energy transferred to the transverse motion. It then happens that, in the diffraction region, $\Delta E_\perp < 0.3$, and in the region where $T_0 \approx 150$ MeV gives a good fit, $\Delta E_\perp > 0.7$. Tentatively using Fig. 19.1 with $\overline{E} \equiv \Delta E_\perp$, one sees that, in the diffraction region, $T < 0.75 T_0$, and in the other region, $T > 0.9 T_0$ would result. This is not of course to be taken too seriously, because in Fig. 19.1 it was supposed that the masses could be neglected. Anyway, the tendency is right.

It is probably not just to demand from our model that it should give even the numerical behaviour in the diffraction region correctly. Indeed, in that region the neglected strong geometrical correlation between the longitudinal and transverse momenta should again become as important as near 90°.

The present remarks are largely just guesswork. It remains to clear up the relation of our model to elastic scattering and peripheral collisions.

A logical difficulty of the model

We have employed statistical thermodynamics of distinguishable particles. This is strictly speaking inconsistent, since nature certainly does not work this way. Indeed, even if we are right in saying that *most* contributions come from states in which all particles (resonances, fireballs) *are* different, there are certainly states in which, for instance, five π^+ are already present in the first generation.

To be really consistent, we should have worked out a statistics of, say, M different species of particles. Particles of the same species must then be considered to be indistinguishable (and a statistics, Bose or Fermi, to be prescribed) and the number N_i of particles of each kind, as well as the total number $N = \sum N_i$ of particles, has to be left open as before. Finally, one lets $M \to \infty$. One sees immediately that the number of particles of each single kind would then tend to zero and we should come back to our model.

In the case of zero masses one finds, however, that this does not work. One simply obtains $\left[Z(V,T)\right]^M$, where $Z(V,T)$ is the usual partition function of *indistinguishable* particles, and in fact that of a massless Fermi or Bose gas, as the case may be. With $M \to \infty$, everything diverges at *any* temperature. That is easily understood if one realizes that, for $M \to \infty$, there is an infinity of states of our gas even if the total number N of particles in it is kept fixed: each single particle may be removed from the gas and be replaced by one of another species. From that operation, a new state with the same energy results, whence the sum over states is infinite.

The situation becomes different if the particles have a mass. Then replacing a particle by one of another species means changing the energy of the system, and one cannot generate an infinity of different states of the same energy. Again, with the number M of kinds of particles going to infinity, one would find that the number of particles of a given kind tends to zero. It is hoped that such a statistics will become equivalent to our present model. This, however, has not yet been worked out. It seems to be an urgent problem in view of the success of the present model. Work is in progress.

19.5 Summary and Conclusions

Our model uses only three basic experimental facts:

1. The strong interactions are strong enough to produce many resonances and even fireballs. We *assume* that the latter are only an 'extrapolation' of the resonances to very high energies.
2. The strong interactions have a range of the order of the Compton wavelength of the pion.
3. In high-energy collisions, the duration of contact is in general so short that a thermodynamical equilibrium (in the sense of Fermi's statistical theory) cannot be reached.

From (1) it follows that particles are to be considered as (quasi) distinguishable, while (2) determines the volume in which the system is enclosed, and (3) allows one to treat the longitudinal and transverse motion as (nearly) independent. All the rest is straightforward and simple statistical thermodynamics with the following results:

- A universal highest temperature $T_0 \approx m_\pi$ (corresponding to $\approx 10^{12}$K) governs all high-energy processes involving strong interactions (and *only* these; no highest temperature exists for gravitational, weak, and electromagnetic interactions since they do not produce the many resonances which make the particles distinguishable).
- The transverse momentum distribution in high-energy collisions of hadrons is a Boltzmann distribution with constant temperature $T_0 \approx m_\pi$ independent
  - of the primary energy ($1 \leq E_{\text{lab}} \leq 10^6$ GeV),
  - of the number of particles involved – for two particles it gives elastic scattering, for many particles the jets,
  - of the centrality (= degree of thermal equilibrium) of the collision.
- Almost all energy is contained in the longitudinal component as kinetic energy. Only a small fraction is used for the excitation of fireballs. The total transverse energy fluctuates strongly but is on average practically independent of the primary energy.
- The multiplicities of particles produced fluctuate strongly (dispersion of order 1) about a slowly increasing mean value, whereas the fluctuations about the Boltzmann law for transverse momenta tend to zero for high primary energies.
- An apparent increase with primary energy of the temperature T_0 needed to fit the experimental distributions of transverse momenta between 1 and 10^6 GeV is qualitatively explained to be a kinematical effect due to a chain of decays leading from the fireballs of the 'first generation' to the observed pions and others.
- Taking the masses into account, the Boltzmann law becomes $\mathrm{e}^{-\sqrt{p_\perp^2+m^2}/T_0}$. For elastic scattering at high energies and outside the diffraction region, this fits the experimental differential cross-section well. If one stresses the formula, one gets into the diffraction region, where the form $\mathrm{e}^{-p_\perp^2/2mT_0}$ is correct, but where $T_0 \approx m_\pi$ should be replaced by $\approx 0.4T_0$. If one accepts that, the fit of the diffraction peak is also good. This lower temperature is not inconsistent with the present theory.

These are the results of our paper.

Let us now add some speculations (= wishful thinking). It is possible that the most interesting consequences of our model lie not in explaining so simply the well known but so far completely obscure characteristics of high-energy interactions above 1 GeV – it is possible that the most interesting consequences are to be found in astrophysics and in elementary particle physics. For astrophysics, it is rather obvious: whenever under the influence of gravitational pressure, strong interactions and kinetic energies per particle of the order of m_π come into play in the centre of a star, the appropriate statistics for a thermodynamical treatment is not Fermi statistics, but the statistics of distinguishable particles. The picture of a 'neutron star' would be inadequate. No work on this question has been done, however. (I thank Dr G. Cocconi for drawing my attention to this point.)

For elementary particle physics, the following possibility arises. We have seen that the Boltzmann distribution $\mathrm{e}^{-\sqrt{p_\perp^2+m^2}/T}$ fits even the diffraction peak if we have $T \approx 0.4m_\pi$. Since this holds down to $p_\perp \to 0$, where no energy is transferred to the transverse motion and where the collision constitutes a very bad 'position measurement', we feel tempted to conjecture that the incoming particles might already have an 'a priori temperature' (of the order of $0.4m_\pi$ for protons) $T \lesssim T_0$ and only a sufficient energy transfer would raise the temperature (for the transverse motion in the c.m. of the collisions) to T_0. Since the differential elastic cross-section and the total cross-section are related by the optical theorem and since all total cross-sections of hadrons are of the same order at high energies, we would conclude that they all have an 'a priori temperature' of the same order. Then the pion, kaon, nucleon, and hyperons would be thermodynamical systems with a temperature $\approx m_\pi/2$. The immediate question is why, if they are hot, they do not radiate. Why are they all stable against strong decay and the proton even against electromagnetic and weak decay? There are the conservation laws, of course, but they merely state the known facts. They do not explain them. That the proton is stable is not the result of any theory. It has been built in as a postulate in all theories from the beginning. Considering the family of all states with nucleon number 1 and charge 1, the proton is the ground state of that family not because the results of field theory teach us so, but because we put it in by defining the operators of asymptotic fields that way. Therefore, whether the nucleon is hot or cold, we do not understand why it is stable. Let us therefore ignore this difficulty, which is not typical to the picture which we wish to try.

We would then have a sequence of different states π, K, pion resonances, K resonances, N, nucleon resonances, hyperons and their resonances, and so on, leading with ever increasing mass into a continuum called 'fireballs'. All of these systems are more or less well described by thermodynamics of distinguishable particles with temperatures between roughly $0.5m_\pi$ and m_π. For the lowest temperatures, we obtain the stable particles ('strongly' stable) and some new unknown principle is necessary to explain why these states are stable. The missing new principle is analogous to Bohr's quantum condition, which 'explained' why the electrodynamical system called the hydrogen atom had a stable ground state in which the rapidly circulating electron did not radiate – in contradiction to all the then known laws. What we need would be a 'quantum condition' which 'explains' similarly why the thermodynamical system called the proton does not – in contradiction to all the now known laws of the present quantum mechanics – radiated off the rapidly moving particles of which it consists. Maybe the statement that the proton is stable is already the proper form of that postulate, but maybe it could be stated in a more illuminating form as deep as Bohr's $\oint p_\varphi \mathrm{d}\varphi = nh$. In any case we cannot be content with that postulate. Bohr's quantum condition for the stability of the proton may later be explained by some generalization of quantum mechanics. It is clear that this condition should give us the mass spectrum of the hadrons, whether the condition itself can be derived from present quantum theory (which is not likely) or from a future generalization of it. It is rather puzzling that not only the spectrum of masses is known to us, but even what corresponds to Wigner's classification of states by group theory, namely, the symmetry schemes SU_3, etc., of strong interactions – and in this respect we are far beyond Bohr's $\oint p_\varphi \mathrm{d}\varphi = nh$ – but the quantum condition proper is still unknown.

The proton, if our picture should turn out to be true, would then seem to be a straightforward extrapolation of ideas which have been familiar for quite some time. In the early days of renormalization theory, Welton [19] proposed a very intuitive picture of how the vacuum fluctuations shuttle the electron and lead to the observable Lamb shift. It was there considered to be essentially the result of statistical vacuum fluctuations. The interaction, however, is not strong and the γ quanta have no mass – the same result could be calculated more exactly with the first few orders of perturbation theory.

In the case of strong interactions, ordinary perturbation theory would not suffice. The large manifold of resonances and fireballs and the short range cause the proton (and all other hadrons) to behave like a thermodynamical system of a rather high (almost the maximal possible) temperature. The 'bootstrap' mechanism would then correspond to a new type of perturbation treatment, in which only the few lowest masses of the unlimited number of interaction 'fireballs' constituting the hadrons are taken into account. It would be a 'first approximation to thermodynamics'.

It is, of course, possible that the picture drawn here is wrong and that the circumstance that our model works even in the diffraction region is purely accidental. But if it is not wrong, then it would follow that we have basically all the information we can hope for in our hands: the mass spectrum, the selection rules (SU_3, etc.), the decay modes of the lower unstable states (ρ, ω, etc.), and that going to higher and higher collisions energies would be comparable to attempting to learn something about the structure of the hydrogen atom by studying the properties of a highly ionized H gas at higher and higher temperatures.

This latter remark has a good chance to hold even if the above picture of 'a priori hot' hadrons is wrong. The good agreement of our results with experiments indicates that, at least in all collisions with a momentum transfer above $\approx m_\pi$, the colliding particles are heated up to the maximal temperature $T_0 \approx m_\pi$. We have therefore little chance of learning much more about the *structure* of hadrons and about the *details* of their interaction than we could learn about the structure and interaction of atoms from high-temperature thermodynamics (ideal gases). On the contrary, when we have learnt such things from thermodynamics, it was at low temperatures (condensation, frozen degrees of freedom, superconductivity). If we draw a parallel, we would would think that our present laboratory energies below 100 GeV would be the most interesting ones for strong interactions. (We stress once more that not a single one of our conclusions applies to weak and electromagnetic interactions.) One extremely interesting question, however, remains to be answered by high-energy experiments, namely, whether basic triplets for SU_3 exist. None of our arguments excludes their being found, e.g., in a *pp* collision with several hundred GeV in the centre of mass.

Acknowledgements

The author is grateful for discussions with A. Bialas, G. Cocconi, B. Escoubès, J. Orear, L. Van Hove and several members of the CERN Theory Division. Mr W. Klein did the numerical calculations for Figs. 19.1 and 19.3.

Appendix A

In the main text [Eqs. (19.18) and (19.19)], it was shown that the relative fluctuations in the number of 'fireballs' tend to one:

$$\frac{\overline{N^2} - \overline{N}^2}{\overline{N}^2} \longrightarrow 1 \quad \text{for } T \to T_0 \,. \tag{19.73}$$

If one defines

$$w_\alpha = \frac{n_\alpha}{N} \,, \tag{19.74}$$

then $\overline{w}_\alpha$ is the probability that a given particle has energy ε_α. We wish to show here that

$$\frac{\overline{w_\alpha^2} - \overline{w}_\alpha{}^2}{\overline{w}_\alpha{}^2} \longrightarrow 1 \quad \text{for } T \to T_0 \,. \tag{19.75}$$

First we note without proof (it is simple) that, if we considered the fluctuations in $\overline{n}_\alpha$, they would tend to one. This is not due to the fluctuations with respect to the Boltzmann law, but to the large fluctuations in their normalizing factor N. Therefore, we have to consider the fluctuations in w_α. It is these latter fluctuations which indicate how much we should expect experimental points to scatter about the Boltzmann distribution of transverse momenta.

We write [see Eq. (19.9)]

$$z = \sum x_\alpha \,, \qquad Z = \sum_N \left\{ \sum_{\sum n = N} \frac{N!}{n_1! \ldots} \prod_\alpha x_\alpha^{n_\alpha} \right\} \equiv \sum_N z^N \,. \tag{19.76}$$

In order to get n_β in front of $\prod_\alpha x^{n_\alpha}$, we should multiply by $x_\beta \partial/\partial x_\beta$. This operator can be written outside Z. Thus $\overline{(n\beta/N)}$ is given by

$$\overline{\left(\frac{n_\beta}{N}\right)} = \frac{1}{Z} x_\beta \frac{\partial}{\partial x_\beta} \sum_n \frac{z^N}{N} = \frac{x_\beta}{z} \,, \tag{19.77}$$

where $\partial/\partial x_\beta = \partial/\partial z$ has been used. Similarly,

$$\overline{\left(\frac{n_\beta}{N}\right)^2} = \frac{1}{Z}\left(x_\beta \frac{\partial}{\partial x_\beta}\right)\left(x_\beta \frac{\partial}{\partial x_\beta}\right)\sum_n \frac{z^N}{N^2} = \frac{1}{Z} x_\beta \left(\sum_N \frac{z^{N-1}}{N} + x_\beta \sum_N \frac{N-1}{N} z^{N-2}\right) . \tag{19.78}$$

Here the first term in the round bracket is $(\log Z)/z$ and the second tends to $x_\beta Z/z^2$. For $T \to T_0$, we can neglect $(\log Z)/Z$ to obtain

$$\overline{\left(\frac{n_\beta}{N}\right)^2} \longrightarrow \left(\frac{x_\beta}{z}\right)^2 . \tag{19.79}$$

This, together with Eq. (19.77), proves Eq. (19.75).

A glance at Figs. 19.3 and 19.4 shows that our result agrees with experiment: large fluctuations about the mean multiplicities and small ones about the Boltzmann distribution of transverse momenta.

Appendix B

We wish to consider here the broadening of the spectrum due to the decay of a fireball. The problem is straightforward as far as kinematics is concerned. Its general treatment offers, however, great computational difficulties. We therefore treat a simple one-dimensional model case.

We suppose a fireball with mass $m^* \gg T_0$ moving along the x axis with a (positive or negative) 'four-velocity'[12] $V = (\gamma, \beta\gamma)$. This fireball can emit another particle with mass m_ℓ, again only in the $\pm x$ direction, with a 'four-momentum' which is described by $P_\ell = (\varepsilon_\ell, p_\ell)$ in the fireball's rest frame F^*, by $P = (\varepsilon, p)$ in the c.m. frame.

We suppose that the momentum distributions of m^* in the c.m. frame and the momentum distribution of m_ℓ in F^* are of our standard form for one-dimensional motion [masses fully taken into account, see Eq. (19.51)]:

$$\begin{aligned} w(p^*)\mathrm{d}p^* &= \mathrm{e}^{-\varepsilon^*/T_0}\mathrm{d}p^* \qquad &&(\text{in c.m. frame}) \,, \\ w(p_\ell)\mathrm{d}p_\ell &= \mathrm{e}^{-\varepsilon_\ell/T_0}\mathrm{d}p_\ell \qquad &&(\text{in } F^* \text{ frame}) \,. \end{aligned} \tag{19.80}$$

The four-velocity of the mass m^* is given by

[12] Strictly speaking, we should say 'two-velocity'.

$$V = (u_0, u) = (\gamma, \beta\gamma) = (\varepsilon^*/m^*, p^*/m^*) , \qquad u_0 = \sqrt{u^2+1} \, . \tag{19.81}$$

To the momentum distribution of m^* corresponds a velocity distribution

$$\overline{v}(u) = w(p^*)\frac{\mathrm{d}p^*}{\mathrm{d}m} = m^* \mathrm{e}^{-m^*\sqrt{u^2+1}/T_0} \, .$$

We normalize this distribution to $\int_0^\infty v(u)\mathrm{d}u = 1$ and obtain

$$v(u) = \frac{1}{K_1(m^*/T_0)} \mathrm{e}^{-m^*\sqrt{u^2+1}/T_0} \, , \tag{19.82}$$

where K_1 is as before a Bessel (modified Hankel) function. For $m^* \gg T$, we have

$$K_1(m^*/T_0) \;\longrightarrow\; \sqrt{\frac{2T_0}{\pi m^*}} \mathrm{e}^{-m^*/T_0} \, .$$

Hence $v(u)$ behaves for small u like $\mathrm{e}^{-m^* u^2/2T_0}$ and for large u like $\mathrm{e}^{-m^* u/T_0}$. Since $m^* \gg T_0$ was supposed, $v(u)$ drops very fast. Its maximum at $u = 0$ increases roughly as $(m^*/T)^{1/2}$.

Now let u be fixed for a moment. Then the number of m_ℓ particles with positive momentum $\{p, \mathrm{d}p\}$ in the c.m. frame is given by

$$W_u(p)\mathrm{d}p = w(p_\ell)\mathrm{d}p_\ell \, , \tag{19.83}$$

where p_ℓ has to be chosen accordingly:

$$p_\ell = p\sqrt{u^2+1} - \varepsilon u \, , \qquad \varepsilon_\ell = \varepsilon\sqrt{u^2+1} - pu \, . \tag{19.84}$$

Hence, with $\mathrm{d}p/\mathrm{d}p_\ell = \varepsilon_\ell/\varepsilon$,

$$\varepsilon W_u(p) = \varepsilon_\ell(u)\mathrm{e}^{-\varepsilon_\ell(u)/T_0} \, . \tag{19.85}$$

This has now to be multiplied by $v(u)\mathrm{d}u$ and integrated from $-\infty$ to ∞ (the negative u values take care of the cases in which the fireball moves in the $-x$ direction and emits a 'meson' in the $+x$ direction with sufficient energy to overcompensate the velocity in the $-x$ direction and to obtain a positive direction of flight). Disregarding the normalization factors, we obtain

$$\varepsilon W(p) = \int_{-\infty}^{\infty} \mathrm{d}u \left[\mathrm{e}^{-m^*\sqrt{u^2+1}/T_0} \mathrm{e}^{-(\varepsilon\sqrt{u^2+1}-up)/T_0} \left(\varepsilon\sqrt{u^2+1} - up\right) \right] \, . \tag{19.86}$$

We know that the first exponential drops first like a Gaussian, later simply exponentially. The exponent in the second exponential is $\varepsilon_\ell(u)$ with the following behaviour:[13]

[13] $\varepsilon_\ell(u)$ has the maximum possible value $m^*/2$, namely, when $m^* \to m_\ell + m_\ell$. In general, ε_ℓ is much smaller. In the integral, we forget about this fact, since large u values contribute nothing.

$$\varepsilon_\ell(u) \longrightarrow \begin{cases} |u|(\varepsilon+p) & \text{for } u \to -\infty\,, \\ \varepsilon - up & \text{for } u \approx 0\,, \\ m_\ell \ \text{(minimal)} & \text{for } u = p/m_\ell\,, \\ u(\varepsilon - p) & \text{for } u \to +\infty\,. \end{cases}$$

Therefore $\exp[-\varepsilon_\ell(u)/T]$ has a maximum at $u = p/m_\ell$ and drops exponentially on both sides. We shall assume that m^* is large enough to ensure that the term $\exp\left(-m^*\sqrt{u^2+1}/T\right)$ can be considered to vary more rapidly than $\exp[-\varepsilon_\ell(u)/T]$, whence the main contributions to the integral will come from $u \approx 0$. Then in the algebraic factor $\varepsilon_\ell(u)$ which multiplies the exponentials, we put $u = 0$. We expand the square roots in the exponentials and integrate these expressions since the resulting error for large $|u|$ is negligible. Thus omitting all constant factors,

$$\varepsilon W(p) \approx \varepsilon \mathrm{e}^{-\varepsilon/T_0} \int_{-\infty}^{\infty} \exp\left(-u^2 \frac{\varepsilon + m^*}{2T_0} + u\frac{p}{T_0}\right) \mathrm{d}u$$

and

$$W(p) \approx \sqrt{\frac{2\pi T_0}{\varepsilon + m^*}}\, \mathrm{e}^{-\varepsilon/T_0} \mathrm{e}^{p^2/2T_0(\varepsilon+m^*)}\,,$$

or again,

$$W(p) \approx \frac{\text{const.}}{\sqrt{\varepsilon + m^*}} \exp\left[-\frac{\varepsilon}{T_0}\frac{2(\varepsilon+m^*) - p^2/\varepsilon}{2(\varepsilon+m^*)}\right]\,.$$

We may then write

$$W(p) \approx \frac{\text{const.}}{\sqrt{\varepsilon+m^*}} \mathrm{e}^{-\varepsilon/T_{\text{eff}}}\,, \tag{19.87}$$

where

$$T_{\text{eff}}(\varepsilon) = T_0 \frac{2(\varepsilon+m^*)}{2(\varepsilon+m^*) - p^2/\varepsilon}\,, \qquad p^2 = \varepsilon^2 - m_\ell^2\,. \tag{19.88}$$

When ε varies between m_ℓ and ∞, $T_{\text{eff}}(\varepsilon)$ varies between T_0 and $2T_0$, the latter value is approached only for ε values $\gg m^*$.

In cosmic ray jets, transverse momenta of pions up to the order $p = \varepsilon = 1$ to 1.2 have been reliably measured [11]. Assume then $p = \varepsilon = 1$ and $m^* = 1$. Then $T_{\text{eff}} = 4T_0/3$ and with $m^* = 2$, we find $T_{\text{eff}} = 6T_0/5$. This is at the upper end of the measured spectrum. At the lower end, $T_{\text{eff}} \to T_0$.

The present analysis is very rough and incomplete. It illustrates only the mechanism. It is not inconsistent with the assumption that nothing serious would happen in reality (the two-dimensional case). If that turned out to be so and if the general case gave a similar result, then it would allow one to conclude from the transverse momentum distribution [namely, the deviations from a pure $\exp(-\varepsilon/T_0)$] something about the average or most frequent mass m^* of fireballs. The heavier the fireballs, the less the actual distribution will deviate from an exponential. On the other hand, the chain of decays will then contain more members and this may increase the deviations again. In any case it will at least cause a larger effective temperature. Actually, the T which is needed to fit the spectra seems to increase somewhat with the primary energy, although not more than by a factor of two, when the primary energy varies by a factor of one million.

In spite of this very crude analysis, we believe that it is sufficient to make it very likely that the apparent increase in the temperature is entirely due to kinematics and that our T_0 is indeed independent of the primary energy.

A more careful and more realistic (two-dimensional) discussion of this problem is highly desirable.

References

1. E. Fermi: Progr. Theor. Phys. **5**, 570 (1950)
2. G. Fast, R. Hagedorn: Nuovo Cimento **27**, 208 (1963)
3. G. Fast, R. Hagedorn, L.W. Jones: Nuovo Cimento **27**, 856 (1963)
4. R. Hagedorn: Nuovo Cimento: **35**, 216 (1965)
5. G. Auberson, B. Escoubés: Nuovo Cimento: **36**, 628 (1965)
6. A. Bialas, V.F. Weisskopf: Nuovo Cimento: **35**, 1211 (1965)
7. J. Vandermeulen: Bull. Soc. Roy. Sci. Liège (Belgium) **34**, 455 (1965)
8. H. Satz: The high energy limit of the statistical model. August 1964, Summer Institute for Theoretical Physics, University of Wisconsin, Madison, Wisc., USA; now at DESY, Hamburg, Germany
9. G. Cocconi: Nuovo Cimento **33**, 643 (1964). Cocconi says: "[. . .] corresponds to the case of a system in which, for *E* increasing, the number of possible kinds of particles increases so as to keep the energy per particle, and hence the temperature, constant." In this remark, our whole theory is implicitly contained.
10. T.T. Wu and C.N. Yang: Phys. Rev. **137**, B708 (1965)
11. P.H. Fowler, D.H. Perkins: Proc. Roy. Soc. A **278**, 401 (1964)
12. D. Amati, A. Stanghellini, S. Fubini: Nuovo Cimento **26**, 896 (1962)
13. K.A. Wilson: Acta Phys. Anstriaca **17**, 37 (1963)
14. J. Orear: Phys. Rev. Lett. **12**, 112 (1964); Phys. Lett. **13**, 190 (1965)
15. G. Cocconi, L.J. Koester, D.H. Perkins: Lawrence Radiation Laboratory Rep. UCID-1444 (1961)
16. J. Orear: private communication
17. See, e.g., M.L. Goldberger, K.M. Watson: *Collision Theory*, Wiley, New York (1964)
18. T.D. Newton, E.P. Wigner: Rev. Mod. Phys. **21**, 400 (1949)
19. T. Welton: Phys. Rev. **74**, 1157 (1948)

Chapter 20
On the Hadronic Mass Spectrum

Rolf Hagedorn

Abstract We argue that the sole requirement of a self-consistent bootstrap including all hadrons up to infinite mass leads to asymptotically exponential laws for the hadron mass spectrum, for momentum distributions, and for form factors (and to a highest temperature).

Over the last few years an increasing number of hadron mass formulas and, recently, of speculations about the whole hadronic mass spectrum have been published, all of them based on group theoretical considerations, quark models, or the like. We present here a different approach, a kind of asymptotic bootstrap, resulting from the 'thermodynamical model' and dealing only with the spectral density $\rho(m)$. The model has been described in three papers [1] entitled *Statistical Thermodynamics of Strong Interactions at High Energies I, II, and III*. The present consideration is a small but basic part of it.

In the thermodynamical model we describe highly excited hadronic matter by relativistic quantum statistical thermodynamics, allowing arbitrary absorption and creation of hadrons (and antihadrons) of all kinds, including all resonances. As the spectrum of resonances cannot be limited, we take into account all of them, even the not yet discovered ones. It goes as follows: we introduce one common name 'fireballs' for all hadrons and postulate (the feedback arrow is most important!)

Preprint CERN-TH-827 dated 12 September 1967

published in: Il Nuovo Cimento, Vol LII A, N. 4 of 21 December 1967

(deceased) CERN-TH 1211 Geneva 23, Switzerland

Postulate. A fireball is

$$\begin{array}{lcr} \rightarrow & \text{a statistical equilibrium of an undetermined number of all} & \\ \uparrow & \text{kinds of fireballs, each of which is in turn considered to be} & \rightarrow \\ \uparrow & & \downarrow \\ \leftarrow & \leftarrow \quad \leftarrow \quad \leftarrow \quad \leftarrow \quad \leftarrow \quad \leftarrow \quad \leftarrow \quad \leftarrow \quad \leftarrow \quad \leftarrow \quad \leftarrow & \leftarrow \end{array} \tag{20.1}$$

We forget about complications like collective motions (in non-central collisions) and imagine ideal equilibrium (realistic fireballs are discussed in Part II of [1]). One writes down the partition function $Z(V,T)$ for a gas consisting of an undetermined number of all kinds of particles (fireballs) which must be labeled, for instance, by their mass m. In calculating Z, one has to sum over all single-particle momentum states, over all possible numbers of particles (bosons $0,\ldots,\infty$, fermions 0,1), and over all possible kinds of particles (hadrons and anti-hadrons). The latter is done by introducing the number of hadron states between m and $m+\mathrm{d}m$, namely, $\rho(m)\mathrm{d}m$. With this (unknown) function $\rho(m)$, the partition function becomes (see Part I of [1])

$$Z = \exp\left[\int_0^\infty \rho(m)F(m,T)\mathrm{d}m\right] , \tag{20.2}$$

with a known function $F(m,T)$. On the other hand, Z can be written (see any book on statistical mechanics)

$$Z = \int_0^\infty \sigma(E)\mathrm{e}^{-E/T}\mathrm{d}E , \tag{20.3}$$

where $\sigma(E)$ is the number of states between E and $E+\mathrm{d}E$ of the fireball considered. As for this fireball $E=m$ (we stay in its rest frame), we can say as well that we have for our 'main' fireball $\sigma(m)\mathrm{d}m$ states in the mass interval $\{m,\mathrm{d}m\}$. Now $\rho(m)$ is the number of hadron states in the interval $\{m,\mathrm{d}m\}$ and if our postulate (20.1) above is applied, it follows that asymptotically $\rho(m)$ and $\sigma(m)$ must somehow become the same. A detailed discussion (see Part I of [1]) reveals that one cannot require more than that

$$\frac{\log\rho(m)}{\log\sigma(m)} \underset{m\to\infty}{\longrightarrow} 1 , \tag{20.4}$$

which says that, for $m\to\infty$, the entropy of a fireball is the same function of its mass as the entropy of the fireballs of which it is composed. This implies that *all fireballs are on an equal footing*.

We now equate the two expressions (20.2) and (20.3) and require simultaneously that Eq. (20.4) should be valid. It is shown in Part I of [1] that $F(m,T)$ falls off asymptotically as $m^{3/2}\exp(-m/T)$ and that therefore

$$Z \longrightarrow \exp\left[\int_0^\infty m^{3/2}\rho(m)\mathrm{e}^{-m/T}\mathrm{d}m\right] \longleftrightarrow \int_0^\infty \sigma(m)\mathrm{e}^{-m/T}\mathrm{d}m . \tag{20.5}$$

This is consistent with the bootstrap requirement (20.4) if and only if[1]

$$\rho(m) \underset{m\to\infty}{\longrightarrow} \frac{\text{const.}}{m^{5/2}} \mathrm{e}^{m/T_0} \,. \tag{20.6}$$

It follows that T_0 is *the highest possible temperature* – a kind of 'boiling point of hadronic matter' in whose vicinity particle creation becomes so vehement that the temperature cannot increase further, no matter how much energy is fed in.

An immediate consequence is a Boltzmann-type momentum distribution [asymptotically $\sim \exp(-p_\perp/T)$] with $T \lesssim T_0$, but never larger than T_0! This explains why the transverse momentum distribution in high energy jets is practically energy independent (for all details and possible deviations, see Part II of [1]).

Back to the mass spectrum: $\rho(m)$ counts each state (spin, etc.) separately and includes antiparticles. If one smooths out the experimental mass spectrum [2], one obtains Fig. 20.1, in which an exponential increase is seen in the region $\lesssim$ 1000 MeV, i.e., in that region where we know almost all resonances. Extrapolating the experimental curve with an expression having the required asymptotic behavior, Eq. (20.6) yields

$$T_0 = 160 \pm 10 \text{ MeV}\,, \tag{20.7}$$

and with this value, excellent fits (ranging over 10 orders of magnitude) to the momentum spectra and multiplicities in high energy production processes are obtained (see Part II of [1]). It is then only natural to expect [3] the form factors to decrease as $\sim \exp\left(-|t^2|^{1/2}/4T_0\right)$.

We treat hadrons as self-consistently infinitely composed of all other hadrons – this is what the postulate (20.1) says. If all hadrons are virtually contained in each of them, it is natural to assume that all phase relations between the infinitely many contributing amplitudes wash out and that therefore statistical thermodynamics is adequate to treat this asymptotic bootstrap. Although the technique is unconventional, it is not so far from the usual ones as one might think. An intimate relation between the mass spectrum and the momentum distribution in multiparticle production seems unavoidable in any theory, and the Gibbs ensemble description with fixed T somehow resembles off-shell effects because the masses of fireballs present at temperature T extend to infinity (with exponentially falling weight).

It will be impossible to prove or disprove our mass density in Eq. (20.6) by direct experiments, because the density increases exponentially and the production cross-section for each individual resonance decreases exponentially with m – the two mechanisms act in common against the experimenter.

Any 'proof' of Eq. (20.6) will be indirect, but the internal consistency of the thermodynamic model and the good agreement of its predictions with an enormous amount of experimental data (see Parts II and III of [1]) provides strong indirect support. In this respect, it is relevant that, in the applications of the model (see Parts

[1] It is not possible to have this $\rho(m)$ cut off somewhere because this would imply two types of essentially different fireballs: one with almost exponential density of states, the other with asymptotically vanishing density of states, and both would contribute and exist on an equal footing. This is inconsistent.

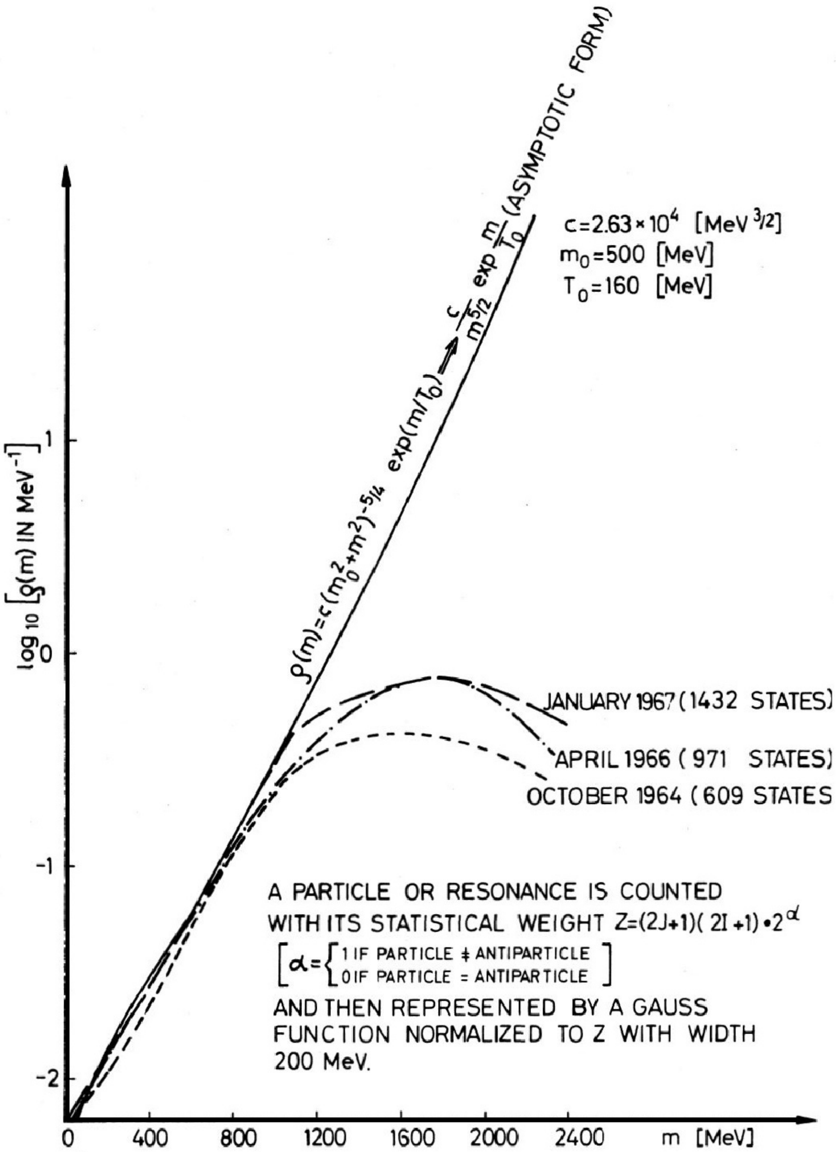


Fig. 20.1 The experimental mass spectrum smoothed by Gauss functions as indicated in figure, experimental spectra for three (1964,1966, 1967) sets of particle data, and a fit by a simple function with the asymptotic behavior required by Eq. (20.6). The normalization constant c is a fitted parameter, m_0 is an estimated value.

II and III of [1]), the asymptotic mass spectrum is used explicitly in integrals over m extending to infinity.

Recently, two papers [4] have been presented which use conventional quantum mechanical techniques to construct infinitely composed, self-consistent hadrons. A variety of different model assumptions were shown to lead to one common behavior: the form factors fall off asymptotically as $\exp(-\text{const.} \times |t|^{1/2})$ in complete analogy with our result on momentum spectra. It seems then that the sole requirement of self-consistent infinite compositeness is sufficient to produce these asymptotically exponential laws for mass spectra, momentum distributions, and form factors – at least this is strongly suggested by the fact that the thermodynamical model does not make any other assumption and that, in the papers by Stack and Harte, this assumption was the only one common to their various models.

In future, one should distinguish the 'vicinity of the boiling point of hadronic matter',[2] where $T \to T_0$ and $E \to \infty$, and where literally all hadrons merge into each other. It follows from the small value $T \approx 160$ MeV (1.86×10^{12}K) that $E \to \infty$ means in this respect E above 10 GeV (for quantitative relations, see Part II of [1]).

We conclude this letter with a curiosity – or perhaps not a curiosity. Consider a class of fireballs $f_n^{(i)}$ with roughly equal mass $m_n^{(i)}$, composed of quarks and antiquarks (n of them altogether, with n large). As the quark has 12 states [$\mathrm{SU}(3) \times \mathrm{SU}(2) \times$ antiparticle conjugation], this class of fireballs will have 12^n states ($i = 1, \ldots, 12^n$) if one assumes that each quark is in the ground state relative to all others (contrary to current models where, e.g., orbital momenta are discussed: here too they might be built in if one tries harder). Assume further that (as in nuclear physics) each of them contributes roughly the same and N independent amounts Δm to the average mass $\langle m \rangle_n$ of these fireballs. Then,

$$\langle m \rangle_n = \Delta m \times n \,. \tag{20.8}$$

For large n, the number of fireballs of mass $\approx \langle m \rangle_n$ becomes

$$z(m) = 12^n = \exp(n \log 12) = \exp\left(\langle m \rangle \frac{\log 12}{\Delta m}\right) \,. \tag{20.9}$$

The quantity Δm can be estimated by using the meson 35-plet and/or the baryon 56-plet, taking the average mass of each of these multiplets

$$\Delta m = \frac{\langle m_{35} \rangle}{2} = \frac{\langle m_{56} \rangle}{3} \,. \tag{20.10}$$

We find with $\langle m_{35} \rangle \approx 700$ MeV and $\langle m_{56} \rangle \approx 1050$ MeV,

$$z(m) = \exp \frac{\langle m \rangle}{140} \qquad (\langle m \rangle \text{ in MeV}) \,. \tag{20.11}$$

---

[2] For symmetries, etc., one had better look at the 'vicinity of the freezing point', so to speak, namely, where most channels are frozen in.

It might be an accident that this is the leading term of Eq. (20.6) with a reasonable value of T_0. (It might be no accident.)

There is no contradiction in considering a fireball as built of fireballs and at the same time as built of quarks – superfluid helium is understood only if considered as a boson liquid, but after all it 'really' consists of fermions. Such pictures are complementary.

References

1. R. Hagedorn: Suppl. Nuovo Cimento **3**, 147 (1965) (Part I);
   R. Hagedorn, J. Ranft: Nuovo Cim. Suppl. **6**, 169 (1968) (Part II);
   R. Hagedorn: Nuovo Cim. Suppl. **6**, 311 (1968) (Part III), see also Chapters 17, 23 and 25
2. A.H. Rosenfeld, A. Barbaro-Galtieri, J. Podolsky, L.R. Price, P. Söding, C.G. Wohl, M. Roos, W.J. Willis: Rev. Mod. Phys. **39**, 1 (1967)
3. T.T. Wu, C.N. Yang: Phys. Rev. **137**, B708 (1965)
4. J. D. Stack, Phys. Rev. **164**, 1904 (1967)
   J. Harte, Phys. Rev. **165**, 1557 (1968)

Chapter 21
On the Hadronic Mass Spectrum – 2014

Johann Rafelski

Abstract The understanding of Hagedorn's pivotal discovery, the exponential mass spectrum, evolved rapidly. Some of the insights have since been lost from view – I recall the relevance of the preexponential power index a. Moving forward to current lattice QCD computation of QGP properties I describe an emerging relationship.

21.1 Data and Hadron Mass Spectrum

Fits of hadron mass spectrum

The number of known hadronic states more than tripled since Hagedorn performed his analysis of the shape of the mass spectrum, see Chapter 20. This provides an opportunity for an important cross-check of the Hagedorn analysis. I will now briefly describe the key new insights.

The Krakow group [1, 2] considered the integrated ('accumulated') spectrum

$$R(M) = \int_0^M \rho(m) dm \tag{21.1}$$

and they also break the large set of hadron resonances into different classes, e.g. non-strange/strange hadrons, or mesons/baryons. While Hagedorn-type approach requires smoothing of the spectrum, adopting an effective Gaussian width for all hadrons, the integrated spectrum Eq. (21.1) allows one to address directly the step function arising from integrating the discrete hadron mass spectrum, i.e. avoiding the Hagedorn smoothing.

One could think that the Hagedorn smoothing process loses information that is now available in the new approach. However, it turns out that a greater information loss comes from the consideration of the integrated 'signal'. This is seen in the results of Krakow group by noting that the fitted value of T_H is strongly varying in dependence on supplementary hypotheses made about the procedure, with the

Department of Physics, The University of Arizona, Tucson, AZ 85721, USA

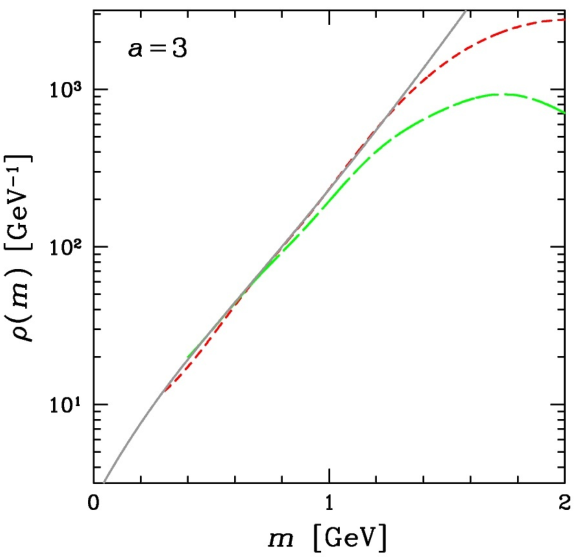


Fig. 21.1 Contemporary Mass Spectrum Fit (short dashed) compared to 1968 fit of Hagedorn (long dashed): The case of power law scaling $a = -3$ is shown for parameters and also other values of a see table 21.1.

value of T_H changing by 100's MeV. This probably means that the integrated mass spectrum Eq. (21.1), aside of the physical parameter fit, also has other good fits. The likely cause of the T_H instability is that these artifacts produce the best fit at an unphysical parameter set. This situation is not uncommon when considering any integrated signal function.

My own work [3, 4] has been more modest, an 'almost' redo of the original Hagedorn fit and is shown in Fig. 21.1. A comparison of the original Hagedorn fit, long dashed line in Fig. 21.1, with an analysis involving more than 5000 hadron states; short dashed line suggests that results are highly compatible. However, there are a few caveats. The hadron mass spectrum that was fitted is

$$\rho(m) = c\,\frac{e^{m/T_H}}{(m_0^2 + m^2)^{a/2}} \tag{21.2}$$

All three parameters T_H, m_0, c are varied and find their best value. Hagedorn fixed $m_0 = 0.5\,\text{GeV}$ as he was working in the limit $m \gg m_0$, and this is clearly not the case as the mass spectrum available experimentally is limited to a range $m < 1.7\,\text{GeV}$. The introduction of a fitted value m_0 is necessary to improve the spectrum for low values of m.

The pre-exponential power value $a = 2.5$ in Eq. (21.2) corresponds to Hagedorn's original work, Eq. (20.6). However, several years later following further developments described below in Chapter 23, the value $a = 3$ was obtained. Moreover,

Table 21.1 Parameters of Eq. (21.2) fitted for a prescribed pre-exponential power a. results from Ref. [4]. Note that the value of c for $a = 2.5$ corresponds to $c = 2.64 \times 10^4\,\mathrm{MeV}^{3/2}$ in excellent agreement to the value shown in Fig. 20.1.

a	$c[\mathrm{GeV}^{a-1}]$	$m_0[\mathrm{GeV}]$	$T_H[\mathrm{GeV}]$
2.5	0.83479	0.6346	0.16536
3.	0.69885	0.66068	0.15760
3.5	0.58627	0.68006	0.15055
4.	0.49266	0.69512	0.14411

$a = 2.5$ leads to divergent energy density and excludes the phase transformation of HG to a new phase. Thus it must not be used considering existence of QGP.

The results shown in Fig. 21.1 are thus given for $a = 3$. The requirement that T_H is a transformation temperature between phases favors larger values of a and thus a range $a \in [2.5\text{–}4]$ is presented in table 21.1; fits obtained in 1994 were for a slightly smaller set of hadrons [4] than is available today. We see that as the pre-exponential power law a increases, the fitted value of T_H decreases.

The value of the power index 'a'

In the first Hagedorn mass spectrum paper Chapter 20, in Eq. (21.2) the values $m_0 = 0.5\,\mathrm{GeV}$ and the power index $a = 2.5$ are assumed in the fit presented in Fig. 20.1. Upon the exact solution of the bootstrap equation it was recognized that the precise form of singularity that SBM condition generates requires $a = 3$, for references see Chapter 22, entries in Fig. 22.2, bottom left square and presentation of SBM, Eq. (23.12). A further requirement is imposed in order to assure that the Hagedorn Temperature is a phase transition temperature. For this to be true the energy density must remain finite when $\Delta T \equiv T - T_H \to 0$, and this requires $a \geq 7/2$, see table 23.1 and comments below this table, as well as the discussion below Eq. (25.16). Inspection of the table 21.1 shows that the condition $a \geq 7/2$ corresponds to $T_H \leq 151\,\mathrm{MeV}$.

Given the extensive literature within the SBM context pointing at $a = 3$ and the phase transition studies which require $a \geq 7/2$, it is hard to understand why modern studies of the the mass spectrum have all focused on $a = 2.5$, a value which is obsolete. This assumption produces the highest value of T_H but is inconsistent with the physics pictures that emerged in regard to SBM, and later of a phase transformation of HG to QGP.

Elaborate lattice-QCD numerical computations of QGP to HG transformation regime are available today [5–7]: the Hot-QCD collaboration [6] converged for 2+1 flavors towards $T_c = 154 \pm 9$ MeV. One of the works of the Wuppertal-Budapest collaboration [8] suggests a low $T_c \simeq 145\,\text{MeV}$. However, this low value depends on the choice of the phase transformation tracking observable. The latest report of this group [7] mentions $T_c \simeq 150\,\text{MeV}$. All current lattice-QCD T_H results are thus according to tabulation in table 21.1 favoring $a = 3.5$, which is the lowest value allowing for a finite energy density near HG phase boundary, see table 23.1.

In the above table 21.1, a value of a is not preferred experimentally: all fits shown are of comparable quality when all three model parameters T_H, m_0, c are allowed to vary. If, however, a fixed value of m_0 is arbitrarily prescribed, as was done by Hagedorn who was compelled half a century ago by limited experimental information, the quality of the fit to the present day data will diminish. For example Ref. [9] fixes $m_0 = 0.5\,\text{GeV}$ at $a = 2.5$, i.e. Hagedorn's 1968 parameter choices. Applying the Krakow group method approach, this fit produces with present day data $T_H = 0.174\,\text{GeV}$. We keep in mind that the assumed value of a is incompatible with SBM, while the assumption of a relatively small $m_0 = 0.5\,\text{GeV}$ is forcing a relatively large value of T_H.

21.2 Quarks and QCD

Lattice-QCD trace anomaly constraint

Arguably, the most important recent step forward in regard to improving the Hagedorn mass spectrum analysis is the realization that one can infer important information about the hadron mass spectrum from lattice-QCD numerical results [10]. The lattice-QCD effort has a relatively easy numerical access to the trace anomaly of the energy-momentum tensor expressed in units of T^4, the so called 'interaction measure'

$$I(T) = \frac{\varepsilon - 3P}{T^4}. \tag{21.3}$$

This quantity vanishes in scale invariant theory, for example for (effectively) massless and free gas of quarks and gluons. For interacting gas of quarks and gluons the QCD scale parameter generates a non-vanishing result, but asymptotic freedom implies that interaction effects decrease with increasing temperature of QGP. Accordingly, $I(T)$, Eq. (21.3) is seen to decrease from a relatively high value achieved when quarks and gluons turn into hadrons, see the high T domain shown in Fig. 21.2. For low temperature, where we do not expect a deconfined quark-gluon phase, the rise of $I(T)$ with T indicates that with increasing temperature more massive hadron states become relevant.

At temperatures below QGP formation $I(T)$, Eq. (21.3) is derived from the contribution of each hadron species folded into the hadron mass spectrum. In the Boltzmann limit Eq.(10.60) in Ref. [11] reads

$$I_{\text{Boltz}}(T) = \int dm \rho(m) \frac{x^3}{2\pi^2} K_1(x), \quad x = m/T, \tag{21.4}$$

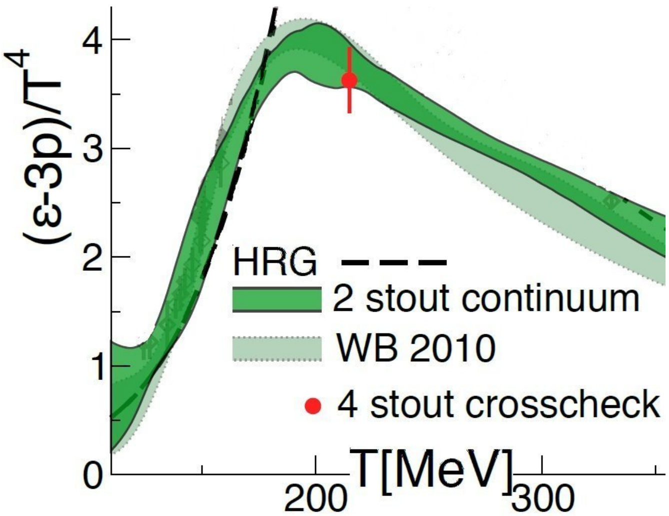


Fig. 21.2 The comparison of Interaction Measure with Hadron Resonance Gas, Latest lattice-QCD results, after Ref. [7].

where the degeneracy g of each state of mass m is renamed $\rho(m)$ and the continuous integral, rather than discrete sum, is introduced.

The Boltzmann limit is, however, not accurate as the power law expansion of the quantum distribution shows. Adopting Eq.(10.68) in Ref. [11], the result is

$$I_Q(T) = \int dm\rho(m) \sum_{n=1}^{\infty} \frac{(f)^{n+1}}{n} \frac{x^3}{2\pi^2} K_1(nx), \quad x = m/T, \tag{21.5}$$

where $f = -/+1$ for Fermions/Bosons. The temperature domain of interest for us is $T < 175$ MeV. The presence of the relatively light boson, the pion, with a mass within this temperature domain means that we must include bosonic quantum corrections. While the results of Ref. [10] suggested the need for some additional undiscovered resonances, the comparison shown in Ref. [7], see Fig. 21.2, suggests that agreement between hadron resonance gas (HRG, dashed line in the figure) and current lattice-QCD results (uncertainty domains shown, techniques will not be discussed here) is achieved just with the known hadron set of states.

In the context of the search for a better understanding of the hadron mass spectrum and the determination of Hagedorn Temperature $T_{\rm H}$, the trace anomaly can fill a very important information gap. Given a psrametrized mass spectrum shape, such as is Eq. (21.2), one must fit both, the experimental hadron mass spectrum, as well as the numerical lattice-QCD trace anomaly. Such a joint approach could produce a more unique phenomenological determination of both T_H and favor a value for the pre-exponential power a.

Quark bags and the hadron mass spectrum

There are additional challenges arising from the prophetic work on the hadron mass spectrum by Hagedorn, Chapter 20. Hagedorn's paper closes with remarks about the possibility that the multitude of quark bound states could relate to an exponential mass spectrum. This indeed is the case. Several quantitative implementations of this idea appeared in literature beginning in 1976 [12]. In 1981 Joe Kapusta [13] writes in his abstract: "A statistical evaluation of the mass spectrum in the (quark) bag model is made ... (which) behaves asymptotically as $\rho(m) \propto c m^{-3} exp(m/T_H)$, ... this model satisfies the strong bootstrap condition. ... The thermodynamics of a system of such composite hadrons naively exhibits a maximum temperature T_0. However, ... first-order phase transition to a gas of free elementary fields is found at a temperature $T_c = 1.05 T_H$."

This work has stimulated continued interest in evaluation of hadron mass spectrum based on quark bag model. However, given that the naive bag model is known to predict hadron states not found in experimental searches, this path cannot be trusted to produce in quantitative terms a result of phenomenological importance. Ref. [14] criticizes also in this context the current widely accepted Hagedorn approach and Hagedorn Temperature. For reasons already described, and in particular in consideration of the loss of information for the integrated mass spectrum, we do not share in any of the views presented in this work.

References

1. W. Broniowski, W. Florkowski and L. Y. Glozman: Phys. Rev. D **70**, 117503 (2004)
2. W. Broniowski and W. Florkowski: Phys. Lett. B **490**, 223 (2000)
3. J. Letessier and J. Rafelski: (unpublished). Evaluation made for the CERN Courier article in honor of R. Hagedorn, see Chapter 6 and Fig. 6.2, based on the method of Ref. [4].
4. A. Tounsi, J. Letessier and J. Rafelski: "Hadronic matter equation of state and the hadron mass spectrum," In proceedings of Divonne 1994, "Hot hadronic matter" NATO-ASI series B **356**, pp 105-116 (1995)
5. G. Endrodi, Z. Fodor, S. D. Katz and K. K. Szabo: JHEP **1104**, 001 (2011)
6. A. Bazavov *et al.*: Phys. Rev. D **85**, 054503 (2012)
7. S. Borsanyi, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg and K. K. Szabo: Phys. Lett. B **730**, 99 (2014)
8. S. Borsanyi: Nucl. Phys. A904-905 **2013**, 270c (2013); and Presentation at 23rd International Conference on Ultrarelativistic Nucleus-Nucleus Collisions (QM 2012), August 13-18, 2012, Washington, DC, USA
9. J. Cleymans and D. Worku: Mod. Phys. Lett. A **26**, 1197 (2011)
10. A. Majumder and B. Muller: Phys. Rev. Lett. **105**, 252002 (2010)
11. J. Letessier and J. Rafelski: "Hadrons and quark-gluon plasma," Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **18**, 1 (2002).
12. J. Baacke, Acta Phys. Polon. B **8**, 625 (1977).
13. J. I. Kapusta: Phys. Rev. D **23**, 2444 (1981).
14. T. D. Cohen and V. Krejcirik: J. Phys. G **39**, 055001 (2012)

Chapter 22
SBM Guide to the Literature as of June 1972

Rolf Hagedorn

Abstract A large number of research papers motivated Rolf Hagedorn to prepare a guide to the Statistical Bootstrap Model literature in mid-1972; he wanted to show the reader a) which publications best prepare the uninitiated, b) how different publications relate to each other, and c) he aimed to connect similar developments into groups. The two figures classify in this fashion the research works Refs. [1-70]. The editor updated all preprint citations for the 2015 printing.

This guide contains two figures and a list of references. It is neither complete nor unbiased.

Figure 22.1 gives the recommended reading sequence. After reading the introductory lectures (partly overlapping), the reader should be able to enter the lower boxes at any place, though it might be advantageous to follow the given sequence. The reader will notice overlaps and inconsistencies, because the reading sequence does not coincide with the historical development.

Figure 22.2 tries to picture the logical (and roughly the historical) connections. The isolated box with the name Koppe indicates that he was the first to contemplate statistical and thermodynamical interpretations of pion production. Unfortunately, his two papers came too early and went unnoticed. When two years later Fermi elaborated the same idea in great detail, he obviously had no knowledge of Koppe's work.

In both figures, the sequence goes along top to bottom lines of connection, unless indicated otherwise by an arrow.

Preprint CERN-TH 1535, dated 20 July 1972, see: http://cds.cern.ch/record/961894?ln=en
not intended as formal publication.

(deceased) CERN-TH, 1211 Geneva 23, Switzerland

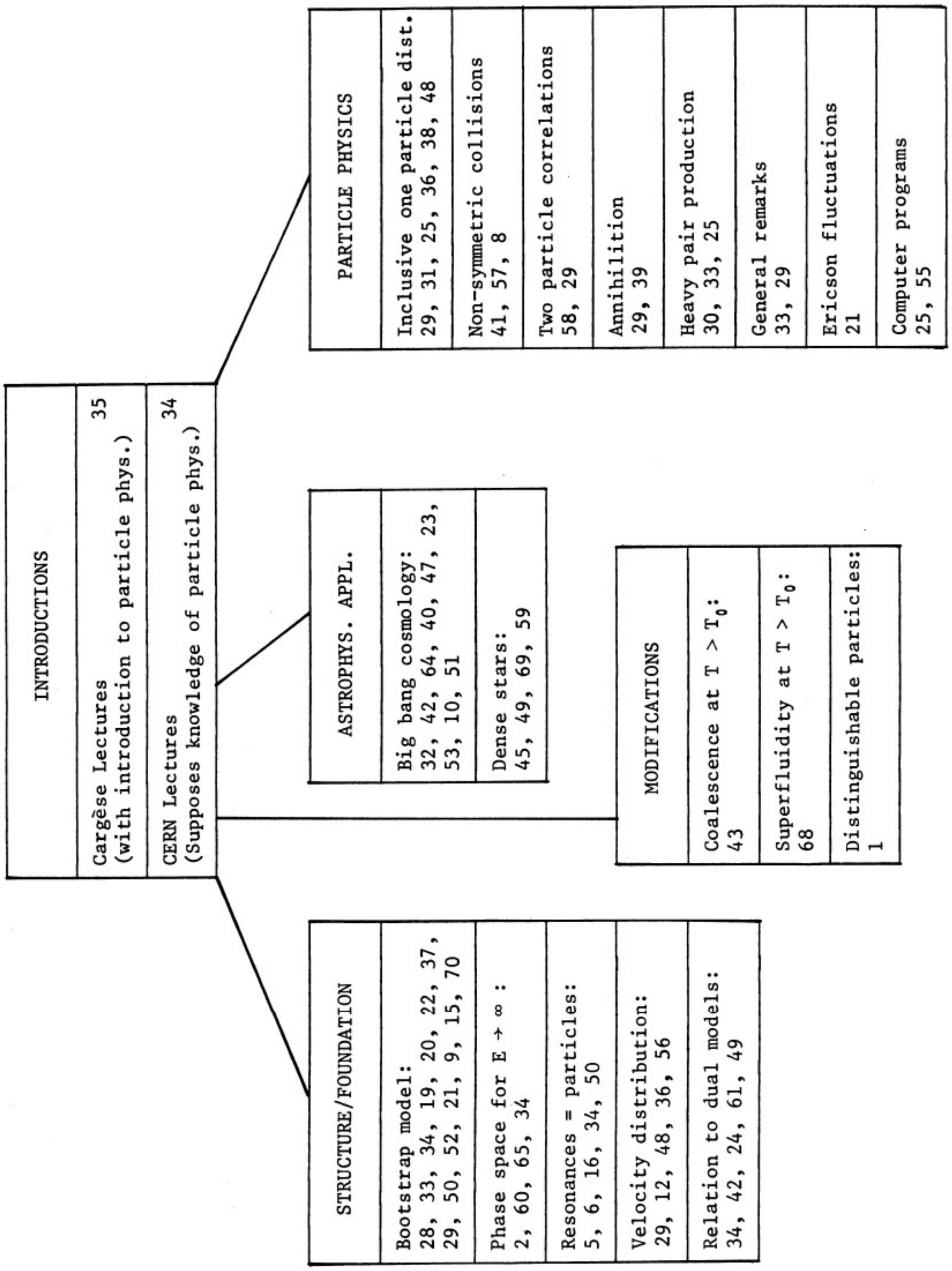


Fig. 22.1 Recommended reading sequence.

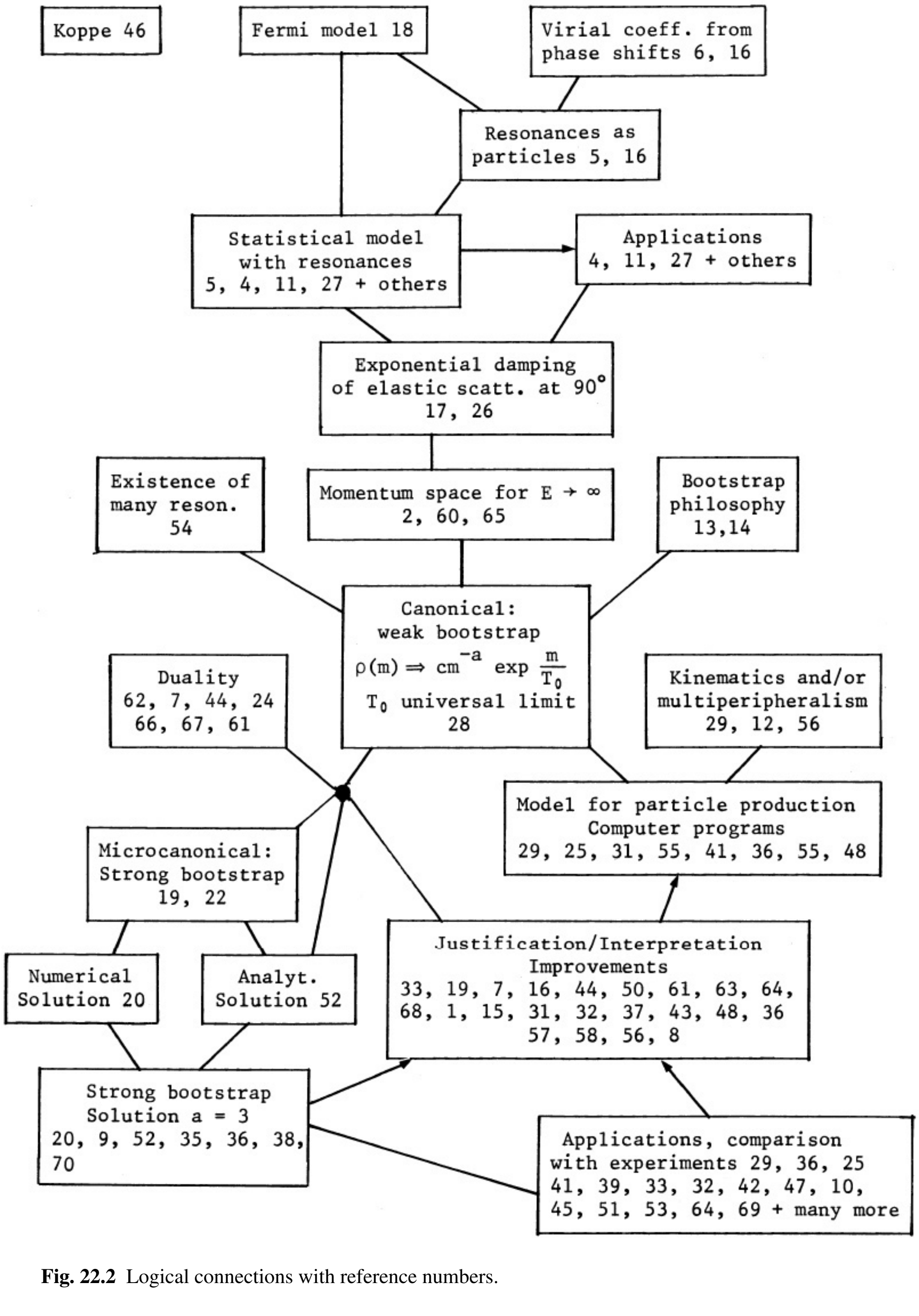


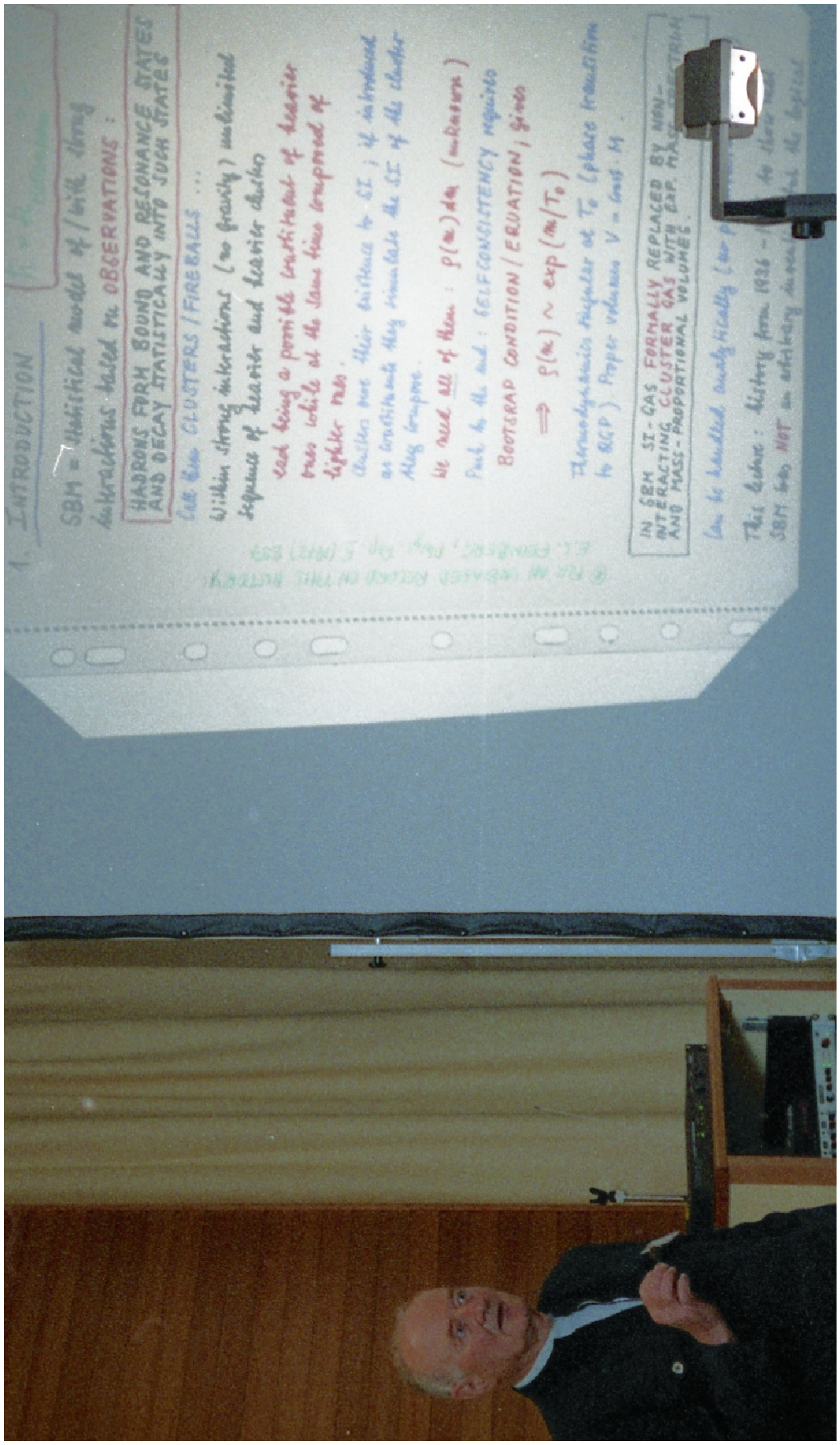
Fig. 22.2 Logical connections with reference numbers.

References

1. M. Alexanian: Phys. Rev. D **5**, 922 (1972)
2. G. Auberson, B. Escoubès: Nuovo Cimento **36**, 628 (1965)
3. J.N. Bahcall, S.C. Frautschi: Astrophys. J. **170**, L81 (1971)
4. V.S. Barashenkov, B.M. Barbashev, E.G. Bubelov, V.M. Maksimenko: Nuclear Phys. **5**, 17 (1958)
5. S.Z. Belenkij: Nuclear Phys. **2**, 259 (1956)
6. E. Beth, G.E. Uhlenbeck: Physica **4**, 915 (1937)
7. R. Brout: Letter to R. Hagedorn, March 1969 (unpublished)
8. B. Buschbeck, J. Hödl: Nucl.Phys. B **48**, 288 (1972)
9. R.D. Carlitz: Phys.Rev. D **5**, 3231 (1972)
10. R.D. Carlitz, C.S. Frautschi, W. Nahm: Astron. Astrophys. **26**, 171 (1973)
11. F. Cerulus: Nuovo Cimento **14**, 827 (1959)
12. F. Cerulus: Proceedings of the Colloquium on High Multiplicity Hadron Interactions, Ecole Polytechnique, Paris 1970
13. G.F. Chew, S. Mandelstam: Nuovo Cimento **19**, 752 (1961)
14. G.F. Chew: Physics Today **23**, 23 (1970)
15. C.B. Chiu, R.L. Heimann: Phys. Rev. D **4**, 3184 (1971)
16. R. Dashen, S. Ma, H.J. Bernstein: Phys. Rev. **187**, 349 (1959)
17. G. Fast, R. Hagedorn: Nuovo Cimento **27**, 208 (1963); G. Fast, R. Hagedorn, L.W. Jones: Nuovo Cimento **27**, 856 (1963)
18. E. Fermi: Progr. Theor. Phys. **5**, 570 (1950)
19. S.C. Frautschi: Phys. Rev. D **3**, 2821 (1971)
20. S.C. Frautschi, C.J. Hamer: Phys. Rev. D **4**, 2125 (1971)
21. S.C. Frautschi: Nuovo Cimento A **12**, 133 (1972)
22. S.C. Frautschi: Lectures given at the Torini Colloquium on Thermodynamics of Strong Interactions; forthcoming report of the Institute of Theoretical Physics, Univ. Torino, Italy (1972)
23. S.C. Frautschi, G. Steigmann, J. Bahcall: Astrophys. J. **175**, 307 (1972)
24. S. Fubini, D. Gordon, G. Veneziano: Phys. Lett. B **29**, 679 (1969); S. Fubini, G. Veneziano: Nuovo Cimento A **64**, 811 (1969)
25. H. Grote, R. Hagedorn, J. Ranft: Atlas of Particle Spectra (CERN, 1970)
26. R. Hagedorn: Nuovo Cimento **35**, 216 (1965)
27. R. Hagedorn: Nuovo Cimento **15**, 434 (1960)
28. R. Hagedorn: Suppl. Nuovo Cimento **3**, 147 (1965)
29. R. Hagedorn, J. Ranft: Suppl. Nuovo Cimento **6**, 169 (1968)
30. R. Hagedorn: Suppl. Nuovo Cimento **6**, 311 (1968)
31. R. Hagedorn: Nuovo Cimento A **56**, 1027 (1968)
32. R. Hagedorn: Astron. and Astrophys. **5**, 184 (1970)
33. R. Hagedorn: Nuclear Phys. B **24**, 93 (1970)
34. R. Hagedorn: "Thermodynamics of Strong Interactions" Yellow Report CERN 71-12 (1971)
35. R. Hagedorn: Lectures to be published in the Proceedings of the Cargèse Summer School on Physics of the Early Universe and Cosmology, Cargèse 1971
36. R. Hagedorn, J. Ranft: Nucl. Phys. B **48**, 157 (1972)
37. R. Hagedorn: Lectures given at the Torino Colloquium on Thermodynamics of Strong Interactions; forthcoming report of the Institute of Theoretical Physics, Univ. Torino, Italy (1972)
38. R. Hagedorn: Lecture to be published in the Proceedings of the Third International Colloquium on Many-Body Reactions, Zakopane 1972

39. C.J. Hamer: Nuovo Cim. A **12**, 162 (1972)
40. E.R. Harrison: Phys. Rev. D **1**, 2726 (1970)
41. Htun Than, J. Ranft: Lett.Nuovo Cim. **5**, 655 (1972)
42. K. Huang, S. Weinberg: Phys. Rev. Lett. **25**, 895 (1970)
43. K. Imaeda: Nuovo Cimento Lett. **1**, 290 (1971)
44. A. Krzywicki: Phys. Rev. **187**, 1964 (1969)
45. K. Koebke, E. Hilf, R. Ebert: Nature **226**, 625 (1970)
46. H. Koppe: Z. Naturforsch. A **3**, 251 (1968); H. Koppe: Phys. Rev. **76**, 688 (1949)
47. W. Kundt: Springer Tracts of Mod. Phys. **58**, 1 (1971)
48. J. Letessier, A. Tounsi: Nuovo Cim. A **11**,353 (1972)
49. Y.C. Leung, G.G. Wang: Astrophys. **170**, 499 (1971); H. Lee, Y.C. Leung, C.G. Wang: Astrophys. J. **166**, 387 (1971)
50. G. Matthiae: Nuclear Phys. B **7**, 142 (1948)
51. C. Mollenhoff: Astron. and Astrophys. **7**, 488 (1970)
52. W. Nahm: Nucl.Phys. B **45**, 525 (1972)
53. R. Omnès: Phys. Reports C **3**, 1 (1972)
54. Particle Data Group: Phys. Letter. B **33**, No. 1 (1970)
55. J. Ranft: Leipzig Univ. Report TUL 37 (1970), reprinted as ‘Reprint 5’ in [25]
56. G. Ranft, J. Ranft: Phys. Lett. B **32**, 207 (1970)
57. J. Ranft, E. Matthäus: Phys. Lett. B **40** 230 (1972)
58. G. Ranft, J. Ranft: Phys. Lett. B **40**, 131 (1972); G. Ranft, J. Ranft: Preprint CERN-TH-1532 “Inclusive Two-particle Correlations From The Thermodynamic Model”
59. C.E. Rhoades, R. Ruffini: Astrophys. Lett. **163**, L83 (1971)
60. H. Satz: Nuovo Cimento **37**, 1407 (1965)
61. H. Satz: Lecture to be published in Proceedings of the Colloquium on High Multiplicity Hadron Interactions, Zakopane, Poland (1972)
62. C. Schmid: Phys. Rev. Lett. **20**, 628 (1968)
63. L. Sertorio, M. Toller: Nuovo Cimento A **14**, 21 (1973)
64. D. Stauffer: Phys.Rev. D **6**, 1797 (1972)
65. J. Vandermeulen: Bull. Soc. Royale Sciences, Liège (Belgium), 34^{e} année, No. 1/2 p.34 (1965)
66. G. Veneziano: Nuovo Cimento A **57**, 190 (1968)
67. G. Veneziano: Physics Today **22**, 31 (1969)
68. A. Mann, R. Weiner: Nuovo Cimento Lett. **2**, 248 (1971); A. Mann, R. Weiner: Phys.Lett. B **40**, 383 (1972)
69. J.C. Wheeler: Astrophys. J. **169**, 105 (1971)
70. J. Yellin: Nucl. Phys. B **52**, 583 (1973)

Next page: Rolf Hagedorn presents a SBM retrospective, June 1994, 22 years after this SBM guide is issued. *Image credit: CERN Image 1994-06-63-022.*


1. INTRODUCTION
SBM = statistical model of / with strong interactions based on OBSERVATIONS :
HADRONS FORM BOUND AND RESONANCE STATES AND DECAY STATISTICALLY INTO SUCH STATES
Call them CLUSTERS / FIREBALLS ...
Within strong interactions (no gravity) unlimited sequence of heavier and heavier clusters
each being a possible constituent of heavier ones while at the same time composed of lighter ones.
Clusters owe their existence to SI ; if introduced as constituents they simulate the SI of the cluster they compose.
We need ALL of them : ρ(m)dm (unknown)
Push to the end : SELFCONSISTENCY requires BOOTSTRAP CONDITION / EQUATION ; gives
⟹ ρ(m) ~ exp(m/T0)
Thermodynamics singular at T0 (phase transition to QGP). Proper volumes V = const · M.
IN SBM SI - GAS FORMALLY REPLACED BY NON-INTERACTING CLUSTER GAS WITH EXP. MASS SPECTRUM AND MASS-PROPORTIONAL VOLUMES.

Chapter 23
Thermodynamics of Hot Nuclear Matter – 1978 in the Statistical Bootstrap Model

Johann Rafelski and Rolf Hagedorn

Abstract We formulate the statistical bootstrap model for nuclear matter, and study its resulting thermodynamic properties at nuclear densities below the saturation density. We discuss the relevance of limiting temperature and the phase transition gas–'liquid' when the volume of the fireball grows with its energy.
Editor's comment: *The numerical results shown are obtained neglecting antibaryons. In obtaining these results we did find that as temperature rises this effect causes an unphysical rise of baryon density for* $120\,\mathrm{MeV} \leq T \leq T_0$ *MeV. Moreover, the net baryon density does not vanish along with baryon chemical potential. Within months we solved the problem numerically without this approximation; the results are presented in Chapter 27 "Extreme States of Nuclear Matter", and in Ref. [1]. However, this report is the most detailed available document describing the theory behind the statistical bootstrap model of hot nuclear matter. Being distracted by the rise of the relativistic heavy ion research program, this more theoretical work was shelved to be part of a Physics Reports article, and was never formally published, not even as a TH-preprint. It is, however available (see footnote) as a CERN library archived document. The Physics Reports review that would have contained this material was never completed.*

23.1 Introduction

Properties of nuclear matter have inspired much of the theoretical work in many-body theory during the last decades. While initially attention was focused on the saturation properties of cold nuclear matter, more recently the advent of high-energy

Presented by JR at the XVIIth International Winter Meeting on Nuclear Physics, Bormio, Italy, 22–27 January, 1979, pp. 531-563 in 'home' press proceedings: Editor: I. Iori, Instituto di Fizica Universita Milano, available at: http://cds.cern.ch/record/134674/files/CM-P00055555.pdf

JR: Department of Physics, The University of Arizona, Tucson AZ
RH: (deceased) CERN-TH, 1211 Geneve 23, Switzerland

heavy ion accelerators has stimulated work on the high temperature and density domain of the phase diagram.

There exist several main lines of approach to this complicated theoretical problem in which substantial simplifications of the actual physical circumstances are supposed. We will not review these approaches here except to say that they can be divided into two categories: 1) the nuclear matter is considered to be a non-interacting ideal gas; or 2) nuclear interactions are considered at the level of classical particle scattering.

It is immediately apparent that the interesting features of nuclear matter, such as density isomerism at high temperatures, phase transitions, condensation phenomena, etc., can hardly be discussed in the framework of the ideal gas equations of state. The fact that some kind of agreement of inclusive particle spectra in heavy ion collisions is found between theory and experiment is in fact only indicative that a thermal equilibrium is achieved in a fireball created in the collisions. To find out more about the properties of these fireballs, one has to perform more refined experiments and consider a more elaborate theory. This aim is achieved in a non-thermodynamical way in the approaches that deal with the $A_1 + A_2$ many-body problem, in which each particle is followed during the collision; but it becomes virtually impossible to identify the relevant collective motion that is characteristic of phase transitions and critical phenomena.

In order to derive the physical properties of hot nuclear matter which are independent of a particular choice of the two-body and multibody interaction, we employ a technique ('bootstrap') developed for similar problems in elementary particle physics – here, however, sufficiently modified to suit the different physical environment. An additional motivation in this direction is the recent recognition that the understanding of nuclear matter at the saturation point depends very sensitively on the character of the two-body potential at short distances, which is not well defined by two-body reactions. It is possible to view the bootstrap technique only as a convenient way of introducing the relevant physical properties which cannot be so easily defined by the choice of a specific potential, but which globally might even be more important than details of the two-body force. We will concentrate on the gross features of nuclear matter that follow when we incorporate into the description the following aspects of nuclear interactions:

i. conservation of baryon number and clustering of nucleons (i.e., attractive forces leading to many-body clusters with well-defined baryon number);
ii. nucleon (isobar) excitations and internal cluster excitations (i.e., internal degrees of freedom that can absorb part of the energy of the system at finite temperature, thus transforming kinetic energy into mass);
iii. approximate extensivity of nuclear matter (volume roughly proportional to baryon number, i.e., effectively a short-range repulsion);
iv. co-existence of a pion gas when the temperature is not equal to zero (and behaving properly even in the presence of nuclear matter);
v. baryon–antibaryon pair creation;

vi. ‘chemical’ equilibrium between all constituents of the system (nucleons, isobars, clusters, pions, etc.).

Our present work [2] should be most trustworthy in the domain of high temperatures and moderately high density, where details of the interaction, of Fermi and Bose statistics, as well as of the quark structure of nucleons, are most likely negligible. Also not considered explicitly here is the isospin of the nuclei.

An important new feature of the Statistical Bootstrap Model as we introduce it here is that the energy density in fireballs,

$$\varepsilon_{\mathrm{FB}} := \frac{m}{V(m)} =: A \,, \tag{23.1}$$

is finite, constant, and of the order of the rest-energy density of a proton. Therefore it occurs to us that it is not reasonable to apply the thermodynamics derived from the ‘bootstrap equation’ beyond the point where the energy density $\varepsilon(T)$ becomes much larger than $1/A$.

Plan of the paper

Section 23.2. We discuss the bootstrap hypothesis first in the context of a strongly interacting pion gas. The bootstrap equation of the pion gas is solved and discussed. We write down, discuss, and solve the bootstrap equation for nuclear matter. It is much richer than that of the pion gas, which it contains as a special case.

Section 23.3. The mass spectrum and its Laplace transform are used to obtain a thermodynamic description of the system. We compute the partition functions for clustered nuclear matter.

Section 23.4. We study the properties of nuclear matter in the thermodynamic limit. Two main properties of our model are:

i. there exists a maximum temperature, which is of the order of that of the pion gas ($T_0 \approx m_\pi$);
ii. there exists at all temperatures $0 \leq T \leq T_0$ a critical baryon number density separating a low-density ‘gas phase’ from a state where a condensate and its vapor exist in equilibrium.

A numerical study is presented in which the simplest non-trivial input spectrum is assumed; the corresponding model is solved explicitly and the results are displayed graphically. This case shows all essential features, but it is still too far from reality to be taken as more than a qualitative prediction.

Section 23.5. Summary.

Our notation and units:

- $\hbar = c = k$ (Boltzmann constant) $= 1$;
- the only dimensional unit is 1 GeV = 1 000 MeV $\approx 5\ \mathrm{fm}^{-1}$;
- metric: $a \cdot b \equiv a_\mu b^\mu \equiv a_0 b_0 - \mathbf{a} \cdot \mathbf{b}$.

Thus for example $m^2 := p^2 = p_0^2 - \mathbf{p}^2 = E^2 - \mathbf{p}^2$.

Remark. Throughout this paper we use only Boltzmann statistics. As the bootstrap approach leads to an extremely rich mass spectrum, it is almost irrelevant whether a particular cluster or particle is a boson or a fermion or a Boltzmannion: it (almost) never happens that two equal clusters occupy the same state.

23.2 The Statistical Bootstrap Method in Particle and Nuclear Physics

The Statistical Bootstrap Model in particle physics

The motivation for a statistic bootstrap model in particle physics comes from two sources:

i. the abundant production of particles in high-energy pp collisions, and a momentum distribution of these particles which suggests that there might be some analogy to black-body radiation emitted from moving sources;
ii. the apparent existence of intermediate states in which lumps of highly excited hadronic matter ('fireballs') are staying together before decaying.

Thus it was tempting to describe the particle production process as pion black-body radiation emitted from one or several fireballs with a volume $v_0 \approx 4\pi/3m_\pi^3$. This idea was first proposed by Koppe [3] and it is called the Fermi statistical model [4]. As for a statistical–thermodynamical description, the density of states $\sigma(E)$ is necessary and sufficient; we may express the Koppe–Fermi approach as follows:

$$\frac{v_0 \sigma_0(E)}{(2\pi)^3} = \sum_{n=1}^{\infty} \frac{1}{n!} \int \delta\left(E - \sum_{i=1}^{n} \sqrt{\mathbf{p}_i^2 + m_\pi^2}\right) \delta^3\left(\sum_{i=1}^{n} \mathbf{p}_i\right) \prod_{i=1}^{n} \frac{v_0 \mathrm{d}^3 p_i}{(2\pi)^3} \ . \tag{23.2}$$

This is nothing else than the phase-space density of a pion gas with free particle creation. If we put $m = 0$ and multiply by 2 for the two helicity states of a light quantum, we obtain from Eq. (23.2) all the usual formulas of the electromagnetic black-body radiation (Planck's law) in the Boltzmann limit.

The next important idea was to admit particles other than just pions, and in particular resonant states of pions, just as if they were stable particles [5]. Not knowing which ones should be admitted and how many there are, we might put them in a mass spectrum of admissible input particles $\rho_{\mathrm{in}}(m)$. The pion contributes to $\rho_{\mathrm{in}}(m)$ a δ-function $\delta(m - m_\pi)$; resonances contribute smeared-out δ-functions. For the moment, $\rho_{\rm in}(m)$ is a function which represents our (incomplete) knowledge of the true mass spectrum $\rho(m)$.

Introducing also relativistically invariant notation with a four-volume $V^\mu = v_0 u^\mu$, $u^2 = 1$, and $\delta_0(p^2 - m^2) = \delta(p^2 - m^2)\theta(p_0)$,

$$\begin{aligned}\sigma(p^2, p\cdot V) = {} & \frac{2p\cdot V}{(2\pi)^3}\delta_0(p^2 - m_\pi^2) \\ & + \sum_{n=2}^{\infty}\frac{1}{n!}\int \delta^4\Big(p - \sum_{i=1}^{n} p_i\Big)\prod_{i=1}^{n}\frac{2p_i\cdot V}{(2\pi)^3}\rho_{\rm in}(m_i)\delta_0(p_i^2 - m_i^2)\mathrm{d}^4 p_i \mathrm{d}m_i\,.\end{aligned} \tag{23.3}$$

This is a relativistically invariant equation for the density of states $\sigma(p^2, p\cdot V)$ in $\{p, \mathrm{d}^4 p\}$ of a gas in which the interaction manifests itself via creation and absorption of Boltzmann pions and their excited states contained in $\rho_{\rm in}(m)$.

Note that we have restricted the one-particle state to have the pion mass. Higher mass 'one-particle states' are already contained in the sum, namely when in any of its terms all $p_i \to m_i$. Our new equation for $\sigma(p^2, p\cdot V)$ describes the density of states of a many-component gas: each species of particle contained in $\rho_{\rm in}(m)$ is present in the gas. All these components are in 'chemical' equilibrium; neither the total particle number nor that of any of the various components is fixed.

The key idea that leads to the hadronic bootstrap is the observation that the quantity $\sigma(p^2, p\cdot V)$ can be related to the mass spectrum $\rho(m)$. Suppose we could insert the true mass spectrum $\rho(m)$ into Eq. (23.3). Then $\sigma(p^2, p\cdot V)$ would be the density of states of a 'fireball' of hadronic dimension built up from *all* strongly interacting particles in statistical equilibrium. Such a fireball is itself a highly excited hadron with mass $m = \sqrt{p^2}$. For reasons of consistency, it should then be admitted as a constituent particle in fireballs of larger mass. Hence it should already be present in the true $\rho(m)$. As both $\sigma(p^2, p\cdot V)$ and $\rho(m)$ are densities of states, it follows that if $\rho(m)$ is the true mass spectrum, $\sigma(p^2, p\cdot V)$ is itself (apart from some minor kinematical differences) the true mass spectrum at $m = \sqrt{p^2}$. This statement establishes a new relation between ρ and σ, leading to an integral equation, the bootstrap equation. Physically, it is equivalent to the postulate that resonances and fireballs are one and the same and that fireballs consist of fireballs.

In order to find the precise relation between $\rho(m)$ and $\sigma(p^2, p\cdot V)$, we consider the conceptual differences between them as exhibited by Eq. (23.3): while $\rho(m)$ counts all hadrons (as given, for example, in the tables of the Particle Data Group [6]) as being at rest in their own confining volume, $\sigma(p^2, p\cdot V)$ is the density of states of an object with mass $m = \sqrt{p^2}$ allowed to move freely in its confining volume instead of being at rest. This fact is also reflected by the dependence of σ on the scalar product $p\cdot V$. Thus σ counts more states than ρ (and contains more information). In order to relate σ to ρ, we restrict this freedom by requiring that p^μ and V^μ be parallel four-vectors, i.e., have a common rest frame. Then $\sigma(p^2, p\cdot V) \to \overline{\sigma}(p^2 = m^2, v_0)$ and the left-hand side represents the internal density of states of a system of mass m at rest in its own volume v_0; this density begins with $[v_0/(2\pi)^3]\delta(m - m_\pi)$ and has a continuum for $m > 2m_\pi$. It therefore might be considered as proportional to an *averaged* mass spectrum (the true one is not yet continuous at $m \geq 2m_\pi$), which asymptotically becomes physically equivalent to $\rho(m)$. We thus have

$$\sigma(p^2, p\cdot V)\Big|_{p\|V} = \overline{\sigma}(m^2, v_0) := \frac{v_0}{(2\pi)^3}\rho_{\rm av}(m) \underset{m\to\infty}{\longrightarrow} \frac{v_0}{(2\pi)^3}\rho(m)\ . \tag{23.4}$$

The precise relationship between $\rho_{\rm av}$, $\rho_{\rm in}$, and ρ will not concern us here – indeed, taking $\rho_{\rm av} = \rho_{\rm in} = \rho$, we will find, solving Eq. (23.3), the result

$$\rho(m) \underset{m\to\infty}{\sim} \frac{c}{m^a}\mathrm{e}^{m/T_0}\ , \quad T_0 \approx m_\pi\ , \quad \frac{3}{2} < a < \frac{7}{2}\ , \tag{23.5}$$

where T_0 is a 'limiting temperature' and where the values of a and T_0 depend on the version of Eq. (23.3) chosen. We will now show how to solve bootstrap equations (23.3) and (23.4) and prove Eq. (23.5).

Solution of the bootstrap equation

For illustrative purposes, let us here consider the bootstrap equation in its simplest form, as proposed by Yellin [7]:

$$B\tau(p^2) = B\delta_0(p^2 - m_\pi^2) + \sum_{n=2}^{\infty}\frac{1}{n!}\int \delta^4\big(p - \sum_{i=1}^{n} p_i\big)\prod_{i=1}^{n} B\tau(p_i^2)\mathrm{d}^4 p_i\ , \tag{23.6}$$

where the relation between τ and ρ is

$$\tau(m^2)\mathrm{d}m^2 = \rho(m)\mathrm{d}m\ , \tag{23.7}$$

and B is a parameter of the model, related to the volume v_0 by a dimensional relation $B \sim v_0 m_\pi$.

The standard method of solving Eq. (23.6) is by Laplace transformation. We introduce two Lorentz-invariant functions:

$$\Phi(\beta) := \int B\tau(p^2)\mathrm{e}^{-\beta_\mu p^\mu}\mathrm{d}^4 p \tag{23.8a}$$

$$\varphi(\beta) := \int B\delta_0(p^2 - m_\pi^2)\mathrm{e}^{-\beta_\mu p^\mu}\mathrm{d}^4 p = 2\pi B m_\pi^2 \frac{K_1(\beta m_\pi)}{\beta m_\pi}\ , \tag{23.8b}$$

where $\beta^0 > 0$, $\beta = (\beta_\mu \beta^\mu)^{1/2}$.

Taking the Laplace transform, as defined by Eq. (23.8), of Eq. (23.6), we obtain

$$\Phi(\beta) = \varphi(\beta) + \mathrm{e}^{\Phi(\beta)} - \Phi(\beta) - 1\ . \tag{23.9}$$

Equation (23.9) can be written

$$\varphi = 2G - \mathrm{e}^G + 1\ , \tag{23.10}$$

and the problem is to invert this equation; that is, to find $G(\varphi) = \Phi(\beta)$. The easiest way to do this is a graphical solution by first plotting $\varphi(G)$ and then considering the curve as $G(\varphi)$. By expanding $\exp(G)$, we see that $\varphi(G) = G + \cdots$; with growing G, the exponential function takes the lead and $\varphi(G)$ goes exponentially to $-\infty$. The maximum lies at $G_0 = \ln 2$ and has the value $\varphi_0 = \ln 4 - 1$; $\varphi''(G_0) \neq 0$ (see Fig. 17.5a). The graphical solution is presented in Fig. 17.5b. From the figure and $\varphi''(G_0) \neq 0$, it follows that $G(\varphi)$ has a square root branch point at φ_0 and is complex for $\varphi > \varphi_0$ [8]. We note that $\varphi_0 = \ln 4 - 1$ corresponds to the value $\beta_0 \approx 1/m_\pi$ in Eq. (23.8b); we also note that φ increases monotonically with $1/\beta$[1].

Thus in Fig. 17.5b, the interval $\varphi \in \{0, \varphi_0\}$ corresponds uniquely to $\beta \in \{\infty, \beta_0\}$. Given φ, we could invert Eq. (23.8a); however, we can obtain the physically interesting information about τ without an explicit inversion.

Since $\int \delta_0(p^2 - m^2)\mathrm{d}m^2 = 1$, we may write

$$\begin{aligned}\Phi(\beta) &= \int \tau(m^2)\mathrm{d}m^2 \int B\delta_0(p^2 - m^2)\mathrm{e}^{-\beta_\mu p^\mu}\mathrm{d}^4 p \\ &= \frac{2\pi B}{\beta}\int \tau(m^2) m K_1(\beta m)\mathrm{d}m^2\ .\end{aligned} \tag{23.11}$$

As we have just seen, $G(\varphi)$ has a square root branch point at φ_0, and so has $\Phi(\beta)$ at β_0, since φ is monotonic in β. Since $K_1(m\beta)$ behaves like $\exp(-\beta m)$ for $m \to \infty$, equation (23.11) can yield a singularity of $\Phi(\beta)$ only if $\tau(m^2)$ grows asymptotically like $\exp(\beta_0 m)$; a square root branch point requires

$$\tau(m^2) \sim \frac{c}{m^3}\mathrm{e}^{\beta_0 m}\ , \tag{23.12}$$

which illustrates the relation (23.5) for $a = 6/2$.

From Eq. (23.10), a Taylor expansion for G around $\varphi = 0$ can be found, with the convergence radius φ_0:

$$G(\varphi) = \sum_{n=1}^{\infty} g_n \varphi^n\ , \tag{23.13}$$

which leads to an analytic form of $\tau(p^2)$:

$$B\tau(p^2) = \sum_{n=1}^{\infty} g_n \Omega_n(p^2, B)\ , \tag{23.14}$$

where Ω_n is the n-pion invariant momentum space (IMS) integral

$$\Omega_n(p^2, B) := \int \delta^4\big(p - \sum_{i=1}^{n} p_i\big)\prod_{i=1}^{n} B\delta_0(p_i^2 - m_\pi^2)\mathrm{d}^4 p_i\ . \tag{23.15}$$

[1] The surprisingly complex analytical structure of this seemingly simple bootstrap function is further explored in: R. Hagedorn and J. Rafelski, "Analytic Structure and Explicit Solution of an Important Implicit Equation," Commun. Math. Phys. **83**, 563 (1982).

The IMS integrals are well-known functions for which powerful computer programs exist. Therefore, Eq. (23.15) is very useful at not too large p^2, since the sum has actually only a finite number of terms – it is cut off at $n \leq \sqrt{p^2}/m_\pi$ by the momentum δ^4-function and by the condition $p_0 \geq m_\pi$.

Had we used the IMS measure in Eq. (23.3), the density of states of the pion gas would have read

$$\sigma_{\mathrm{IMS}}(p^2) = \sum_{n=1}^{\infty} \frac{1}{n!} \Omega_n(p^2, B) \,, \tag{23.16}$$

while now we have Eq. (23.14).

It can be seen that the rapidly decreasing $1/n!$ has been replaced by the (exponentially increasing!) g_n. Thus the Ω_n in Eq. (23.16) have been multiplied by $n!g_n$, which is the total number of possible ways to cluster n objects recursively (admitting clusters of clusters).

It remains to determine the coefficients g_n. This is done most simply by considering the first-order differential equation that G satisfies:

$$1 = \frac{\mathrm{d}G}{\mathrm{d}\varphi}(\varphi + 1 - 2G) \,. \tag{23.17}$$

Inserting Eq. (23.13), we find the recursion relation

$$g_n = -\frac{n-1}{n} g_{n-1} + \sum_{k=1}^{n-1} g_k g_{n-k} \,, \qquad g_0 = 0 \,, \qquad g_1 = 1 \,. \tag{23.18}$$

Given Eqs. (23.13), (23.14), and (23.18), the bootstrap equation (23.6) can be considered as solved.

The nuclear matter bootstrap equation

According to the aims described in Sect. 23.1, we now generalize the bootstrap equation (23.6) to the case of nuclear matter. We postulate the following bootstrap equation for the level density of 'nuclear clusters' with baryon number $b \in (-\infty, \infty)$:

$$\begin{aligned}\frac{2p \cdot V}{(2\pi)^3}\sigma(p,v,b) = {} & \delta^4(V - V_b) C_b \frac{2p \cdot V_b}{(2\pi)^3} \delta_0(p^2 - M_b^2) \\ & + \sum_{k=2}^{\infty} \frac{1}{k!} \int \sum_{\{b_i\}} \delta_k\big(b - \sum_{i=1}^{k} b_i\big)\big(p - \sum_{i=1}^{k} p_i\big)\big(V - \sum_{i=1}^{k} V_i\big) \\ & \times \prod_{i=1}^{k} \frac{2p_i \cdot V_i}{(2\pi)^3} \sigma(p_i, V_i, b_i) \mathrm{d}^4 p_i \mathrm{d}^4 V_i \,. \end{aligned} \tag{23.19}$$

Equation (23.19) is not a single bootstrap equation, but a member (with baryon number b) of an infinite set of coupled integral equations, each having its own input term. The $(k!)^{-1}$ is necessary for correct counting. The non-vanishing pion and nucleon mass ensure that, for any finite p^2, the set (23.19) has only a finite number of equations: $|b_{\max}| \leq \sqrt{p^2}/m_p$. Therefore the solutions for any finite p^2 can (in principle) be built up iteratively by starting with $4m_\pi^2 \leq p^2 \leq m_p^2$ and by increasing this interval stepwise to include higher and higher $|b|$. This, incidentally, also allows us to prove that, for any p, V, b, equations (23.19) have a physical solution.

This equation fulfills the requirements set up in Sect. 23.1:

1. *Conservation of baryon number and clustering of nucleons*. The baryon number (number of baryons minus number of antibaryons) is conserved with the help of the Kronecker $\delta_K(b - \sum b_i)$ function. The infinite set of density functions $\sigma(p,V,b)$ corresponds to the admission of nucleon clusters with any baryon number b, four-momentum p, and four-volume V.
2. *Nucleon (isobar) excitation and internal cluster excitation*. Internal cluster excitation is contained in the $p^2 = m^2$ dependence of $\sigma(p,V,b)$, and single-nucleon (isobar) excitation is contained in the same way in $\sigma(p,V,b=1)$.
3. *Extensivity of nuclear matter*. This is ensured by the volume δ^4-function.
4. *Co-existence of a pion gas*. This is contained in the equation with $b = 0$, and in all others by the presence of factors $\sigma(p_i,V_i,b_i = 0)$ on the right-hand side.
5. *Baryon–antibaryon pair creation (and annihilation)*. This is built in by allowing $-\infty < b_i, b < \infty$. Then on the right-hand side an arbitrary number of clusters $(\sum b_i)$ and anticlusters $(-\sum \overline{b}_i)$ may occur.
6. *'Chemical equilibrium' between all constituents*. This is expressed by the infinite set of coupled integral equations (23.19), which allows all multibody reactions between clusters Q_i,

$$Q_1 + Q_2 + \cdots + Q_n \rightleftarrows Q'_1 + Q'_2 + \cdots + Q'_n$$

   compatible with b and p conservation.

The input terms, except that for $b = 0$ (pion) and for $b = 1$ (nucleon), specify particular features of the model, namely:

i. Details of nuclear interaction may be represented by giving clusters (e.g., alpha particles) a special weight.
ii. Equations (23.19) deal with Boltzmann particles without charge and spin. Introducing spin, isospin, and statistics would be possible but complicated. We can obtain a similar physical effect by assigning to an input nucleus of baryon number b and volume V_b a mass M_b which is different from $b \cdot m_p$.

The mass spectrum for nuclear matter

We introduce

$$\frac{2p \cdot V}{(2\pi)^3} \sigma(p,V,b) = \delta^4 \left(V - V(m,b) \frac{p}{m} \right) \tilde{B}(p^2,b) \tilde{\tau}(p^2,b) \,, \tag{23.20}$$

where the function $\tilde{B}$ describes the p^2, b dependence of the $V \cdot p$ term, while $\tilde{\tau}$ describes that of σ. We now rewrite Eq. (23.19), integrate over the $\prod \mathrm{d}^4 V$, and require that all volume δ^4-functions have the same argument. We find the condition

$$\sum_{i=1}^{n} \left[\frac{V(m_i, b_i)}{m_i} - \frac{V(m)}{m} \right]_\mu p_i^\mu = 0 , \tag{23.21}$$

from which it follows that, for all i,

$$V(m_i, b_i) = \mathscr{A} m_i . \tag{23.22}$$

The constant $\mathscr{A}$ is independent of i and is therefore a parameter of the theory; $\mathscr{A}^{-1}$ is the constant energy density in the natural volume V : thus it is about $1/7\ m_{\mathrm{N}}/\mathrm{fm}^3 =$ 130 MeV/fm^3. We further find that

$$\tilde{B}(m^2, b) := \frac{2V(m,b)}{(2\pi)^3} m = \frac{2\mathscr{A} m^2}{(2\pi)^3} = \tilde{B}(m^2) \tag{23.23}$$

is independent of b.

The volume δ^4-function can now be factored out on both sides of Eq. (23.19), and what remains is a new bootstrap equation for the function $\tilde{B} \cdot \tilde{\tau}(p^2, b)$:

$$\begin{aligned} \tilde{B}(p^2)\tilde{\tau}(p^2,b) &= C_b \tilde{B}(M_b^2)\delta_0(p^2 - M_b^2) \\ &+ \sum_{k=2}^{\infty} \frac{1}{k!} \int \sum_{\{b_i\}} \delta_k\big(b - \sum_{i=1}^{k} b_i\big)\, \delta^4\big(p - \sum_{i=1}^{k} p_i\big) \prod_{i=1}^{k} \tilde{B}(p_i^2)\,\tilde{\tau}(p_i^2, b_i)\mathrm{d}^4 p_i \, . \end{aligned} \tag{23.24}$$

The essential step now consists in the proper extraction of the mass spectrum $\tau(p^2, b)$ from the function $\tilde{\tau}$. Motivated by the form (23.6) of the bootstrap equation, we chose here

$$\tau(p^2, b) := \frac{\tilde{B}(p^2)}{\tilde{B}(M_b^2)} \tilde{\tau}(p^2, b) , \tag{23.25}$$

since we can write the bootstrap equation for the mass spectrum as

$$\begin{aligned} B_b \tau(p^2,b) &= C_b B_b \delta_0(p^2 - M_b^2) \\ &+ \sum_{k=2}^{\infty} \frac{1}{k!} \int \sum_{\{b_i\}} \delta_k\big(b - \sum_{i=1}^{k} b_i\big)\, \delta^4\big(p - \sum_{i=1}^{k} p_i\big) \prod_{i=1}^{k} B_b \tau(p_i^2, b_i)\mathrm{d}^4 p_i \, , \end{aligned} \tag{23.26}$$

where

$$B_b := \tilde{B}(M_b^2) = \frac{2\mathscr{A} M_b^2}{(2\pi)^3} \, .$$

We would like to emphasize that the choice (23.25) leading to Eq. (23.26) is arbitrary. Another very likely choice is to take $\tilde{\tau}$ as the physical mass spectrum. As we have found recently, this significantly simplifies our final formula but complicates the numerical evaluation. Throughout this paper, we will constrain our work to the mass spectrum defined through Eq. (23.25).

The bootstrap equation (23.26) is much richer than that for the pion gas; we have allowed the presence of arbitrarily complicated clusters characterized by the baryonic number b_i. For $b=0$, we have a description of meson fireballs; but in order to understand these fireballs properly, especially when baryon–antibaryon clusters are among their constituents, we have to obtain a solution for the function τ for all values of b.

Laplace and L-transforms of the mass spectrum

In order to solve the nuclear bootstrap equation, a treatment of the b dependence is necessary. This is done by defining the 'L-transform':[2]

$$L\big[f(b)\big] := \sum_{b=-\infty}^{\infty} \lambda^b f(b) := f_\lambda(\lambda)\ . \tag{23.27}$$

Hence $f_\lambda(\lambda) = L\big[f(b)\big]$ is the generating function of $f(b)$. We multiply the entire bootstrap equation by λ^b and sum over b. Defining the L-transform of $\tau(p^2,b)$ and of the input term, respectively,

$$B_\mathrm{N}\tau_\lambda(p^2,\lambda) := \sum_{b=-\infty}^{\infty} \lambda^b B_b \tau(p^2,b)\ , \tag{23.28a}$$

$$B_\mathrm{N}\tau_{0\lambda}(p^2,\lambda) := B_\pi \delta_0(p^2-m_\pi^2) + \sum_{b=1}^{\infty} C_b B_b (\lambda^b+\lambda^{-b})\delta_0(p^2-M_b^2)\ , \tag{23.28b}$$

where

$$B_1 = B_\mathrm{N}\ , \qquad M_1 = m_\mathrm{N}\ ,$$

we find that the bootstrap equation takes the form of the pion bootstrap equation (23.6), but with a much more involved input function $\tau_{0\lambda}$,

$$\tau_\lambda(p^2,\lambda) = \tau_{0\lambda}(p^2,\lambda) + \sum_{k=2}^{\infty} \frac{1}{k!} \int \delta^4\Big(p-\sum_{i=1}^{k} p_i\Big) \prod_{i=1}^{k} \tau_\lambda(p_i^2,\lambda)\mathrm{d}^4 p_i\ . \tag{23.29}$$

That illustrates the general bootstrap philosophy that the input function characterizes the 'raw material', while the integral equation imposes the dynamics on it. The dynamics should be more or less independent of what the raw material is (but it will depend on kinematics, statistics, etc.).

In order to solve Eq. (23.29), we introduce the Laplace transforms of τ_λ and $\tau_{0\lambda}$:

[2] We use the expression 'L-transform' to stress the formal analogy with the Laplace transform: L is the discrete counterpart of $\mathscr{L}$.

$$
\begin{tikzcd}
& B_b\tau(p^2,b) \arrow[dl, "\mathrm{L}"'] \arrow[dr, "\mathscr{L}"] & \\
B_{\mathrm{N}}\tau_\lambda(p^2,\lambda) \arrow[dr, "\mathscr{L}"'] & & \varPhi_b(\beta,b) \arrow[dl, "\mathrm{L}"] \\
& \varPhi(\beta,\lambda) &
\end{tikzcd}
$$

Fig. 23.1 Relations between the mass spectrum $\tau(p^2,b)$ and its Laplace ($\mathscr{L}$) and L-transforms.

$$
\begin{aligned}
\varPhi(\beta,\lambda) &:= \int B_{\mathrm{N}}\tau_\lambda(p^2,\lambda)\mathrm{e}^{-\beta\cdot p}\mathrm{d}^4p \\
&=: \sum_{b=-\infty}^{\infty}\lambda^b\varPhi_b(\beta,b)\,,
\end{aligned}
\tag{23.30}
$$

$$
\varphi(\beta,\lambda) := \int B_{\mathrm{N}}\tau_{0\lambda}(p^2,\lambda)\mathrm{e}^{-\beta\cdot p}\mathrm{d}^4p\,. \tag{23.31}
$$

In analogy with the case of pionic bootstrap, we now find the bootstrap equation [see Eq. (23.9)] considering the Laplace transform of Eq. (23.29):

$$
\varPhi(\beta,\lambda) = \varphi(\beta,\lambda) + \exp\left[\varPhi(\beta,\lambda)\right] - \varPhi(\beta,\lambda) - 1\,. \tag{23.32}
$$

For the input function φ, we have explicitly

$$
\varphi(\beta,\lambda) = \sum_{b=-\infty}^{\infty}\lambda^b\varphi_b(\beta,b)\,, \tag{23.33}
$$

$$
\varphi_b(\beta,b) = \varphi_b(\beta,-b) = C_bB_b2\pi M_b^2\frac{K_1(\beta M_b)}{\beta M_b}\,. \tag{23.34}
$$

In Fig. 23.1, we give a short summary of the relations between the functions arising from τ through application of $\mathscr{L}$ and L-transforms.

The bootstrap equation (23.32) for the doubly transformed function $\varPhi(\beta,\lambda)$ has a real solution wherever in the (β,λ) plane the input function $\varphi < \varphi_0 = \ln 4 - 1$ (see Figs. 17.5a and b). Thus along a curve $\beta_{\mathrm{c}} = f(\lambda_{\mathrm{c}})$ in the (β,λ) plane defined as the boundary of this domain $(\beta_{\mathrm{c}},\lambda_{\mathrm{c}}) = \varphi_0$, a qualitative change in the behavior of the properties of nuclear matter may occur. Quite aside from the physical questions, we have to ask for a mathematical solution of the bootstrap equation beyond this boundary line. As we have previously argued by a recursive argument, a physical solution for $\tau(p^2,b)$ exists for any p^2. Our $\varPhi(\beta,\lambda)$ is the Laplace-L-transform of $B_b\tau(p^2,b)$, which does not exist in this form everywhere in $(0 \le \beta < \infty)\otimes(1 \le \lambda < \infty)$. However, once defined in a domain where it does exist, it fulfills Eq. (23.4), which then permits analytical continuation of $\Phi(\beta,\lambda)$ beyond the limit $\varphi = \varphi_0$ in the whole (complex β) $\otimes$ (complex λ) domain. Thus using the methods of complex analysis, we will be in a position to study the new phases in the future.

We remark here that the analytical continuation beyond φ_0 has never been considered in the case of pionic bootstrap, since there this limit on φ led to a limiting temperature; the energy of fireballs diverged at this point and made a transition from our world to the new domain impossible. Now the presence of baryons changes this – the introduction of λ leads to the existence of a new region with $T < T_0$ but $\varphi > \ln 4 - 1$. We will find in our present model again a boundary $T = T_0$, at which the energy density diverges – but this limit is not at $\varphi = \ln 4 - 1$, except when $\lambda = 1$.

23.3 Thermodynamics

In Sect. 23.2, we solved the bootstrap equation with the help of the Laplace transformation. The same mathematical procedure is used in statistical thermodynamics to obtain the partition function from the density of states. This coincidence has the effect that the Laplace transform $\Phi(\beta)$ of the mass spectrum $\tau(p^2)$ and the Laplace transform $Z(\beta,V)$ of the density of states of a thermodynamical system containing particles with the mass spectrum $\tau(m^2)$ can easily be confounded. We expect a relation between $\Phi(\beta)$ and $Z(\beta,V)$ – and we will exploit it below – but conceptually these two quantities are different.

The partition functions of the one-component ideal gas

Consider an ideal relativistic Boltzmann gas with one sort of particle of mass m enclosed in an arbitrary, macroscopic external volume V_μ^{ex}. The number of states $\{p, \mathrm{d}^4 p\}$ of one particle in the four-volume V_μ^{ex} is

$$\frac{2V_\mu^{\text{ex}} p^\mu}{(2\pi)^3}\delta_0(p^2 - m^2)\mathrm{d}^4 p\ .$$

Thus the one-particle partition function $Z_1^{(0)}$ (in which the superscript denotes 'non-interacting') is defined by

$$Z_1^{(0)}(\beta, V^{\text{ex}}) := \int \frac{2V_\mu^{\text{ex}} p^\mu}{(2\pi)^3}\delta_0(p^2 - m^2)\mathrm{e}^{-\beta_\mu p^\mu}\mathrm{d}^4 p\ . \tag{23.35}$$

Here the four-volume V_μ^{ex} is an arbitrary external parameter[3] [a box of arbitrary volume $V^{\text{ex}} = (V_\mu V^\mu)^{1/2}$ having an arbitrary four-velocity], while before, in Sect. 23.2, we took the volume to be the dynamically determined proper comoving volume of

[3] We will often drop the superscript 'ex' on V when the meaning is unambiguous.

the particle. β_μ has now the meaning of the inverse temperature four-vector, where (the Lorentz invariant) $T = (\beta \cdot \beta)^{-1/2}$ is the temperature in the rest frame of the thermometer. $Z_1^{(0)}$ is by construction a function of the invariants β^2, $V_{\rm ex}^2$, and $\beta^\mu V_\mu^{\rm ex}$. As it seems not very useful to consider a description where the thermometer moves (fast) with respect to the container of a gas, we take here β^μ parallel to $V_{\rm ex}^\mu$. We then obtain in the common rest frame of β^μ and V^μ,

$$Z_1^{(0)}(\beta, V) = \frac{Vm^3}{2\pi^2}\frac{K_2(\beta m)}{\beta m} \ . \tag{23.36}$$

[Notice the difference with Eq. (23.8b).]

From the one-particle partition function, the N-Boltzmann-particle partition is found:

$$Z_N^{(0)}(\beta, V) = \frac{1}{N!} Z_1^{(0)}(\beta, V)^N \ . \tag{23.37}$$

The grand canonical partition function is then

$$Z^{(0)}(\beta, V, \lambda) := \sum_{n=0}^{\infty} \lambda^n Z_n^{(0)}(\beta, V)^n = \mathrm{e}^{\lambda Z_1^{(0)}(\beta, V)} \ , \tag{23.38}$$

with λ being the fugacity. From $Z^{(0)}(\beta, V, \lambda)$ nearly all relevant quantities can be found by logarithmic differentiation, in particular

$$\begin{aligned} \varepsilon^{(0)}(\beta, V, \lambda) &:= -\frac{1}{V}\frac{\partial}{\partial \beta} \ln Z^{(0)}(\beta, V, \lambda) = \text{energy density}\,, \\ P^{(0)}(\beta, V, \lambda) &:= \frac{T}{V} \ln Z^{(0)}(\beta, V, \lambda) = \text{pressure}\,, \\ n^{(0)}(\beta, V, \lambda) &:= \frac{\lambda}{V}\frac{\partial}{\partial \lambda} \ln Z^{(0)}(\beta, V, \lambda) = \text{particle number density}\,, \end{aligned} \tag{23.39}$$

and so on. We introduce the relativistic chemical potential (equal to $\mu_{\text{non-rel}} + m$) by $\lambda = \mathrm{e}^{\beta\mu}$; $\mu = 0$ $(\lambda = 1)$ corresponds to black-body radiation of quanta with rest mass m.

The strongly interacting pion gas

The basic hypothesis is that in many instances an assembly of strongly interacting particles (of one kind[4]) enclosed in an arbitrary volume at arbitrary temperature and chemical potential may be described

- either as a multitude of particles of one kind with a complicated interaction,
- or as a non-interacting phase consisting of an infinity of different species with a mass spectrum appropriate to the interaction in question.

---

[4] The generalization to several different species is straightforward.

This implies that, if the mass spectrum of the interaction is known, replacing the interacting particles by an ideal infinite-component phase and weighting the different components according to the mass spectrum generates the same distortion of phase space as the interaction would do. An example is, for instance, a dilute He gas. Usually, this is not described as an assembly of protons, neutrons, and electrons with a Hamiltonian containing QED and strong interactions; instead, one uses the mass spectrum (here essentially one state with mass, spin, etc., of ^{4}He) and calculates the properties of an ideal Bose gas of He atoms, considering the latter as elementary.

Taking now the attitude that the statistical bootstrap model has provided us with the correct spectrum, the corresponding statistical thermodynamics of strongly interacting particles follows from the formulas of the ideal gas given in Sect. 23.3, now generalized to include the mass spectrum. The one-particle phase-space measure (23.35) now becomes the 'one-fireball' phase-space measure:

$$\mathrm{d}^4 p\sigma_1(p,V) = \frac{2V_\mu^{\mathrm{ex}} p^\mu}{(2\pi)^3}\mathrm{d}^4 p \int \mathrm{d}m^2 \tau(m^2)\delta_0(p^2-m^2) = \frac{2V_\mu^{\mathrm{ex}} p^\mu}{(2\pi)^3}\tau(p^2)\mathrm{d}^4 p\ . \tag{23.40}$$

Accordingly, we find the 'one-fireball' partition function

$$Z_1(\beta,V) = \int \frac{2V_\mu^{\mathrm{ex}} p^\mu}{(2\pi)^3}\tau(p^2)\mathrm{e}^{-\beta_\mu p^\mu}\mathrm{d}^4 p\ . \tag{23.41}$$

Recalling Eq. (23.8a), we find in the common rest frame of V_μ^{ex} and B_μ,

$$Z_1(\beta,V) = -\frac{2V^{\mathrm{ex}}}{B(2\pi)^3}\frac{\partial\Phi(\beta)}{\partial\beta}\ . \tag{23.42}$$

We can now proceed in the same manner as in Eqs. (23.37) to (23.39), which follow now for the interacting particles, dropping the upper index (0). However, in Eq. (23.39), n is now the average number of fireballs present. For this $n(\beta,V,\lambda)$, we have the ideal gas equation (due to the linearity of $\ln Z$ in λ):

$$P = nT\ , \tag{23.43}$$

while the corresponding equation in terms of the average number of pions (contained in all these fireballs together) would look horribly complicated. This result (23.43), which in the framework of this model is exact, shows once more how simple things become once the interaction is hidden in the mass spectrum.

Physics near T_0

We have seen how the bootstrap equation provides us with the function $\Phi(\beta)$, from which $Z_1(\beta,V,\lambda) = \ln Z(\beta,V,\lambda)$ can be calculated; $Z_1(\beta,V,\lambda)$ then serves as the generating function for physical quantities. In all versions of the statistical bootstrap model, we find an exponential mass spectrum

$$\rho(m) \sim \frac{C}{m^a} \mathrm{e}^{m/T_0} \,, \tag{23.44}$$

with T_0 of order m_π. While the small variation of $T_0 \approx m_\pi$ from version to version is of no physical importance, the nature of the system when $T \to T_0$ depends critically on the power a of m in Eq. (23.44). We now study this in order to determine how the behavior of ρ determines the physical properties of fireballs.

Inserting $1 = \int \delta_0(m^2 - p^2)\mathrm{d}m^2$ and replacing $\tau(m^2)\mathrm{d}m^2$ by $\rho(m)\mathrm{d}m$ in Eq. (23.41), we find

$$Z_1(\beta, V) = \frac{V^{\mathrm{ex}} T}{2\pi^2} \int m^2 \rho(m) K_2(m\beta) \mathrm{d}m \,. \tag{23.45}$$

As we are interested in the behavior at $T \to T_0$ ($\beta \to \beta_0$), we denote all quantities which are constant in this limit by the symbol C (at each place where it occurs, C may have a different value and/or dimension). Using Eq. (23.44) and the asymptotic formula $K_2(x) \sim \sqrt{\pi/2x}\,\mathrm{e}^{-x}$, we obtain

$$Z_1(\beta, V) \underset{T \to T_0}{\sim} C \int_M^\infty m^{3/2-a} \mathrm{e}^{-(\beta-\beta_0)m} \mathrm{d}m + C \,. \tag{23.46}$$

Here M is a mass large enough to justify the use of the asymptotic form of K_2 and Eq. (23.44), while $+C$ stands for the non-singular integral from m_π to M. With $\beta - \beta_0 \sim C(T - T_0) = C\Delta T$, we find

$$Z_1(\beta, V) \underset{T \to T_0}{\sim} \begin{cases} C + C\Delta T^{a-5/2} \,, & a \neq 5/2 \,, \\ C - \ln \dfrac{\Delta T}{T_0} \,, & a = 5/2 \,. \end{cases} \tag{23.47}$$

In Table 23.1, we list the most interesting quantities for $a = 1/2, 2/2, \ldots, 8/2$, namely, pressure P, fireball number density n, energy density ε, mean relative fluctuations $\delta\varepsilon/\varepsilon$ of ε, and specific heat $C_V = \mathrm{d}\varepsilon/\mathrm{d}T$. We notice that, as $T \to T_0$ ($\Delta T \to 0$), the energy density diverges for $a < 7/2$. Thus only for $a < 7/2$ can we expect T_0 to be a maximum temperature. For all cases we find for the velocity of sound: $v_s^2 := dP/d\varepsilon \propto \Delta T$.

Thermodynamics of clustered matter

Let us consider a cluster with baryonic number b enclosed in an 'external' four-volume V_μ^{ex}. Then the one-cluster partition function $Z_{1,b}(\beta, V, b)$ is given by Eq. (23.41), the only change being the dependence of the mass spectrum on the baryonic number b. When n such clusters are present, but each with the same b, we find for the n cluster function the usual expression (23.37). When clusters with different b are present, then we have to compute the product of the different contributions. Assume

Table 23.1 Thermodynamic quantities calculated from Eq. (23.47)

a	P	n	ε	$\delta\varepsilon/\varepsilon$	$C_V = \mathrm{d}\varepsilon/\mathrm{d}T$
1/2	$C/\Delta T^2$	$C/\Delta T^2$	$C/\Delta T^3$	$C + C\Delta T$	$C/\Delta T^4$
1	$C/\Delta T^{3/2}$	$C/\Delta T^{3/2}$	$C/\Delta T^{5/2}$	$C + C\Delta T^{3/4}$	$C/\Delta T^{7/2}$
3/2	$C/\Delta T$	$C/\Delta T$	$C/\Delta T^2$	$C + C\Delta T^{1/2}$	$C/\Delta T^3$
2	$C/\Delta T^{1/2}$	$C/\Delta T^{1/2}$	$C/\Delta T^{3/2}$	$C + C\Delta T^{1/4}$	$C/\Delta T^{5/2}$
5/2	$C\ln(T_0/\Delta T)$	$C\ln(T_0/\Delta T)$	$C/\Delta T$	C	$C/\Delta T^2$
3	$P_0 - C\Delta T^{1/2}$	$n_0 - C\Delta T^{1/2}$	$C/\Delta T^{1/2}$	$C/\Delta T^{1/4}$	$C/\Delta T^{3/2}$
7/2	$P_0 - C\Delta T$	$n_0 - C\Delta T$	ε_0	$C/\Delta T^{1/2}$	$C/\Delta T$
4	$P_0 - C\Delta T^{3/2}$	$n_0 - C\Delta T^{3/2}$	$\varepsilon_0 - C\Delta T^{1/2}$	$C/\Delta T^{3/4}$	$C/\Delta T^{1/2}$

that l clusters are present. Then the sum over all possible partitions of b nucleons into l clusters gives us the partition function of b baryons assembled into l clusters:

$$Z_b(\beta,V,b,l) := \sum_{\{n_j\}}^{(l;b)} \prod_{j=-\infty}^{\infty} \frac{1}{n_j!} Z_{1,b}(\beta,V,j)^{n_j} \,. \tag{23.48}$$

The sum is over all partitions of b baryons into l clusters, with n_j being the number of clusters having baryon number j:

$$n_j \in \left\{ n_j \geq 0 \,, \quad \sum_{j=-\infty}^{\infty} n_j = l \,, \quad \sum_{j=-\infty}^{\infty} j n_j = b \right\} .$$

In order to obtain the partition function of an arbitrary number of clusters having together b baryons, we have to compute in Eq. (23.48) the sum over all possible numbers of clusters l, since each such configuration is possible. This has the net effect that the restriction $\sum n_n = l$ is removed:

$$Z_b(\beta,V,b) = \sum_{\{n_j\}} \delta_k\big(b - \sum_{j=-\infty}^{\infty} j n_j\big) \prod_{j=-\infty}^{\infty} \frac{1}{n_j!} Z_{1,b}(\beta,V,j)^{n_j} \,. \tag{23.49}$$

We have made the constraint on baryonic number explicit.

The grand canonical partition function Z is the L-transform of Z_b in Eq. (23.49):

$$Z(\beta,V,b) = \sum_{b=-\infty}^{\infty} \lambda^b Z_b(\beta,V,b) \,. \tag{23.50}$$

It is straightforward to carry out the sum over b when Eq. (23.49) is inserted into Eq. (23.50), and we obtain

$$Z(\beta,V,b) = \sum_{\{n_j\}} \prod_{j=-\infty}^{\infty} \frac{1}{n_j!} \Big[\lambda^j Z_{1,b}(\beta,V,j)\Big]^{n_j} . \tag{23.51}$$

All values of n_j are allowed and the set $\{n_j \geq 0\}$ depends on j only through the fact that there are j members of the set. Since all j are permitted, the order in which the infinite sum and product are evaluated is irrelevant, provided that the sum converges. Under this assumption, we obtain

$$\begin{aligned} Z(\beta,V,b) &= \prod_{j=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} \Big[\lambda^j Z_{1,b}(\beta,V,j)\Big]^{n} \\ &= \exp\left[\sum_{j=-\infty}^{\infty} \lambda^j Z_{1,b}(\beta,V,j)\right] , \end{aligned}$$

which very much resembles the results of Sects. 23.3 and 23.3 [see Eq. (23.38)]:

$$\ln Z(\beta,V,\lambda) = Z_1(\beta,V,\lambda) , \tag{23.52a}$$

$$Z_1(\beta,V,\lambda) = \sum_{j=-\infty}^{\infty} \lambda^j Z_{1,b}(\beta,V,j) . \tag{23.52b}$$

Note that the existence of $Z_1(\beta,V,\lambda)$, the one-cluster grand canonical partition function, is not assured. In fact, often only the canonical partition function Z_b in Eq. (23.49) exists. When an analytical expression for Z can be found, then we can recover the physically relevant quantity Z_b by the inverse L-transform.

Partition function of nuclear matter

Thus we see that we need only to compute the one-cluster grand canonical partition function Z_1 to determine the grand canonical partition function Z in Eq. (23.52). This is an easy task – we recall the definition of the function Φ_b in Eq. (23.30) and find

$$Z_{1,b}(\beta,V,\lambda) = -\frac{2V^{\text{ex}}}{B_b(2\pi)^3} \frac{\partial}{\partial\beta} \Phi_b(\beta,b) \tag{23.53}$$

in the common rest frame of the volume and the 'thermometer', in complete analogy to Eq. (23.42). Consequently,

$$Z_1(\beta,V,\lambda) = -\frac{V}{(2\pi)^3} \frac{\partial}{\partial\beta} \sum_{b=-\infty}^{\infty} \frac{\lambda^b}{B_b} \Phi_b(\beta,b) . \tag{23.54}$$

Were it not for the b dependence of the function B_b [see Eq. (23.26)], $B_b \sim M_b^2$, we would already have the analogue of Eq. (23.42).

In order to proceed further, we have to make an assumption about the b dependence of the cluster mass M_b. For the present, we choose to consider the case

$$M_b = \begin{cases} b m_{\mathrm{N}} \,, & |b| \geq 1 \,, \\ m_\pi \,, & b = 0 \,, \end{cases} \tag{23.55}$$

where m_π and m_{N} are the pion and nucleon masses, respectively. Here we have assumed that the mass of a ground-state cluster is proportional to the baryonic number. We now find for the grand canonical partition function

$$\begin{aligned} \ln Z(\beta,V,\lambda) &= Z_1(\beta,V,\lambda) \\ &= -\frac{V}{\mathscr{A}}\frac{1}{m_\pi^2}\frac{\partial}{\partial\beta}\Phi_b(\beta,0) - \frac{V}{\mathscr{A}}\frac{1}{m_{\mathrm{N}}^2}\frac{\partial}{\partial\beta}\sum_{b\neq 0}\frac{\lambda^b}{b^2}\Phi_b(\beta,b) \,. \end{aligned} \tag{23.56}$$

In order to sum the expression (23.56), we can generate b^{-2} in the sum by a double integral over λ.

While we can sum the general formula (23.56), we will be interested here in properties of bulk nuclear matter: that is, the case when a certain number of nucleons is already present in a given volume. Unless $T \sim T_0$, we expect only moderate contributions from baryon–antibaryon pair production, since $m_{\mathrm{N}} \gg T_0$. Therefore we further simplify our model and *neglect now antibaryon production*. We can implement this by restricting b to be positive in Eq. (23.56). We note that in doing so we allow uncompensated baryon production, which is, for $T \leq T_0$, a small effect[5], since $m_{\mathrm{N}}/T_0 \gtrsim 7$.

The bootstrap equation is then as it was before, viz., (23.9), but the input term that describes only 'raw' pions and nucleons takes the form

$$\varphi(\beta,\lambda) = \varphi_\pi(\beta) + \lambda\,\varphi_{\mathrm{N}}(\beta) \,. \tag{23.57}$$

The sum in Eq. (23.56) can now be obtained by integrating from zero to λ :

$$\begin{aligned} -\frac{\mathscr{A}}{V}\ln Z(\beta,V,\lambda) &= \frac{1}{m_\pi^2}\frac{\partial}{\partial\beta}\Phi_b(\beta,0) \\ &\quad + \frac{1}{m_{\mathrm{N}}^2}\frac{\partial}{\partial\beta}\int_0^\lambda \frac{\mathrm{d}\lambda'}{\lambda'}\int_0^{\lambda'}\frac{\mathrm{d}\lambda''}{\lambda''}\Big[\Phi(\beta,\lambda'') - \Phi_b(\beta,0)\Big] \,. \end{aligned} \tag{23.58}$$

Given the grand canonical partition function $Z(\beta,V,\lambda)$, we want to obtain the quantities of physical interest for nuclear matter. The energy density, pressure, and baryon number density are, respectively,

[5] However, we did find that as temperature rises this effect causes the rise of baryon density, a complete solution is presented in Chapter 27.

$$\varepsilon(\beta,V,\lambda) \;=\; -\frac{1}{V}\frac{\partial}{\partial\beta}\ln Z(\beta,V,\lambda)\;, \tag{23.59a}$$

$$P(\beta,V,\lambda) \;=\; \frac{T}{V}\ln Z(\beta,V,\lambda)\;, \tag{23.59b}$$

$$\frac{\langle b\rangle}{V} \;=:\; \nu(\beta,V,\lambda) = \frac{\lambda}{V}\frac{\lambda}{\partial\lambda}\ln Z(\beta,V,\lambda)\;. \tag{23.59c}$$

Of further physical interest is the energy per baryon $\varepsilon_b = \varepsilon/\nu$.

In the next section, we illustrate our model by some numerical results obtained by studying Eqs. (23.57) to (23.59).

23.4 Properties of Nuclear Matter in the Bootstrap Model

The different phases

In this section, we will study the physical properties of our model. We begin by considering in more detail the point $\varphi_0 = \ln 4 - 1$, where the function $G(\varphi)$ [see Eq. (23.10)] has a square root singularity. This point corresponds to a curve $\lambda_c = f(\beta_c)$ in the (λ,β) plane, defined implicitly by the equation

$$\varphi_0 = \ln 4 - 1 = \varphi_\pi(\beta_c) + \lambda_c\varphi_N(\beta_c)\;.$$

Thus,

$$\lambda_c = \frac{\ln 4 - 1 - \varphi_\pi(\beta_c)}{\varphi_N(\beta_c)}\;, \qquad \varphi_i(\beta) = \frac{\mathscr{A}m_i^4}{2\pi^2}\frac{K_1(\beta m_i)}{\beta m_i}\;, \quad i = \pi, \mathrm{N}\,. \tag{23.60}$$

We introduce the chemical potential μ by $\lambda = \mathrm{e}^{\beta\mu}$ and consider the function $\mu_c = f(T_c)$, where $T_c = \beta_c^{-1}$, as follows from Eq. (23.60). As shown in Fig. 23.2, this line divides the (μ,T) plane into two parts. For $\mu < \mu_c(T_c)$, we have $\varphi < \varphi_0$ and we know that the grand canonical description is valid there. At $\varphi = \varphi_0$, we are on the critical curve corresponding to a singularity of $\ln Z$. We record the interesting behavior of $\mu_c(T_c)$ for small T_c (large β_c). This can be found analytically, employing the asymptotic expansion for the Hankel functions; but we will not pursue this point here. From Fig. 23.2, we see that μ_c increases initially as a function of T_c. Beyond a certain point it drops continuously until $\mu_c = 0$ at $T_c = T_0$. Here $\mathscr{A}$ was chosen to give $T_0 \sim 150$ MeV, which is a phenomenologically good value for a hadronic bootstrap. We note that the behavior of the chemical potential for $T \neq T_0$ is similar even when the pion term is switched off entirely (dashed line in Fig. 23.2).

The limiting temperature T_0 is now a solution of Eq. (23.60) with $\lambda_c = 1$. However, since the nuclear term is exponentially small at $\beta_c \approx 1/m_\pi \approx 1/T_0$, we expect that the limiting temperature is but little changed from that of pionic bootstrap. The change of T_0 induced by the possible baryon production is obtained by expanding

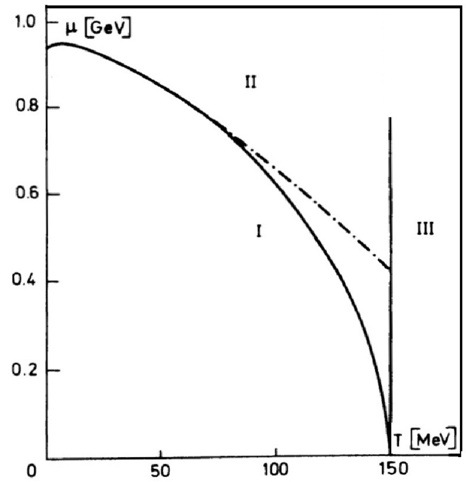


Fig. 23.2 The critical curve $\mu_c = \mu_c(T_c)$ in the (μ, T) plane, separating the gaseous phase (I) from the 'liquid' phase (II). The *dot-dashed line* would be the critical curve if pions were excluded. Region III is inaccessible ($T > T_0$): infinite energy density. For $T = 0$, the critical chemical potential equals the nucleon mass; note that this is not its maximum value.

Eq. (23.60) around β_0. We find that the change of T_0 is negative: the limiting temperature is slightly lowered (by about 10 MeV) by the presence of nucleons.

There are three domains shown in Fig. 23.2. In domain I, enclosed by the function $\mu_c(T_c)$, the grand canonical description is valid; in domain II, above the critical curve, we have $\varphi > \ln 4 - 1$, but $T < T_0$. In this region, the description of physical quantities should be canonical, since the grand canonical partition function does not exist for $\varphi < \varphi_0$. It is possible, however, to consider the analytical continuation of the grand canonical function into this domain – inverse L-transform can then be used to find the canonical quantities. Henceforth, we will call region I the gaseous phase (because it contains the region of small density), and region II the 'liquid' phase (because it is approached if at fixed temperature the baryon density, i.e., λ or μ, increases). Region III, characterized by $T > T_0$, is a domain that cannot be reached from the physical phases in those bootstrap models that give divergent energy density at $T = T_0$. We have found, however, other versions of the nuclear bootstrap model which allow a transition even to this region – however, we will not discuss this possibility here.

We cannot exclude that, in models with more general input functions φ, a further phase develops for large baryon densities. However, this is not so within our simple model of pions and nucleons, where we neglect most of the details of nuclear structure. In particular, for $T \to 0$ and for μ corresponding to $v/v_0 \sim 1$, we might need more detailed input than we have considered in the present simplified study.

Baryon density in the gaseous phase

We begin with a short description of the numerical methods. We need to compute the different derivatives with respect to β and λ of Eq. (23.58). Since $\Phi(\beta,\lambda) = G\big[\varphi(\beta,\lambda)\big]$, we need only to have the function $G(\varphi)$ and its derivatives with sufficient precision in order to calculate the quantities of physical interest. This is done by considering the expansion of $G(\varphi)$ at φ_0:

$$G(\varphi) = G_0 - (\varphi_0 - \varphi)^{1/2} - \frac{1}{6}(\varphi_0 - \varphi)^{2/2} - \frac{1}{36}(\varphi_0 - \varphi)^{3/2} + \Delta G(\varphi)\ . \quad (23.61)$$

This equation defines the remainder ΔG, which can be taken to have the polynomial form. Since we know the inverse function $\varphi = \varphi(G)$, we can easily fit ΔG. We find that, even for a quite small degree of the polynomial N $(= 3)$, already a very satisfactory result is obtained. This is partly due to the fact that Eq. (23.61) with ΔG neglected is, in itself, a very good approximation of G since the maximum error occurs at $\varphi = 0$ and is $\Delta G(\varphi) = 5.7 \times 10^{-4}$. Also, at $\varphi = \varphi_0$, the proper analytic behavior is obtained from Eq. (23.61) for $G(\varphi)$ *and* its first and second derivatives. Thus to one per mille accuracy, the expansion (23.61) is already quite adequate; however, in numerical calculations, we have included the remainder ΔG in order to achieve a relative accuracy of 10^{-8}. Another merit of the expansion (23.61) is its analytical integrability in Eq. (23.58). Thus we have succeeded in obtaining $\ln Z$ in terms of known functions. The computation of the different physical quantities, though tedious, is an elementary exercise now. The results were obtained and graphically processed by the CERN Interactive Computing System SIGMA [9]. An independent check of our calculations has been done with the Yellin expansion (23.13), wherever this was possible.

We begin the discussion of our results by considering the baryon number density ν [see Eq. (23.59c)] along the gas–'liquid' phase limit. As a unit of V, we will choose the 'elementary' volume of one baryon, $V_{\mathrm{N}} = m_{\mathrm{N}} A$, as introduced in Eq. (23.22), along with the constant $\mathscr{A}$ (not related to atomic and/or baryon number here denoted as b). The baryon number contained in the elementary volume V_{N} now follows from Eqs. (23.59c) and (23.58):

$$V_{\mathrm{N}}\nu = -\frac{1}{m_{\mathrm{N}}}\frac{\partial}{\partial\beta}\int_0^{\lambda}\frac{\mathrm{d}\lambda'}{\lambda'}\Big[G(\varphi_\pi + \lambda'\varphi_{\mathrm{N}}) - G(\varphi_\pi)\Big]\ . \quad (23.62)$$

We find, upon differentiation,

$$\begin{aligned} V_{\mathrm{N}}\nu \;=\; &-\frac{1}{m_{\mathrm{N}}}\frac{\partial\varphi_\pi}{\partial\beta}\frac{\partial}{\partial\varphi_\pi}\int_0^{\lambda}\frac{\mathrm{d}\lambda'}{\lambda'}\Big[G(\varphi_\pi + \lambda'\varphi_{\mathrm{N}}) - G(\varphi_\pi)\Big] \\ &-\frac{1}{m_{\mathrm{N}}}\frac{\partial\varphi_{\mathrm{N}}}{\partial\beta}\frac{1}{\varphi_{\mathrm{N}}}\Big[G(\varphi_\pi + \lambda\varphi_{\mathrm{N}}) - G(\varphi_\pi)\Big]\ . \end{aligned} \quad (23.63)$$

At the critical line, we just have $\varphi_\pi + \lambda\varphi_{\mathrm{N}} = \varphi_0$, so

$$V_{\mathrm{N}}\nu\Big|_{\mathrm{crit}} = -\left[\ln 2 - G(\varphi_\pi)\right]\frac{1}{m_\pi}\frac{\partial}{\partial\beta}\ln\varphi_{\mathrm{N}}\Big|_{\mathrm{crit}} - \frac{1}{m_{\mathrm{N}}}\frac{\partial\varphi_\pi}{\partial\beta}\frac{\partial}{\partial\varphi_\pi}\int_0^\lambda\frac{\mathrm{d}\lambda'}{\lambda'}\left[G(\varphi_\pi+\lambda'\varphi_{\mathrm{N}})-G(\varphi_\pi)\right]\Big|_{\mathrm{crit}}. \tag{23.64}$$

The first term is the only one remaining in the absence of pions and is shown as a dash-dotted line in Fig. 23.3. Since for $T \le T_0$ we have $m_{\mathrm{N}}/T \gg 1$, the asymptotic form for the Bessel function in φ_{N} can be used to determine ν. Therefore, we find

$$V_{\mathrm{N}}\nu\Big|_{\text{crit, no pions}} = \left(1+\frac{3T}{2m_{\mathrm{N}}}\right)\ln 2\,. \tag{23.65}$$

Even including pions, this expression is correct for low temperatures since, as before, $\varphi_{\pi,\mathrm{crit}} \ll \lambda_{\mathrm{crit}}\varphi_{\mathrm{N,\,crit}}$. The values obtained from Eq. (23.64) are shown in Fig. 23.3. We see that the onset of the pion component lowers the phase transition density, but at high temperatures, the density again increases sharply[6].

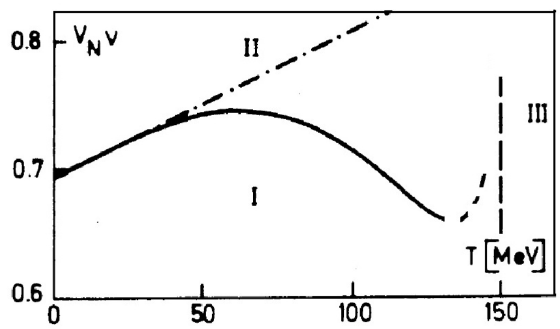


Fig. 23.3 Critical baryon number per nucleon volume V_{N} as a function of the temperature. The *dot-dashed line* results if pions are excluded. The unexpected shape of the critical curve is seen to be due to the coexistence of pions and nucleons. Region I is gaseous, while region II is fluid.

We notice that, for $T < T_0 - \delta$ (with δ a few MeV), the transition from gaseous to 'liquid phases' occurs always below one (one baryon per unit volume is by definition the normal nuclear density). This justifies a posteriori our choice for the names of the different phases.

In Fig. 23.4, we show the baryon density in the gaseous phase: in Fig. 23.4a as a function of chemical potential with temperature being the parameter (isotherms), in Fig. 23.4b as a function of temperature, with the chemical potential as a parameter. In Figs. 23.4c and d, we have eliminated the chemical potential from Fig. 23.4a and

---

[6] This mirrors the behavior of the rapidly changing factor $\exp[(m-\mu)/T]$; hadronic matter at phase boundary is meson dominated for $T > m_\pi/2$ MeV. Moreover, after we allwed for antimatter production (see solution presented in Chapter 27) the net baryon density continues to decrease for $T \to T_0$.

replaced it by the pressure [see Eq. (23.59b)] in units of $P_0 = m_N/V_N = \mathscr{A}^{-1}$. In Fig. 23.4a, $\mu = 0$ implies a finite baryon density, particularly noticeable for $T \gtrsim 120$ MeV[7].

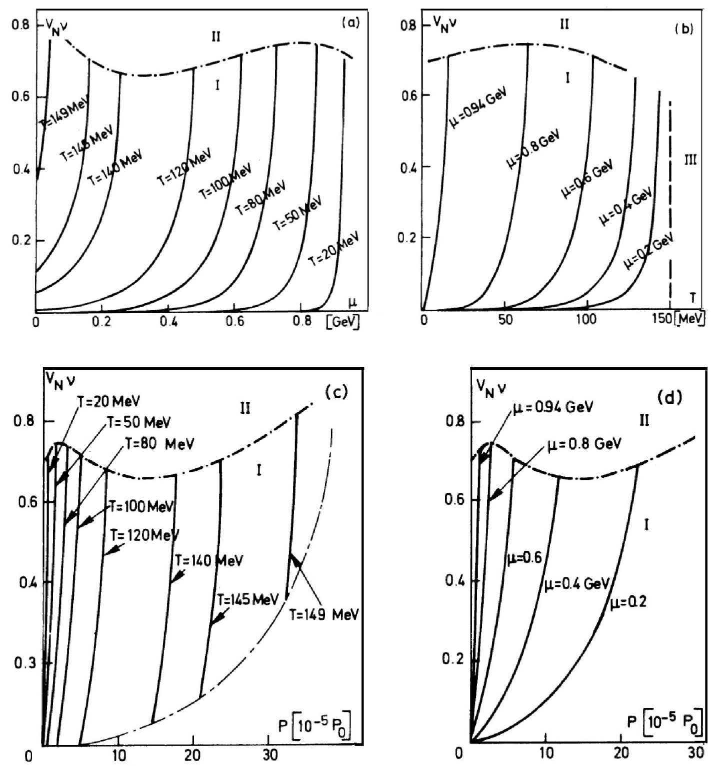


Fig. 23.4 Baryon number per nucleon volume V_N in various representations up to the critical curve: (**a**) against chemical potential with isotherms, (**b**) against the temperature with μ as parameter, (**c**) against the pressure with isotherms, (**d**) against the pressure with μ as parameter. $P_0 = \mathscr{A}^{-1} \approx$ proton rest energy density ('internal proton pressure'). The *dash-dotted curve* is the critical curve, and region II the liquid phase. The white lower right corner in (**c**) is due to the impossibility of having antibaryons at high temperature (asymmetry of our input term).

Baryon energy in the gaseous phase

The energy contained in the unit volume V_N can easily be obtained from Eqs. (23.59a) and (23.58):

[7] This is another artifact of the approximation to ignore antibaryons.

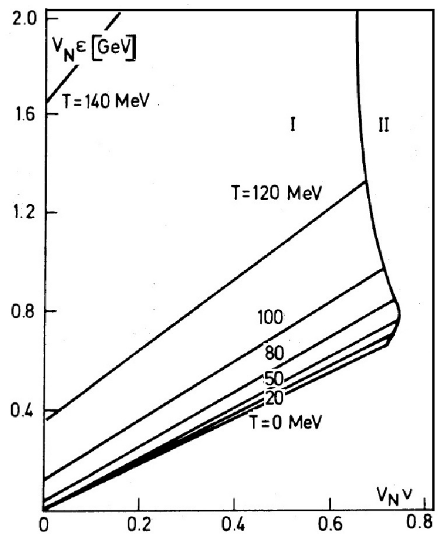


Fig. 23.5 Energy per nucleon volume V_N as a function of baryon number per nucleon volume. Isotherms up to the critical curve separating gas (I) from liquid (II). As the rest mass is included in the energy per nucleon volume, the lower part of the diagram remains empty.

$$V_N\varepsilon(\beta,\lambda) = \frac{m_N}{m_\pi^2}\frac{\partial^2}{\partial\beta^2}G(\varphi_\pi)+\frac{1}{m_N}\frac{\partial^2}{\partial\beta^2}\int_0^\lambda\frac{d\lambda'}{\lambda'}\int_0^{\lambda'}\frac{d\lambda''}{\lambda''}\Big[G(\varphi_\pi+\lambda''\varphi_N)-G(\varphi_\pi)\Big]\,. \tag{23.66}$$

Both equations (23.66) and (23.62) are functions of μ and T, and we can eliminate numerically either one of these physical parameters in Eq. (23.66) and replace it by ν [see Eq. (23.62)]. Since T has a better intuitive meaning, we eliminate the chemical potential from Eq. (23.66) and consider

$$\varepsilon(\beta,\lambda) = \varepsilon\big(\beta,\lambda(\beta,\nu)\big) = \varepsilon_\nu(\beta,\nu)\,. \tag{23.67}$$

dropping henceforth the lower subscript ν. The results are shown in Fig. 23.5. Here the isotherms $T=$ constant are shown for $V_N\varepsilon$ as a function of $V_N\nu$. We record the nearly linear behavior (in the gas phase) of the energy density: $\varepsilon \sim C_1 + C_2\nu$, with temperature-dependent constants C_1, C_2. We recall that, for very small $\nu(T)$, our neglect of antibaryons is not justified. But above $V_N\nu = 0.1$ and $T \lesssim 120$ MeV, our results should be independent of this approximation.

Even better insight can be obtained by inspecting the energy per baryon, excluding the rest mass,

$$\mathscr{E}_b^{nr} := \frac{\varepsilon(\beta,\nu)}{\nu} - m_N = \mathscr{E}_b - m_N\,, \tag{23.68}$$

shown in Fig. 23.6. For small temperatures *and* densities, this should be just the usual $3T/2$, which we actually find for $T = 20$ MeV. For higher temperatures, as we can see in Fig. 23.6b, this is the lower limit of the thermal and interaction energy $\mathscr{E}_b^{nr} \equiv$. For $T = 50$ MeV and higher, we have a large pion component; thus the energy per baryon (total energy divided by total baryon number), which also includes the energy of the pions, stays high above the lower limit $3T/2$. We note that our interaction energy is, by definition, positive. Our nuclear mass m_{N} for the input nucleon should, in principle, include all the binding effects at saturation, and thus be really $m_{\mathrm{N}} - E_{\mathrm{B}}$. Therefore, at densities lower than the saturation density in the gaseous phase, the thermal energy $3T/2$ is the lower limit on the energy per baryon. Furthermore, we note that, within our model, the thermal energy dominates the picture between ~ 20 and ~ 60 MeV, at which point the onset of pion and resonance excitation becomes important.

It is a straightforward matter to isolate the thermal term from Eq. (23.66). In fact, recalling the rules of chain differentiation, we obtain from Eqs. (23.59a), (23.59c), and (23.58),

$$\varepsilon = -\frac{1}{V}\frac{\partial \varphi_\pi}{\partial \beta}\frac{\partial}{\partial \varphi_\pi}\ln Z - \nu \frac{\partial}{\partial \beta}\ln \varphi_{\mathrm{N}} \,. \tag{23.69}$$

The first term expresses the pion–nucleon interacting component and, as discussed in Sect. 23.4, it is small at temperatures below 60 MeV. The second term is just the 'free' nucleon term at density ν, which in the non-relativistic limit gives us the usual $3T\nu/2$.

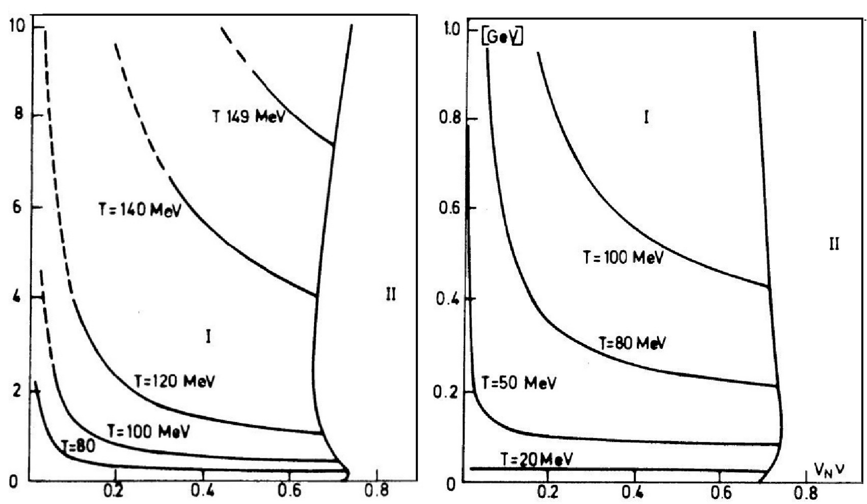


Fig. 23.6 Energy per baryon minus rest mass $\mathscr{E}_b^{nr} = E/b - m_{\mathrm{N}}$ as a function of baryon number per nucleon volume $V_{\mathrm{N}}\nu$: (Left:from 0 to 10 GeV: right: from 0 to 1 GeV with isotherms up to the critical curve separating gas (I) from fluid (II).

23.5 Summary

We have generalized the Statistical Bootstrap Model in a suitable way, which allows for the description of clustering hadron matter with constant energy density and a conserved quantum number. We apply our theory to the particular case of nuclear matter which, in the thermodynamic equilibrium, consists at finite temperature of nuclear clusters and their excitations, pions, and mesonic and baryonic resonances. Although in the general theoretical part of our work we have maintained baryon number conservation, in the numerical part, we study the properties of nuclear matter, neglecting the antibaryon production.

At many stages of our model, relatively arbitrary assumptions have been made, which can only be justified a posteriori by a comparison with the experimental results; or, perhaps, by the beauty and simplicity of the theoretical relation and intuition and experience. In particular:

1. we assume an ad hoc ansatz for the bootstrap equation for the nuclear level density [see Eq. (23.19)];
2. we assume a relation of σ with the mass spectrum [see Eqs. (23.25) and (23.20)];
3. we take the proper volume as being parallel to the momentum of the fireball;
4. we assume that the natural volumes of the input particles grow with their mass.

As a consequence of (iii) and (iv), we have found that the energy density of fireballs is constant [see Eq. (23.22)] and equal to $\mathscr{A}^{-1}$, which is the only arbitrary parameter of our model. Although our calculations are more in the nature of an exploratory study than a final result, we believe that some of the general features we find in our model are relatively model-independent and could survive further elaboration.

1. Considering the grand canonical partition function, depending on the chemical potential and temperature, we find three different situations:
   i. a gaseous state (containing the empty vacuum for $\mu \to 0$, $T = 0$), characterized by the presence of easily movable but strongly interacting nuclei and pions, all in arbitrary states of excitation;
   ii. a 'liquid' phase at larger baryon densities; and
   iii. a supercritical (unphysical) region above $T = T_0 = 150$ MeV, where the energy density becomes infinite.
2. The transition to the 'liquid' phase occurs at about 0.65–0.75 of the normal nuclear number density and at finite energy density, except when T approaches T_0, where the pure gaseous phase persists through high density and where the energy density becomes very large. We would like to mention now that what we have called throughout this paper the 'liquid' phase is really the coexistence of two phases, vapor and liquid, in equilibrium. We are currently working on a description of the high density region beyond the phase transition from gas to liquid.

3. In our actual description, we find a limiting temperature $T_0 \approx 150$ MeV. At this temperature, the energy density diverges. We have noted, however, that this is a subtle point which touches on the limits of validity of our present interpretation of the mass spectrum. In this respect, we recall that the volume of fireballs now grows with the fireball mass – thus the average density should be finite for $T \to T_0$. In a consistent model, we expect a finite energy density at T_0, so that the presently forbidden region beyond T_0 will now become accessible.
4. Below 60 MeV, we find that the energy per baryon obeys roughly the simple relation $\sim 3T/2$; however, below 20 MeV, our model includes too little nuclear structure to have enough predictive power. Above 60 MeV, we find that pion degrees of freedom absorb an increasing amount of the total energy, so that the 'energy per baryon' (the total energy/number of baryons) exceeds more and more the energy which the baryons themselves carry.

Looking ahead, we hope to enlarge our model by making the input more elaborate, by maintaining the particle–antiparticle symmetry, and by considering the particular importance of alpha clusters. It seems that a profound study of the 'liquid' phase will be rewarding since much of the structure of the liquid (maybe even the existence of a new 'solid' phase) depends on the amount of nucleon structure we include in the input terms. An obvious first step in this direction is the possible introduction of effective masses ($<$ free masses) of the bound nucleons, a feature that is very likely relevant to the understanding of the saturation of nuclear matter in the bootstrap description. We must also incorporate Fermi and Bose statistics and investigate models leading to a finite energy density at T_0.

References

1. R. Hagedorn and J. Rafelski: Phys. Lett. B **97** 136 (1980)
2. R. Hagedorn, I. Montvay, and J. Rafelski: "Thermodynamics of nuclear matter from statistical bootstrap model." Preprint CERN-TH-2605 (1979); In *Hadronic Matter at Extreme Energy Density*, pp:49-148 (Plenum Press 1980) Eds: N. Cabibbo and L. Sertorio
3. H. Koppe: Phys. Rev. **76**, 688 (1949)
4. E. Fermi: Progr. Theor. Phys. **5**, 570 (1950)
5. S.Z. Belenkij: Nucl. Phys. **2**, 259 (1956)
6. Particle Data Group: Phys. Lett. **75B**, No. 1 (1978)
7. J. Yellin: Nucl. Phys. B **52**, 583 (1973)
8. W. Nahm: Nucl. Phys. B **45**, 525 (1972)
9. R. Hagedorn, J. Reinfelds, C. Vandoni, and L. Van Hove: *SIGMA, a new language for interactive array-oriented computing* Yellow Report CERN 78-12 (1978)

Chapter 24
On a Possible Phase Transition Between Hadron Matter and Quark-Gluon Matter – 1981

Rolf Hagedorn

Abstract We employ the technique of the analytically continued grand canonical pressure partition function to show that, under physically meaningful boundary conditions (non-existence of external confining vessels, i.e., no fixed volumes), the energy density and similar intensive quantities do indeed have, in a statistical bootstrap model of extended hadrons (van der Waals type volume corrections), the singularity claimed in previous papers. Earlier results obtained with an entirely different technique (which had been criticized) are recovered and shown to be correct. The technique used here is useful in all cases where the volume is not imposed from the outside but results from the internal dynamics of the system, as is generally the case in high energy physics and astrophysics.

24.1 Introduction

Hadrons have finite sizes and consist internally of quarks, antiquarks, and gluons, though none of these constituents has ever been observed as a free particle. They seem to be confined to the inside of hadrons.

Consider a hadron gas at temperature T. At low T, it behaves more or less as an ideal gas, if T increases to the order of $\lesssim$ 100 MeV, pion creation sets in and if T is increased further, heavier resonances and baryon–antibaryon pairs are produced. When T becomes sufficiently large, particle production becomes so strong that the energy density of the 'gas' reaches the value of the internal energy density of its constituents. In other words, the hadrons (having finite volume) begin to overlap and finally might form one single hadron. At that stage we have no longer to do with

---

Preprint CERN-TH-3207 dated 30 November 1981, unpublished; See also the expanded version Preprint CERN-TH-3392 dated 18 August 1982 with title "The Pressure Ensemble as a Tool for Describing the Hadron-Quark Phase Transition" published in Z. Phys. C **17**, pp. 265-281 (1983)

(deceased) CERN-TH, 1211 Geneva 23, Switzerland

a hadron gas, but with an interacting quark-gluon gas. Increasing the temperature further would lead to a free quark-gluon gas confined to a macroscopic hadron [1].

The following question arises: is this transition from a hadron gas to a quark-gluon gas a smooth transition (like ionization) or a phase transition? There are two approaches to this problem: experimental and theoretical. Experimental facts suggest a phase transition: mean transverse momenta of particles produced in high energy hadron collisions may be interpreted in terms of a temperature which, with increasing collision energy, rapidly reaches a limit of the order of 160 MeV [2] (the exact value is difficult to determine because many secondary effects arising from the spacetime history of a collision disturb the ideal picture of a phase transition at a certain critical temperature). Limiting temperatures indicate phase transitions.

The theoretical approach is suffering from lack of a theory. We only have models. There is a choice of models describing the hadron side and another choice of models for the quark-gluon side, but no analytical model which contains both [3]. A single, closed and consistent analytical model unifying both aspects would be ideal. If we had one which described hadronic as well as quark-gluon systems, we could use it to find theoretical support for either a phase transition or for a smooth transition. A phase transition would be indicated by a singularity (pole, branch point) in the grand canonical partition function at some real temperature, while a smooth transition would require the absence of singularities on the real T axis.

In this situation, the general habit is to take some hadron gas (or nuclear matter) model and some quark-gluon model and try to fit them together. This procedure leads then to two different partition functions, one for low T (hadron side) and one for high T (quark-gluon side). If the two pressure curves thus obtained cross at some temperature, it is often claimed that a phase transition has thereby been established and located. This is unjustified, as one easily sees from a counterexample: a dilute hydrogen gas might, according to this philosophy, be described as an ideal gas of $2N$ protons plus $2N$ electrons. The pressure curves do cross, but we know that, in this system, there is no phase transition; instead a smooth shift of the chemical equilibrium between molecules, atoms, ions, and electrons takes place when the temperature changes.

Only the explicit exhibition of a singularity (in at least one of the two models to be fitted together) proves that the model under consideration has a phase transition (in the vicinity of the singularity).

The statistical bootstrap model of hadronic matter in its most recent form has been claimed to have a singular curve in the μ, T diagram, along which the energy density is constant and equal to the bag energy density, i.e., the energy density of the hadronic clusters constituting the gas, while the pressure vanishes there [4]. Taking this singularity as indicating a phase transition to a quark-gluon phase [5] seems most natural; the more so as the description of the other side, in terms of a free quark-gluon gas with perturbative corrections [6], leads to vanishing pressure and to the usual bag energy density in the same μ, T region where the hadron critical curve lies.

The model of [4] is a statistical bootstrap model with baryon number conservation and proper volumes of the constituent hadrons and hadron clusters. These proper volumes grow in proportion to the cluster mass. As the strong interaction is in this model represented by all possible particle reactions (hadron chemistry), the number of particles is not conserved and, in calculating the partition function, summed from 0 to infinity. The mass spectrum (listing all possible hadrons and clusters) turns out to be exponentially increasing with the cluster mass. It is this exponential increase which generates the singularity via an integration over masses up to infinity.

In a recent paper [7], the results of [4] were criticized on the basis of the following argument: if particle volumes grow in proportion to the mass, the mass integration is necessarily cut off when the sum of all particle volumes reaches the externally given volume V. Likewise the sum over particle numbers is cut off. Thus, trivially, no singularity can occur. What is not trivial is that, as the authors show, the thermodynamic limit

$$\lim_{V\to\infty}\frac{1}{V}\ln Z(\beta,V) \tag{24.1}$$

exists for all β. Hence, even in this limit, still no singularity exists in spite of the exponential spectrum and in spite of the fact that now integrations and sums do go to infinity. This proof does not, however, apply to the situation under which the singularity was found. In [4], the limit was not taken in the usual way, viz., as in Eq. (24.1): first calculate $\ln Z$ for fixed V, then let $V\to\infty$. Instead, the 'available volume' $\Delta = V-\sum V_i$, where V_i is the proper volume of the ith particle, was used as a volume parameter and kept constant. Thus $V=\Delta+\sum V_i$, so that, when sums over particle numbers and integrals over masses were done, V was pushed to infinity. Then expectation values $\langle V(\beta,\Delta,\lambda)\rangle$, $\langle E(\beta,\Delta,\lambda)\rangle$, etc., could be calculated and densities could be defined by $\langle E(\beta,\Delta,\lambda)\rangle/\langle V(\beta,\Delta,\lambda)\rangle$, etc., which did indeed show a singularity. Since therefore the existence of a singularity depends on the limiting procedure, it seems important to clear this up.

A simple example shows that there is nothing like a universally 'correct' limiting procedure, but that different limiting procedures correspond to different physical situations. Imagine a high pressure container of volume V filled completely with water at room temperature and atmospheric pressure and then hermetically closed. One may heat it up to any temperature and the water will not boil; putting infinitely many such boxes together and removing interior walls ($V\to\infty$) will change nothing. If, on the other hand, one closes the vessel by a movable piston, one sees the water boil if pressure and temperature fall in a certain interval. In this last case, the water pushes the volume to ever larger values similarly to the situation considered in [4].

We believe that at temperatures and densities where hadron matter changes into quark-gluon matter, no fixed volumes should be used in theoretical considerations, since boxes do not exist in this regime. Forces keeping a system together (the tendency to cluster is just such an internal force, while gravity might be considered as an external one) control pressure and densities rather than volume.

The method of the 'available volume' Δ used in [4] seems therefore to be adapted to reality. Nevertheless, one may argue that even such a Δ cannot be controlled and therefore should not be used as an external variable.

In this paper, the results of [4] will be rederived in a different way which does not make use of a volume-like variable. The technical tool is the grand canonical pressure partition function.

Our units are $\hbar = c = k = 1$, mass unit MeV, metric $(1,-1,-1,-1)$. Notation is as in [4].

24.2 The Grand Canonical Pressure Partition Function

Introduction

Given the grand canonical partition function $Z(\beta,V,\lambda)$, where $\beta = 1/T$, V a fixed external volume, and $\lambda = \exp(\mu/T)$ a fugacity ensuring the conservation of some charge-type quantum number Q, the grand canonical pressure partition function $\Pi(\beta,\xi,\lambda)$ is defined by [8]

$$\Pi(\beta,\xi,\lambda) := \int_0^\infty \mathrm{d}V \mathrm{e}^{-\xi V} Z(\beta,V,\lambda) \,, \tag{24.2}$$

where ξ is a new, intensive parameter related to the volume in a similar way as β is related to the energy and μ to some conserved quantity. The larger ξ, the stronger is the exponential volume suppression in the integral of Eq. (24.2). Thus ξ is a measure for the pressure and hence the name of this partition function.

Rewriting Eq. (24.2),

$$\Pi(\beta,\xi,\lambda) = \int_0^\infty \mathrm{d}V \mathrm{e}^{-V\left[\xi - \ln Z(\beta,V,\lambda)/V\right]} \,, \tag{24.3}$$

we can read off for which values of ξ the integral converges, provided that the thermodynamic limit

$$\lim_{V\to\infty} \frac{1}{V} \ln Z(\beta,V,\lambda)$$

exists:

$$\xi > \xi_0(\beta,\lambda) := \lim_{V\to\infty} \left[\ln Z(\beta,V,\lambda)/V\right] \equiv \beta P(\beta,\lambda) \,, \tag{24.4}$$

where P is the pressure.

We define a function $g(\beta,V,\lambda)$ as the difference between $\ln Z/V$ and its limit:

$$g(\beta,V,\lambda) := \xi_0(\beta,\lambda) - \frac{1}{V} \ln Z(\beta,V,\lambda) \,. \tag{24.5}$$

It can be shown under very general conditions [8, item (d)] that, if the thermodynamic limit exists, this limit commutes with the differential operators $\partial/\partial\beta$ and $\partial/\partial\lambda$. We assume these conditions to be fulfilled. Thus,

$$\left.\begin{array}{l}\lim\limits_{V\to\infty} g(\beta,V,\lambda) \\ \lim\limits_{V\to\infty} \dfrac{\partial g}{\partial\beta}(\beta,V,\lambda) \\ \lim\limits_{V\to\infty} \dfrac{\partial g}{\partial\lambda}(\beta,V,\lambda)\end{array}\right\} = 0\,. \tag{24.6}$$

$\Pi(\beta,\xi,\lambda)$ has a singularity at $\xi_0(\beta,\lambda)$. Its nature (pole, branch point) depends on $g(\beta,V,\lambda)$. In principle, it is possible to continue the analytic function $\Pi(\beta,\xi,\lambda)$, defined by the integral representation in Eq. (24.2) for $\mathrm{Re}\,\xi > \xi_0$, into the whole complex plane beyond the convergence domain of the integral. Therefore it might well be possible that quantities derived from $\Pi(\beta,\xi,\lambda)$ have a physical meaning for ξ values where the integral representation of $\Pi(\beta,\xi,\lambda)$ does not exist.

That this is indeed the case and that the singularity at ξ_0 is absent in meaningful physical quantities will now be shown. It implies that *the singularity at ξ_0 has nothing to do with a phase transition*, in contradistinction to singularities of $Z(\beta,V,\lambda)$.

It is convenient to define a new function whose limit is $\Pi(\beta,\xi,\lambda)$:

$$\begin{aligned}\Pi_W(\beta,\xi,\lambda) &:= \int_0^W \mathrm{d}V \mathrm{e}^{-\xi V} Z(\beta,V,\lambda) \\ &= \int_0^W \mathrm{d}V \mathrm{e}^{-V(\xi-\xi_0)-Vg(\beta,V,\lambda)}\,, \end{aligned} \tag{24.7}$$

$$\lim_{W\to\infty} \Pi_W(\beta,\xi,\lambda) = \Pi(\beta,\xi,\lambda)\,.$$

We now calculate the energy density. First we define the expectation value of the total energy:

$$\left\langle E_W(\beta,\xi,\lambda)\right\rangle := -\frac{1}{\Pi_W}\frac{\partial \Pi_W}{\partial\beta} = \frac{\displaystyle\int_0^W \mathrm{d}V \mathrm{e}^{-\xi V} Z(\beta,V,\lambda)\left\langle E(\beta,\xi,\lambda)\right\rangle}{\displaystyle\int_0^W \mathrm{d}V \mathrm{e}^{-\xi V} Z(\beta,V,\lambda)}\,, \tag{24.8}$$

where $\langle E(\beta,\xi,\lambda)\rangle = -\partial \ln Z(\beta,V,\lambda)/\partial\beta$ was used. Similarly, we define the expectation value of the volume:

$$\left\langle V_W(\beta,\xi,\lambda)\right\rangle := -\frac{1}{\Pi_W}\frac{\partial \Pi_W}{\partial\xi} = \frac{\displaystyle\int_0^W \mathrm{d}V \mathrm{e}^{-\xi V} V Z(\beta,V,\lambda)}{\displaystyle\int_0^W \mathrm{d}V \mathrm{e}^{-\xi V} Z(\beta,V,\lambda)}\,. \tag{24.9}$$

The energy density is then

$$\mathscr{E}(\beta,\xi,\lambda) := \lim_{W\to\infty} \frac{\left\langle E_W(\beta,\xi,\lambda)\right\rangle}{\left\langle V_W(\beta,\xi,\lambda)\right\rangle}\,. \tag{24.10}$$

Similarly, if Q is the quantum number conserved by λ, the quantum number density $q(\beta,\xi,\lambda)$ is

$$q(\beta,\xi,\lambda) := \lim_{W\to\infty} \frac{\langle Q_W(\beta,\xi,\lambda)\rangle}{\langle V_W(\beta,\xi,\lambda)\rangle}\ , \quad \langle Q_W(\beta,\xi,\lambda)\rangle = -\frac{1}{\Pi_W}\lambda\frac{\partial \Pi_W}{\partial\lambda}\ . \tag{24.11}$$

In this formalism, the usual thermodynamic limit is replaced by the limit $W\to\infty$.

We now use the explicit form given by the last member of Eq. (24.7) to calculate $\mathscr{E}(\beta,\xi,\lambda)$:

$$\begin{gathered}\mathscr{E}(\beta,\xi,\lambda) = \lim_{W\to\infty}\frac{\partial \Pi_W/\partial\beta}{\partial \Pi_W/\partial\xi}\ ,\\ \frac{\partial \Pi_W}{\partial\beta} = \int_0^W V\left(\frac{\partial \xi_0}{\partial\beta} - \frac{\partial g}{\partial\beta}\right)\mathrm{e}^{-V(\xi-\xi_0)-Vg}\mathrm{d}V\ ,\\ \frac{\partial \Pi_W}{\partial\xi} = -\int_0^W V\mathrm{e}^{-V(\xi-\xi_0)-Vg}\mathrm{d}V\ .\end{gathered}\tag{24.12}$$

Since ξ_0 is independent of V, we have

$$\mathscr{E}(\beta,\xi,\lambda) = -\frac{\partial \xi_0(\beta,\lambda)}{\partial\beta} + \lim_{W\to\infty}\frac{\displaystyle\int_0^W \mathrm{e}^{-V(\xi-\xi_0)-Vg}V\frac{\partial g}{\partial\beta}\mathrm{d}V}{\displaystyle\int_0^W \mathrm{e}^{-V(\xi-\xi_0)-Vg}V\mathrm{d}V}\ . \tag{24.13}$$

We now recall Eq. (24.4), viz., $\xi_0 = \beta P$. Hence, if the second term of Eq. (24.13) were absent, we would recover the usual thermodynamic limit definition

$$\mathscr{E} = -\frac{\partial(\beta P)}{\partial\beta} = -\frac{\partial}{\partial\beta}\left[\lim_{V\to\infty}\frac{\ln Z(\beta,V,\lambda)}{V}\right]\ .$$

Next we observe that, for $\xi>\xi_0$, the integrals in Eq. (24.13) converge in the limit so that, unless $g(\beta,V,\lambda)=0$, the second term is a non-vanishing function $C(\beta,\xi,\lambda)$. Indeed, $\xi>\xi_0$ implies exponential suppression of large volumes. Therefore this function $C(\beta,\xi,\lambda)$ represents the corrections to the energy density coming from finite volume effects.

If finite volume effects are neglected already in defining $Z(\beta,V,\lambda)$, then $g\equiv 0$. In that case $\Pi = 1/(\xi-\xi_0)$ has a simple pole at ξ_0 which cancels out in $\mathscr{E}(\beta,\xi,\lambda)$ and the second term of Eq. (24.13) is absent. Furthermore, both $\mathscr{E}(\beta,\xi,\lambda)$ and $q(\beta,\xi,\lambda)$ are trivial analytic functions of ξ, namely, constants in the whole ξ plane. This particularly simple case is a good illustration of what happens. While $\langle E\rangle$ and $\langle V\rangle$ both have a pole at ξ_0 and become negative at $\xi<\xi_0$, the energy density does not care: the pole cancels and with it the whole ξ dependence. The $\mathscr{E}$ calculated from Eq. (24.13) is just the usual one obtained from $-\partial[\ln Z/V]/\partial\beta$.

In the more general case where finite volume effects are not neglected, i.e., $g(\beta,V,\lambda)\neq Q$, the correction term in Eq. (24.13) is present for $\xi>\xi_0$. It vanishes, however, identically for $\xi\leq\xi_0$ due to our assumption that $\lim_{V\to\infty}\partial g/\partial\beta = 0$ [see

Eq. (24.6)]. The simple proof is by de l'Hôpital's rule. Thus, if finite volume effects are included in the definition of $Z(\beta,V,\lambda)$, we recover the usual thermodynamic limit results for $\mathscr{E}(\beta,\xi,\lambda)$ for all $\xi \le \xi_0$ [there $\mathscr{E}(\beta,\xi,\lambda)$ becomes independent of ξ], while for $\xi > \xi_0$, finite volume corrections appear explicitly. All this is physically obvious: for $\xi < \xi_0$, large volumes have an exponentially increasing weight in the integration, whence the main contributions come from 'infinite' volumes where finite volume effects are absent by definition. Once this happens, it does not matter how fast the exponential weight increases. Therefore, $\mathscr{E}(\beta,\lambda)$ is independent of ξ for $\xi \le \xi_0$. Again, $\mathscr{E}(\beta,\xi,\lambda)$ defined by Eq. (24.13) is a meaningful physical quantity which may be evaluated at any ξ, while the individual integrals in Eq. (24.13) go to infinity in the limit $W \to \infty$.

This introduction thus results in two useful conclusions:

- Whatever the singularity of $\Pi(\beta,\xi,\lambda)$ at ξ_0 may be, it has no significance for quantities like $\mathscr{E}(\beta,\xi,\lambda)$ and $q(\beta,\xi,\lambda)$. While $\langle\mathscr{E}(\beta,\xi,\lambda)\rangle$ and $\langle V(\beta,\xi,\lambda)\rangle$ do have a singularity at ξ_0 and may become meaningless for $\xi \le \xi_0$, the singularity (pole, branch point, cut) cancels in calculating the above densities, which may be evaluated at any ξ.
- If one wishes to obtain explicit finite volume corrections, one must evaluate densities at $\xi > \xi_0$. If, on the other hand, one evaluates at $\xi \le \xi_0$, it is irrelevant whether or not finite volume terms, or more precisely, surface terms, have been included in the definition of $Z(\beta,V,\lambda)$: they are suppressed by the exponentially increasing weight of large volumes.

The real power of the pressure partition function formalism is this: it may happen that $\Pi(\beta,\xi,\lambda)$ can be calculated explicitly as an analytic function of ξ, while the direct analytic calculation of $Z(\beta,V,\lambda)$ is impossible. In that case, we can obtain exact results from $\Pi(\beta,\xi,\lambda)$ which we could not obtain from $Z(\beta,V,\lambda)$. This is precisely what happens in our problem of the van der Waals statistical bootstrap model.

In applying the technique introduced here to our explicit problem, the situation will be slightly different from the above, already fairly general case. However, our main conclusion that $\mathscr{E}(\beta,\xi,\lambda)$ and $q(\beta,\xi,\lambda)$ can be continued beyond the singularity ξ_0 remains valid. As this will be seen explicitly, we kept this additional complication out of our discussion above.

How shall we use $\Pi(\beta,\xi,\lambda)$?

Having decided that the usual thermodynamic limit – which requires, at least in a gedanken experiment, the existence of rigid boxes with a fixed volume – does not correspond to the situation where hadron matter goes over to quark matter, we shall not evaluate $\Pi(\beta,\xi,\lambda)$ at $\xi_0(\beta,\lambda)$, but rather consider ξ as an independent thermodynamic variable on the same footing as β and λ.

As the only relevant quantities to be calculated are densities which, as we have seen, ignore the existence of the singularity at ξ_0, we adopt the following philosophy. If we can obtain an analytic expression for $\Pi(\beta,\xi,\lambda)$, then we proceed to calculate from it expressions for the interesting densities which then are also analytic functions (not containing the singularity at ξ_0). We consider these functions as analytic continuations of the functions defined via the integral representation of Eq. (24.2) beyond the region of convergence of the latter. We then evaluate these functions at a ξ value appropriate to the physical situation.

Before we turn to the application to our specific problem, we have to generalize Eq. (24.2) relativistically, since it will be used with a relativistic formulation of $Z(\beta,V,\lambda)$. In this formulation, V is a timelike four-vector, so ξ must also be written as a timelike four-vector. Hence the generalization of Eq. (24.2) is

$$\Pi(\beta,\xi,\lambda) = \int \frac{\mathrm{d}V_\mu \xi^\mu}{\sqrt{\xi_\mu \xi^\mu}} \mathrm{e}^{-\xi_\mu V^\mu} Z(\beta,V,\lambda) \,, \tag{24.14}$$

where $Z(\beta,V,\lambda)$ is already a Lorentz invariant. The integration in Eq. (24.14) goes over the forward cone $V^0 \geq 0$. Going to the rest frame of ξ leads back to Eq. (24.2).

24.3 The Hadron Gas

Introduction

The grand canonical partition function of the strongly interacting hadron gas described by statistical bootstrap is written [4]

$$Z(\beta,V,\lambda) = \sum_{N=0}^{\infty} \frac{1}{N!} \int \prod_{i=1}^{N} \left[\frac{2(V-\mathscr{A}p)_\mu p_i^\mu}{(2\pi)^3} \right]_+ \tau(p_i^2,\lambda) \mathrm{e}^{-\beta_\mu p_i^\mu} \mathrm{d}^4 p_i \,. \tag{24.15}$$

Here V^μ is the external volume (to be integrated away), $\mathscr{A}p^\mu$ is $\mathscr{A} = 1/4\mathscr{B}$ times the sum of all four-momenta $\sum p_i^\mu$, with $4\mathscr{B}$ being the bag energy density [9], $\tau(p^2,\lambda)$ is the hadron cluster mass spectrum with baryon number conservation as follows from the bootstrap equation, and β^μ is the inverse temperature four-vector. The whole expression is written as a Lorentz invariant, following Touschek [10].

Attractive forces are represented by the mass spectrum, repulsive forces by the van der Waals type correction to the volume: from the total volume, the proper volumes of all particles are subtracted:[1]

[1] The factor of 4 multiplying the proper volumes of the constituents in the usual van der Waals correction is omitted, since it is specific to a gas of identical hard spheres, while here the clusters are deformable and of different sizes.

$$V^\mu - \mathscr{A} p^\mu = V^\mu - \mathscr{A} \sum_{i=1}^{N} p_i^\mu = V^\mu - \sum_{i=1}^{N} V_i^\mu \ , \tag{24.16}$$

which is the covariant generalization of what would be $V - \mathscr{A} \sum m_i$ in a non-relativistic formulation. That $\mathscr{A} p_i^\mu$ is the proper four-volume of the ith particle is a byproduct obtained in formulating the bootstrap equation [11]. It agrees (in the particle's rest frame) with the nuclear physics, where the volume is proportional to the mass, and with the bag model [9]. For further information, see [4] and references therein.

The subscript $+$ on the square bracket in Eq. (24.15) indicates that each single bracket is to be ≥ 0. This is guaranteed if

$$\begin{aligned} p_i^2 > 0\ , &\quad (V - \mathscr{A} p)^2 \geq 0\ , \\ p_i^0 > 0\ , &\quad (V - \mathscr{A} p)^0 \geq 0\ . \end{aligned} \tag{24.17}$$

The first two are trivial requirements since we are dealing with physical particles. The last two ensure positivity. Implicitly, they define the limits of the sum over N and of the integrations over p_i.

It is this van der Waals correction in Eq. (24.16) which prevents the integrations in Eq. (24.15) from factorizing into N independent integrals and which, moreover, makes the boundary of the sum and the integrals so complicated that it seems hopeless to calculate $Z(\beta, V, \lambda)$ without using drastic approximations. Introducing the pressure partition function $\Pi(\beta, \xi, \lambda)$ is not only suggested by the physical situation (no boxes), but it also solves the technical problem just mentioned.

Digression: the pointlike hadron gas

For later use, we need to consider the pointlike case. If no volume correction is applied in Eq. (24.15), the integrations factorize. Moreover the sum and integrals are unrestricted. We introduce under the integrals the identity

$$\tau(p_i^2, \lambda) \equiv \int \delta_0(p_i^2 - m_i^2) \tau(m_i^2, \lambda) \mathrm{d}m_i^2 \ . \tag{24.18}$$

The N identical integrals, evaluated in the common rest frame of the volume [$V^\mu = (V, 0, 0, 0)$] and of the thermometer [$\beta^\mu = (1/T, 0, 0, 0)$], can then be summed up over N and yield an exponential function, so that

$$\frac{1}{V} \ln Z_{\mathrm{pt}}(\beta, V, \lambda) := \int_0^\infty \tau(m^2, \lambda) \mathrm{e}^{-\beta \sqrt{p^2 + m^2}} \mathrm{d}m^2 \frac{p^2 \mathrm{d}p}{2\pi^2} \ , \tag{24.19}$$

where Z_{pt} is the 'point particle partition function'. Note that the expression in Eq. (24.19) is independent of the volume V. The remaining integral is, in the Statistical Bootstrap Model, simply related to the 'bootstrap function' $\phi(\beta, \lambda)$ [4, 11]:

$$\frac{1}{V}\ln Z_{\rm pt}(\beta,V,\lambda) = -\frac{2}{H(2\pi)^3}\frac{\partial\phi(\beta,\lambda)}{\partial\beta} =: f(\beta,\lambda)\,, \tag{24.20}$$

where $f(\beta,\lambda)$ is a shorthand notation for later use. The function $\phi(\beta,\lambda)$ is analytically and numerically well known [12] and easy to compute. Thus $f(\beta,\lambda)$ may be considered as a known function.

The function $\phi(\beta,\lambda)$ has, for given λ, a square root singularity at some $\beta^*(\lambda)$, namely [4, 12]

$$\phi(\beta,\lambda) \underset{\beta\to\beta^*}{\longrightarrow} \ln 2 - h(\lambda)\sqrt{\beta-\beta^*(\lambda)}\,. \tag{24.21}$$

The curve $\beta^*(\lambda)$ would thus be a singular curve of the point particle model. We shall see that it will also be a singular curve of the model with nonzero particle volumes. From Eqs. (24.20) and (24.21), we infer

$$\ln Z_{\rm pt}(\beta,\lambda) \underset{\beta\to\beta^*}{\longrightarrow} \sim \frac{1}{\sqrt{\beta-\beta^*}} \longrightarrow \infty\,. \tag{24.22}$$

Consequently, we obtain for the energy density

$$\mathscr{E}_{\rm pt}(\beta,\lambda) = -\frac{1}{V}\frac{\partial \ln Z_{\rm pt}}{\partial\beta} \underset{\beta\to\beta^*}{\longrightarrow} \sim \frac{1}{\sqrt{\beta-\beta^*}^{\,3}}\,, \tag{24.23}$$

for the baryon number density

$$\nu_{\rm pt}(\beta,\lambda) = \frac{1}{V}\lambda\frac{\partial \ln Z_{\rm pt}}{\partial\lambda} \underset{\beta\to\beta^*}{\longrightarrow} \sim \frac{1}{\sqrt{\beta-\beta^*}^{\,3}}\,, \tag{24.24}$$

and for the pressure

$$P_{\rm pt}(\beta,\lambda) = \frac{1}{\beta V}\ln Z_{\rm pt} \underset{\beta\to\beta^*}{\longrightarrow} \sim \frac{1}{\sqrt{\beta-\beta^*}}\,. \tag{24.25}$$

The real hadron gas

The pressure partition function of the real hadron gas is given by the definition (24.14) with $Z(\beta,V,\lambda)$ taken from Eq. (24.15):

$$\Pi(\beta,\xi,\lambda) = \int\frac{\mathrm{d}V_\mu\xi^\mu}{\sqrt{\xi_\mu\xi^\mu}}\mathrm{e}^{-\xi_\mu V^\mu}\sum_{N=0}^{\infty}\frac{1}{N!}\int\prod_{i=1}^{N}\left[\frac{2(V-\mathscr{A}p)_\mu p_i^\mu}{(2\pi)^3}\right]_+\tau(p_i^2,\lambda)\mathrm{e}^{-\beta_\mu p_i^\mu}\,\mathrm{d}^4p_i\,. \tag{24.26}$$

We make the change of variables

$$(V-\mathscr{A}p)^\mu =: x^\mu\,. \tag{24.27}$$

Positivity then requires [see Eq. (24.17)]

$$x_\mu x^\mu \geq 0 , \qquad x^0 \geq 0 . \tag{24.28}$$

With this substitution and with the identity in Eq. (24.18), we obtain

$$\Pi(\beta,\xi,\lambda)=\int \frac{\mathrm{d}x_\mu \xi^\mu}{\sqrt{\xi_\mu \xi^\mu}} \mathrm{e}^{-\xi_\mu x^\mu} \sum_{N=0}^{\infty} \frac{1}{N!} \int \prod_{i=1}^{N} \frac{2x_\mu p_i^\mu}{(2\pi)^3} \mathrm{e}^{-(\beta+\mathscr{A}\xi)_\mu p_i^\mu} \tau(m_i^2,\lambda) \mathrm{d}m_i^2 \frac{\mathrm{d}^3 p_i}{2p_{0i}} , \tag{24.29}$$

where $p^\mu = \sum p_i^\mu$ has been used. Now the positivity condition is automatically satisfied by integrating over the forward light cone of x. The explicit p dependence due to the volume correction in Eq. (24.15) has disappeared from the volume factors $x_\mu p_i^\mu$ and is shifted as $\xi_\mu p_i^\mu$ to the exponent where it factorizes. Thus the integrals over the p_i are again all identical and unrestricted, as if we had a pointlike gas.

We assume the temperature to be measured in the rest frame of the expectation value of the volume $\langle V^\mu \rangle$, whence $\beta \parallel \xi$. As $\Pi(\beta,\xi,\lambda)$ is a Lorentz scalar, we evaluate in the common rest frame of β and ξ.

One of the N identical integrals in Eq. (24.29) is then[2]

$$\int \frac{2(x_0 p^0 - \mathbf{x}\cdot\mathbf{p})}{(2\pi)^3} \mathrm{e}^{-(\beta+\mathscr{A}\xi)\sqrt{p^2+m^2}} \tau(m^2,\lambda) \mathrm{d}m^2 \frac{\mathrm{d}^3 p}{2\sqrt{p^2+m^2}} , \tag{24.30}$$

where $\mathbf{x}\cdot\mathbf{p}$ vanishes upon angular integration, so that, with $x := x^0$, the integral reduces to [see Eqs. (24.19) and (24.20)]

$$x \int \tau(m^2,\lambda) \mathrm{e}^{-(\beta+\mathscr{A}\xi)\sqrt{p^2+m^2}} \mathrm{d}m^2 \frac{p^2 \mathrm{d}p}{2\pi^2} = x f(\beta+\xi\mathscr{A},\lambda) . \tag{24.31}$$

Thus Eq. (24.29) becomes

$$\Pi(\beta,\xi,\lambda) = \sum_{N=0}^{\infty} \frac{1}{N!} \int_0^\infty \mathrm{e}^{-\xi x} x^N \mathrm{d}x \big[f(\beta+\xi\mathscr{A},\lambda) \big]^N . \tag{24.32}$$

The x integration yields $N!/\xi^{N+1}$, so that finally,

$$\Pi(\beta,\xi,\lambda) = \frac{1}{\big[\xi - f(\beta+\xi\mathscr{A},\lambda)\big]} , \tag{24.33}$$

where $f(\beta+\xi\mathscr{A},\lambda)$ is the point particle expression of Eq. (24.19) taken at $\beta+\xi\mathscr{A}$. As mentioned earlier, $f(x,y)$ is a perfectly known function, both numerically and analytically. Thus, within the framework of the Statistical Bootstrap Model with extended particles, we have obtained a simple analytical expression for $\Pi(\beta,\xi,\lambda)$ defined in the whole ξ plane [the difference from the case discussed in Sect. 24.2 is that ξ appears in $f(\beta+\xi\mathscr{A},\lambda)$].

[2] From now on, we write $\beta := \sqrt{\beta_\mu \beta^\mu}$ and $\xi := \sqrt{\xi_\mu \xi^\mu}$.

Interpretation

The usual thermodynamic limit with fixed $V \to \infty$

From Eq. (24.4), we know that the singular point $\xi_0(\beta,\lambda)$ of $\Pi(\beta,\xi,\lambda)$ is equal to $\beta P(\beta,\lambda)$ with the usual thermodynamic limit prescription

$$\beta P_{(V)} = \xi_0 = \lim_{V\to\infty} \frac{1}{V} \ln Z(\beta,V,\lambda) \;,$$
$$\xi_0(\beta,\lambda) = \text{root of equation } \; \xi = f(\beta + \mathscr{A}\xi,\lambda) \;. \tag{24.34}$$

Finding this root is a simple numerical exercise, which we shall not execute here since we are not interested in this pressure, which is irrelevant for our physical problem.

We can now give a simple proof (without using any approximations) of the statement [7] that, in the *usual* thermodynamic limit procedure, the singularity at $\beta^*(\lambda)$ cannot be reached by any $\beta > 0$, or in other words, that $\lim_{V\to\infty} \ln Z(\beta,V,\lambda)/V$ has no singularity on the real T axis. We must show only that $\xi_0(\beta,\lambda)$ has no singularity. Equation (24.34) states that

$$\xi_0(\beta,\lambda) = f\big(\beta + \mathscr{A}\xi_0(\beta,\lambda),\lambda\big) \;. \tag{24.35}$$

The singularity $f(x,\lambda) \sim \left[x - \beta^*(\lambda)\right]^{-1/2}$ [see Eq. (24.22)] cannot be reached. Assume that indeed $x = \beta + \mathscr{A}\xi_0(\beta,\lambda) \to \beta^*(\lambda)$. Then by Eq. (24.35), $\xi_0 \to \infty$ and $\beta + \mathscr{A}\xi_0 \to \infty$, contradicting the assumption. This holds for all $\beta > 0$. Hence $\lim_{V\to\infty} \ln Z(\beta,V,\lambda)/V$ is analytic along the whole T axis, together with all its derivatives.

Having already decided that in our problem the usual thermodynamic limit does not correspond to reality, we do not pursue this line further.

Hot hadron matter: no fixed volume

According to our philosophy stated in Sect. 24.2, we now evaluate $\Pi(\beta,\xi,\lambda)$ and its derivatives at fixed β,ξ,λ. Applying the definitions of Sect. 24.2 to Eq. (24.33), we immediately find

$$\mathscr{E}(\beta,\xi,\lambda) = \frac{\langle E\rangle}{\langle V\rangle} = \frac{\mathscr{E}_{\rm pt}(\beta+\mathscr{A}\xi,\lambda)}{1+\mathscr{A}\mathscr{E}_{\rm pt}(\beta+\mathscr{A}\xi,\lambda)} \;, \tag{24.36}$$

$$\nu(\beta,\xi,\lambda) = \frac{\langle b\rangle}{\langle V\rangle} = \frac{\nu_{\rm pt}(\beta+\mathscr{A}\xi,\lambda)}{1+\mathscr{A}\mathscr{E}_{\rm pt}(\beta+\mathscr{A}\xi,\lambda)} \;, \tag{24.37}$$

where, bearing in mind Eq. (24.20),

$$\mathscr{E}_{\mathrm{pt}}(\beta+\mathscr{A}\xi,\lambda) = -\frac{\partial}{\partial\beta'}f(\beta',\lambda)\Big|_{\beta'=\beta+\mathscr{A}\xi}, \tag{24.38}$$

$$\nu_{\mathrm{pt}}(\beta+\mathscr{A}\xi,\lambda) = \lambda\frac{\partial}{\partial\lambda}f(\beta+\mathscr{A}\xi,\lambda). \tag{24.39}$$

The energy density and baryon number density no longer contain the pole at ξ_0. They are analytic functions of β,ξ,λ for all real values

$$1 \le \lambda < \infty, \qquad \beta+\xi\mathscr{A} > \beta^*(\lambda). \tag{24.40}$$

As ξ is now an independent variable [and no longer related to β and λ by an equation like (24.35)], the singularity $\beta+\xi\mathscr{A}=\beta^*(\lambda)$ can be reached. There $\mathscr{E}_{\mathrm{pt}}$ and ν_{pt} go to infinity [see Eqs. (24.23) and (24.24)] and thus $\mathscr{E}(\beta,\xi,\lambda)_{\mathrm{crit}} = 1/\mathscr{A}$, while $\nu(\beta,\xi,\lambda)_{\mathrm{crit}} \ne 0,\infty$.

As any $\xi > 0$ corresponds to an external force trying to compress the system [see Eq. (24.2)], we consider $\xi = 0$ to be the appropriate value for a system which determines its own volume dynamically. Thus for the hadron gas without external forces

$$\mathscr{E}(\beta,\lambda) = \frac{\mathscr{E}_{\mathrm{pt}}(\beta,\lambda)}{1+\mathscr{A}\mathscr{E}_{\mathrm{pt}}(\beta,\lambda)}, \tag{24.41}$$

$$\nu(\beta,\lambda) = \frac{\nu_{\mathrm{pt}}(\beta,\lambda)}{1+\mathscr{A}\mathscr{E}_{\mathrm{pt}}(\beta,\lambda)}, \tag{24.42}$$

which are the results already derived in [4] with the ‘available volume’ technique.

In astrophysical applications, ξ can be used to take gravitational pressure into account. This has the effect of replacing the singular curve $\beta^*(\lambda)$ by a singular surface $\beta^*(\lambda,\xi)$. The limiting values of $\mathscr{E}$ and ν on this critical surface can then be calculated. For $\mathscr{E}$, it is again $1/\mathscr{A} = \text{const.}$, as seen from Eq. (24.36).

A small conceptual problem arises with the pressure. We have stated often enough that the usual definition (24.34) is useless here. We must therefore define the pressure as in kinetic gas theory: there it is the result of the stochastic cannonade of the wall of the vessel by the gas particles. The pressure is found to be proportional to the average normal component of the momenta of the particles hitting the wall. Here, where we do not have material walls, we may define the pressure as being proportional to the average normal momentum component of particles passing through an imaginary surface from left to right. Then, going through the usual textbook derivation, we find that

$$\beta P = \frac{\langle N\rangle}{\langle V\rangle}, \tag{24.43}$$

where all the dynamics is hidden in $\langle N\rangle$, the average number of clusters, and $\langle V\rangle$, the volume chosen by the system. One should not expect here a van der Waals type of equation, because there N and V are fixed external parameters, so that the ideal gas equation has to be corrected. Here this is not necessary. In the second paper of [4], Eq. (24.43) was indeed derived in the 'available volume' formalism. Here we take it as the definition of the pressure.

It remains to calculate $\langle N(\beta,\xi,\lambda)\rangle$. In Eq. (24.32), we multiply f by a fugacity η and obtain, instead of Eq. (24.33),

$$\Pi(\beta,\xi,\lambda,\eta) := \frac{1}{\xi - \eta f(\beta+\mathscr{A}\xi,\lambda)} . \tag{24.44}$$

Obviously,

$$\big\langle N(\beta,\xi,\lambda)\big\rangle = \frac{\eta}{\Pi}\frac{\partial \Pi(\beta,\xi,\lambda,\eta)}{\partial\eta}\bigg|_{\eta=1} . \tag{24.45}$$

Thus, with Eq. (24.20),

$$\frac{\langle N\rangle}{\langle V\rangle} = \frac{\eta f(\beta+\mathscr{A}\xi,\lambda)}{1+\eta A\mathscr{E}_{\mathrm{pt}}(\beta+\mathscr{A}\xi,\lambda)}\bigg|_{\xi=0,\eta=1} = \frac{\beta P_{\mathrm{pt}}(\beta,\lambda)}{1+\mathscr{A}\mathscr{E}_{\mathrm{pt}}(\beta,\lambda)} . \tag{24.46}$$

Hence, with Eq. (24.43),

$$P(\beta,\lambda) = \frac{P_{\mathrm{pt}}(\beta,\lambda)}{1+\mathscr{A}\mathscr{E}_{\mathrm{pt}}(\beta,\lambda)} , \tag{24.47}$$

which is the same as in [4].

From Eqs. (24.23) and (24.25), we see that $\mathscr{E}_{\mathrm{pt}}$ diverges more strongly than P_{pt} on the critical curve. Thus $P(\beta,\lambda)_{\mathrm{crit}} = 0$. This is not surprising, since on the critical curve our 'gas' has coalesced into one single 'particle' of infinite volume which does not move. In a complete model unifying the hadron side and the quark side, the pressure should not go to zero. In our case, where two different models have to be fitted together at the singularity, the pressure on the quark-gluon side rises steeply. The usual Maxwell construction then gives a region of constant vapour pressure along an isotherm [6, item (d)].

24.4 Conclusions

It has been shown how the thermodynamic limit procedure must be adapted to the real physical situation. Different procedures may give different results: one may exhibit a singularity while the other does not, and yet both are correct – they simply apply to different physical boundary conditions. In the particular problem of the transition from hadron to quark-gluon matter, the usual grand canonical partition function does not lead to a singularity. We consider it (in the context of extended particles and a van der Waals type volume correction) as badly suited to describe our problem, because it assumes that a rigid volume containing the system can exist. At the transition from hadron matter to quark matter, this assumption is principally wrong. In an earlier attempt to do better [4], we introduced the 'available volume' as a new, independent variable in place of the volume. The result was that the thus modified grand canonical partition function had a singularity indicating a phase transition. In the present paper, we have confirmed the results of [4] using the grand canonical pressure partition function, which seems to be tailored to our specific problem. It can be stated that, in the Statistical Bootstrap Model with extended hadrons (volume proportional to mass), a phase transition does occur as claimed earlier [4] and that the objections raised in [7], though correct in themselves, do not apply to physical reality in the temperature and density regime considered here.

Furthermore, we do not accept the conclusions of a recent paper [13], namely that it is important which singularity of $\Pi(\beta,\xi,\lambda)$ is reached first, the (trivial) one at $\xi_0(\beta,\lambda)$ or some other at $\beta^*(\lambda)$ originating from $Z(\beta,V,\lambda)$. These conclusions disregard the disappearance of the singularity at ξ_0 from densities. (We do not claim, however, that there might not be cases to which the analysis presented in [13] is relevant.)

Acknowledgements

I am grateful to J. Rafelski, H. Satz, and L. Van Hove for extensive discussions and constructive criticism (this does not imply that they would agree on everything said here).

References

1. J. Baacke: Acta Phys. Pol. B **8**, 625 (1977)
2. This fact is known already to cosmic ray physicists and has been stated in so many papers that it is impossible to list them all. Let two more recent ones represent the whole lot:
   (a) A.T. Laasanen, C. Ezell, L.J. Gutay, W.N. Schreiner, P. Schübelin, L. von Lindern, F. Turkot: Phys. Rev. Lett. **38**, 1 (1977);
   (b) G.W. van Appeldoorn, S. Barshay, D. Harting, D.J. Holthuizen, P.G. Kuijer, B.J. Pijlgroms, V. Karimäki, R. Kinnunen, M. Korkea-Aho, P. Johnson, D. Michaelides, Ch. Michaelidou, D.G. Patel: Z. Physik C, Particles and Fields **7**, 235 (1981)
3. We do not consider here Monte Carlo lattice calculations which may indeed cover both sides and do suggest a phase transition, but which cannot exhibit singularities in the strict sense. See, e.g., J. Engels, F. Karsch, I. Montvay, H. Satz: Phys. Lett. B **102**, 332 (1981) and references therein
4. (a) R. Hagedorn, J. Rafelski: Phys. Lett. B **97**, 136 (1980);
   (b) R. Hagedorn, J. Rafelski: From hadron gas to quark matter I. In *Statistical Mechanics of Quarks and Hadrons*, ed. by H. Satz, North Holland Publ. Co., Amsterdam (1981)
5. N. Cabibbo, G. Parisi: Phys. Lett. B **59**, 67 (1974)
6. (a) J.I. Kapusta: Nucl. Phys. B **148**, 461 (1979);
   (b) O.K. Kalashnikov, V.V. Klimov: Phys. Lett. B **88**, 328 (1979);
   (c) J. Kuti, J. Polonyi, K. Szalachani: Phys. Lett. B **98**, 199 (1981);
   (d) J. Rafelski, R. Hagedorn: From hadron gas to quark matter II. In *Statistical Mechanics of Quarks and Hadrons*, ed. by H. Satz, North Holland Publ. Co., Amsterdam (1981)
7. V.V. Dixit, F. Karsch, H. Satz: Phys. Lett. B **101**, 412 (1981)

8. (a) E.A. Guggenheim: J. Chem. Phys. **7**, 103 (1939);
(b) I. Prigogine: Physica **16**, 133 (1950);
(c) A. Münster: *Statistical Thermodynamics*, Springer-Verlag, Berlin, Heidelberg (1969), Sects. 3.1 and 3.2;
(d) A. Münster: loc. cit.
9. A. Chodos, R.L. Jaffe, K. Johnson, C.B. Thorn, V.F. Weisskopf: Phys. Rev. D **9**, 3471 (1974)
10. B. Touschek: Nuovo Cimento B **58**, 295 (1968)
11. R. Hagedorn, I. Montvay, J. Rafelski: In *Hadronic Matter at Extreme Energy Density*, ed. by N. Cabibbo and L. Sertorio, Plenum Press, New York (1980)
12. R. Hagedorn, J. Rafelski: Commun. Math. Phys. **83**, 563 (1982)
13. M.I. Gorenstein, V.K. Petrov, G.M. Zinovjev: Phys. Lett. B **106**, 327 (1981)

Chapter 25
How We Got to QCD Matter from the Hadron Side – 1984

Rolf Hagedorn

Abstract Rolf Hagedorn reminsces in 1984 about limiting temperature, the development of the statistical bootstrap model (SBM). He argues that consideration of hadrons of finite size allowed the generalization of SBM into a sophisticated relativistic van der Waals-type gas, leading on to a theory of phase transformation from melting hadrons to boiling quarks.

25.1 Introduction

I have 50 minutes and 25 pages of print available to describe the origin and development of work which took 27 years and was done by perhaps more than 200 people in perhaps more than 300 papers. This leaves me about two minutes of talk per year of work and about three lines of print per paper.

You, and hopefully even those whose work is not reported here, will understand my dilemma: I have to concentrate on the essentials and skip all technical details.

I hope to be able to let you participate in retrospection in the ups and downs of this adventure, which started from a simple statistical model of hadron production without any attempt to understand the underlying dynamics and without any idea of what finally it would lead to: a phase transition to QCD matter.

This whole history can be squeezed into one sentence coined by Helmut Satz [1]: "Hadron thermodynamics defines its own limits." But how, and why? To realize and formalize that took a long time, as I shall now describe.

Invited Lecture at QM3 (Helsinki 1984); Lect. Notes Phys. **221** (1985) pp. 53-76

(deceased) CERN-TH, 1211 Geneve 23, Switzerland

25.2 Pre-Bootstrap

In 1957, F. Cerulus and I were asked by the then CERN-TH-Division leader, B. Ferretti, to do a few calculations ("just a fortnight") of particle production at the future CERN 30 GeV proton synchrotron. We used the Fermi statistical model [2] and realized at once that the phase space integrals could not be calculated with the known approximations ($m \to 0$ and $m \to \infty$) since, on the average, the particles (mostly pions) would be neither relativistic nor non-relativistic. We heard of the Monte Carlo method and adapted it to our problem [3]. We thus were able to compute reliably hundreds of phase space integrals with all the then known hadrons and resonances. We even included two-body phase spaces and archived all our results for some reason or another. This proved fortunate when, five years later, L.W. Jones proposed to compute large-angle elastic pp scattering and asked us whether we had any possibility to do so. The idea was that there might be a statistical intermediate state decaying into two protons isotropically. The cross-section for this process (visible, if existing, only far outside the diffraction region) should be proportional to the ratio

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}E}\right|_{q_0} \sim \frac{\text{2-body phase space}}{\sum_n n\text{-body phase spaces}} \,. \tag{25.1}$$

We dug out our old numerical results and found [4] an energy dependence $\sim \exp(-3.3E_{\mathrm{cm}}\,[\mathrm{GeV}])$, in agreement with experiments. This numerical result could, in simple analytical models (equal particles) only be reproduced *if one suppressed there the factor* $1/N!$ *in front of the phase space integrals*, otherwise one would obtain $\exp(-\text{const.}\,E^{\alpha})$ with $\alpha < 1$ [5–7]. *But omitting* $1/N!$ *amounted to considering the particles to be distinguishable* – a most important observation which triggered all the rest, but which was greeted with pitiful smiles by those to whom I talked about it, although I tried to argue that indeed the particles should be *effectively distinguishable*, since the factor was $1/N!$ only for like particles, whereas when there were several different kinds, then

$$\frac{1}{N!}\int \ldots \longrightarrow \frac{1}{\prod_i N_i!}\int \ldots \,, \tag{25.2}$$

where i goes over all types of particles. We had in our earlier numerical computations used some 40 or 50 *different types* of particles (including some resonances and counting each state: spin, isospin, antiparticle, etc., as a new kind), and it so happened that, on the average, all $\langle N_i \rangle \leq 1$, so that the main contributions to Eq. (25.1) had come from integrals whose factor $1/\prod_i N_i! = 1$. We then worked out a model of massless distinguishable particles [5] which was so simple that I present it here.

Let N massless, distinguishable particles be enclosed in a box of volume V and let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i, \ldots$ denote the single-particle energy levels with occupation numbers $n_1, n_2, \ldots, n_i, \ldots$. Then there will be

$$\frac{N!}{n_1!n_2!\ldots} \text{ states of total energy } E = \sum n_i \varepsilon_i \,, \quad N = \sum n_i \,. \tag{25.3}$$

If particles can be freely created, the partition function will be

$$Z = \sum_{N=0}^{\infty} \sum_{\sum n_i = N} \frac{N!}{\prod_i n_i!} \exp\left(-\frac{1}{T}\sum n_i \varepsilon_i\right) = \sum_{N=0}^{\infty} \left[\sum_i \exp\left(-\frac{\varepsilon_i}{T}\right)\right]^N . \tag{25.4}$$

Replacing as usual

$$\exp\left(-\frac{\varepsilon_i}{T}\right) \longrightarrow \frac{V}{2\pi^2}\int p^2 \exp\left(-\frac{p}{T}\right) \mathrm{d}p = \frac{VT^3}{\pi^2} , \tag{25.5}$$

we find

$$Z = \sum_{N=0}^{\infty} \left(\frac{VT^3}{\pi^2}\right)^N = \frac{1}{1 - VT^3/\pi^2} , \tag{25.6}$$

which has a pole at $T_\mathrm{c} = (\pi^2/V)^{1/3}$, where the energy density would also diverge, implying that T_c is the maximum temperature for this system. Now comes the *miracle*: putting $V = V_\mathrm{hadron} = (4\pi/3)(1/m_\pi)^3$ gives $T_\mathrm{c} \approx 185$ MeV, very near a 'temperature' that has been familiar for many years from cosmic ray physics ($p_\perp$ distribution).

25.3 Early Bootstrap

Quite apart from hadrons not being massless, the above model is pure cheating if you only look at it a little closer. But it had enough cheating power to convince me that something in it must be true. The next move had thus to be to build a better model with real, massive, hadrons while keeping the idea of many different types of them being involved.

For the sake of clarity, I present now a simplified version (Boltzmann statistics, non-relativistic approximation) of the original paper [8]. Consider a mixture of ideal gases, each gas belonging to one species of particle with mass m_i; all particles can be freely created and absorbed. The total partition function is the product of the partition functions of the individual gases labelled by i. We have

$$\begin{aligned} \ln Z_i(T,V) &= \frac{V}{(2\pi)^3}\int \mathrm{e}^{-\sqrt{p^2+m_i^2}/T} \mathrm{d}^3 p = \frac{VT}{2\pi^2} m_i^2 K_2(m_i/T) \\ &\approx V\left(\frac{T}{2\pi}\right)^{3/2} m_i^{3/2} \mathrm{e}^{-m_i/T} . \end{aligned} \tag{25.7}$$

Thus in this approximation,

$$\ln Z = \sum_i \ln Z_i = V\left(\frac{T}{2\pi}\right)^{3/2} \sum_i m_i^{3/2} \mathrm{e}^{-m_i/T} , \tag{25.8}$$

where the sum over i goes over *all hadron species which can in principle participate*, including resonances and counting each spin and isospin state separately. Assuming most possible particles to be as yet unknown, it is convenient to introduce the mass spectrum $\rho(m)$, where

$$\rho(m)\mathrm{d}m = \text{number of different hadron states in } \{m, \mathrm{d}m\} \,, \qquad (25.9)$$

which, of course, is also unknown (except for a few δ-functions at low-lying stable masses and Breit–Wigner functions at the established resonances). We then replace the sum over i by an integral over $\rho(m)$ and obtain

$$Z(T,V) = \exp\left[V\left(\frac{T}{2\pi}\right)^{3/2}\int_0^\infty \rho(m)m^{3/2}\mathrm{e}^{-m/T}\,\mathrm{d}m\right] \,. \qquad (25.10)$$

For the exact relativistic expression with Fermi and Bose statistics, see [8].

What now? We have a formula which tells us exactly nothing as long as $\rho(m)$, i.e., the complete $\rho(m)$, is not given. Here comes the key idea: *statistical bootstrap*.

The bootstrap idea

We can present this in four steps:

- We are after the description of 'fireballs', i.e., highly excited and decaying hadronic systems of hadronic size V_0, composed of hadrons. Our earliest results (phase space, large-angle elastic scattering) had suggested that the set of constituent hadrons should also include all resonances, which therefore must be counted in $\rho(m)$.
- The hadronic system of size V_0 (fireball) is, however, itself a highly excited hadron and is therefore not essentially different from a resonance: there should be just one set of hadrons, viz.,

  $$\left\{\pi,\ \mathrm{K},\ \text{resonances},\ \mathrm{N},\ \Lambda,\ \Sigma,\ \text{more resonances},\ \ldots,\ \text{fireballs ad infinitum}\right\} ,$$

  and all should appear in $\rho(m)$.
- The partition function Eq. (25.10) can be written down, using the density of states of the system it describes (a fireball if $V \approx V_0$ is the hadron volume!):

  $$Z(T,V_0) = \int_0^\infty \sigma(V_0,m)\mathrm{e}^{-m/T}\,\mathrm{d}m \,, \qquad (25.11)$$

  where $\sigma(V_0,m)$ is the density of states of the fireball at mass m.
- Since $\rho(m)$ should include fireballs and since ρ and σ are densities of states of essentially the same objects, the two functions should be essentially the same, at least for large m. A weak requirement ('equal entropy') of this sort was

$$\frac{\log \rho(m)}{\log \sigma(V_0, m)} \quad \underset{m\to\infty}{\longrightarrow} \quad 1\,. \tag{25.12}$$

On the other hand, one has the identity of Eq. (25.10) and Eq. (25.11):

$$\int_0^\infty \sigma(V_0, m) \mathrm{e}^{-m/T} \mathrm{d}m = \exp\left[V \left(\frac{T}{2\pi} \right)^{3/2} \int_0^\infty \rho(m) m^{3/2} \mathrm{e}^{-m/T} \mathrm{d}m \right]\,. \tag{25.13}$$

The two equations Eq. (25.12) and Eq. (25.13) together form the *bootstrap condition*. In words,

$$\begin{array}{llll} \text{fireballs} & & & \\ & \rightarrow & \text{consist of fireballs, which} & \rightarrow \\ & \uparrow & & \downarrow \\ & \uparrow & & \downarrow \\ & \leftarrow & \leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow & \leftarrow \end{array}$$

Consequences

From the bootstrap condition, it can be proved that the mass spectrum must grow exponentially (for large m):

$$\rho(m) \quad \underset{m\to\infty}{\longrightarrow} \quad \frac{\text{const.}}{m^a} \mathrm{e}^{+m/T_0}\,. \tag{25.14}$$

Together with its low mass (empirical) part, this then constituted the so far unknown mass spectrum $\rho(m)$.

Inserting this into (25.10) shows that the integral does not converge if $T > T_0$: the partition function has a singularity at T_0 whose nature depends on the value of a. Thus there were three predictions:

- for sufficiently high collision energy, the $p_\perp$ distribution should be (very approximately) $\sim \exp(p_\perp/T)$ with $T \lesssim T_0$ [often called the $\exp(-6p_\perp)$ law],
- the mass spectrum should grow exponentially,
- T_0 in the $p_\perp$ distribution and in the mass spectrum should be the same.

I presented these predictions and the whole model in a theory seminar in the fall of 1964. The result was disastrous. Nobody would believe it and I was shouted at: "but the mass spectrum *does not* grow exponentially".

True enough, I had not yet even checked it. The next morning, I did so with beating heart and found what is depicted in Fig. 20.1 as the October 1964 situation. The $p_\perp$ distribution for 90°C elastic pp scattering [9] looked as in Fig. 17.4, where the slope required that $T_0 = 158$ MeV, while for the mass spectrum, it was $T_0 = 160 \pm 10$ MeV. After the depressing seminar and an equally depressing night, this was such a relief that I never forgot this experience and henceforth was convinced that, all imperfections notwithstanding, the model was essentially right and that I should continue to work on it no matter what people would say (and they said a lot, mostly based on the misunderstanding that, in the model, a maximal temperature and/or an exponential spectrum were *postulated*, while these were in fact the *results* I was so proud of). The general rejection had one immense advantage: those few who worked with me (and myself) could proceed without fear of competition and without hurry.

After this anecdotal digression, consider now the energy density of such a system:

$$\mathscr{E} = \frac{T^2}{V}\frac{\partial \ln Z}{\partial T} = \left(\frac{T}{2\pi}\right)^{3/2}\left[\frac{3T}{2}\int_0^\infty \rho(m)m^{3/2}\mathrm{e}^{-m/T}\,\mathrm{d}m + \int_0^\infty \rho(m)m^{5/2}\mathrm{e}^{-m/T}\,\mathrm{d}m\right]. \tag{25.15}$$

With $\rho(m) \to \text{const.}\, m^{-a}\exp(m/T_0)$ and at $T = T_0$, we find

$$\mathscr{E}(T_0) = \text{const.}\left[\frac{3T_0}{2}\left(m^{5/2-a}\right)_M^\infty + \left(m^{7/2-a}\right)_M^\infty + \text{finite parts}\right], \tag{25.16}$$

where M is assumed so large that the asymptotic expression Eq. (25.14) is valid for $m \geq M$.

The point I wish to make here is that, from Eq. (25.16), it follows that *the energy density* $\mathscr{E}$ *is finite at* T_0 *if* $a > 7/2$ *and infinite otherwise*. This could have been interpreted as indicating that, since for $a > 7/2$ one could reach the temperature T_0 at finite energy density, one could also pass over it into a region which would need another description. In other words, *for* $a > 7/2$*, there would be a phase transition*, because all the usual features of a phase transition, in particular a critical temperature T_0, were present.

Unfortunately, and at that time for good reasons, I discarded this choice of a and preferred $a = 5/2$, giving infinite energy density at T_0, which therefore should be the highest temperature, approachable only asymptotically. And thus the discovery of the phase transition to the quark-gluon plasma, alas, escaped me (it was a bit early for that though!).

Difficulties

The model was still unsatisfactory in two respects:

- **Inconsistency.** While the fireball described by the partition function $Z(V_0, T)$ had a volume V_0, its constituents were treated as pointlike, although they were hadrons and fireballs, too. This inconsistency was not removed until 1978–80. Then, however, with striking results (see below).
- **Difficulty.** No reason was given for treating resonances and fireballs as if they were particles. This obstacle was removed a few years later by using the arguments of Beth and Uhlenbeck [10] and Belenkij [11]. In fact these arguments put the somewhat shaky bootstrap idea (Sect. 25.3) on rather firm ground and

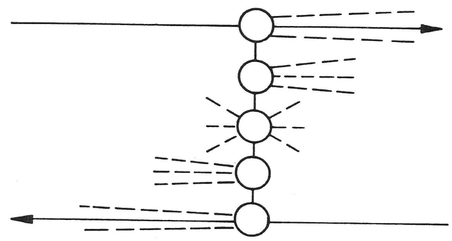


Fig. 25.1 Bootstrap thermodynamics is applied to each bubble in its rest frame. Lorentz transformation to the laboratory (or centre of mass) frame and integration over all possible impact parameters and velocities gives realistic predictions of particle momentum spectra.

> allowed us to state that *a strongly interacting hadron gas can be described alternatively as a mixture of infinitely many ideal gases listed in the complete mass spectrum.*

These arguments were given *in extenso* in three places and cannot be repeated here [12–14].

Early developments

The model could not immediately be applied to hadron collisions, but it was important to do so in order to convince the non-believers (all theoreticians except for a set of measure zero) that it did work. Thus J. Ranft and myself worked out the so-called 'thermodynamical model' in which (see Fig. 25.1) collective velocity distributions were combined with bootstrap thermodynamics in local rest frames [15]. This mammoth paper of 141 pages was too long to be read by anybody, with the consequence that even today people rediscover (independently) results and techniques buried in it. For us, it was a kind of handbook used in further work on the model. In particular, J. and G. Ranft and their collaborators in Leipzig developed it and confronted it with experiment (see later). We produced an 'Atlas of Particle Spectra' (Fig. 25.2) [16], in which we reproduced measured momentum spectra and predicted unmeasured ones up to $p_0 = 800$ GeV/c [for p–nucleus (Be, Al, Cu, Pb) up to 70 GeV/c] and discovered that the model obeyed 'Feynman scaling' and 'limiting fragmentation' [12].

While all these years much talk was about quarks, but experimental searches did not reveal any; confinement and QCD were still far in the future. It was speculated that maybe quarks are so heavy that they would be practically unobservable. Bootstrap thermodynamics was presumably the best way to calculate $q\bar{q}$ production rates as a function of their mass (applying perturbation theory to this problem was a then fashionable nonsense). The model was used to predict [17] $K\overline{K}$, $p\bar{p}$, $d\bar{d}$, and $q\bar{q}$ pair production with the result that the production rates of the first three agreed with experiment and that for quarks a mass of 4–5 nucleon masses would make them practically unobservable (with the techniques known at that time), even if they could, in principle, exist as free particles. If anyone had suggested to me then a $q\bar{q}$ plasma, I would have declared it impossible (alas, again).

One often-heard objection to the model was that resonances (inside a fireball) would not live long enough to justify treating them as 'particles'. G. Matthiae [18] proved, using the principle of detailed balance, that they might just live long enough.

It was tempting to apply the model to astrophysics: the Big Bang and neutron stars [19]. This triggered a number of papers, notably the one by K. Huang and S. Weinberg [20] and by J.C. Wheeler [21]. The then proposed 'limited temperature (T_0) Big Bang' depended entirely on the (at that time reasonable) interpretation of the model as yielding an infinite energy density at T_0.

For an easily readable summary of the situation up to 1972, see [14], where no prior knowledge about statistical bootstrap is required.

Microcanonical bootstrap

The invention of Veneziano's dual resonance model and the discovery that it also predicted an exponential mass spectrum [22, 23] added to the credibility the statistical bootstrap model had won in the meantime by successful applications. A number of people became interested and the following few years saw the appearance of some important papers.

S. Frautschi, who also coined the name 'Statistical Bootstrap Model' (SBM) [24], made a great breakthrough by reformulating the model (grand) microcanonically, instead of (grand) canonically. (Roughly, the original model was the Laplace transform of Frautschi's.) His bootstrap equation (BE) reads in invariant notation [25]

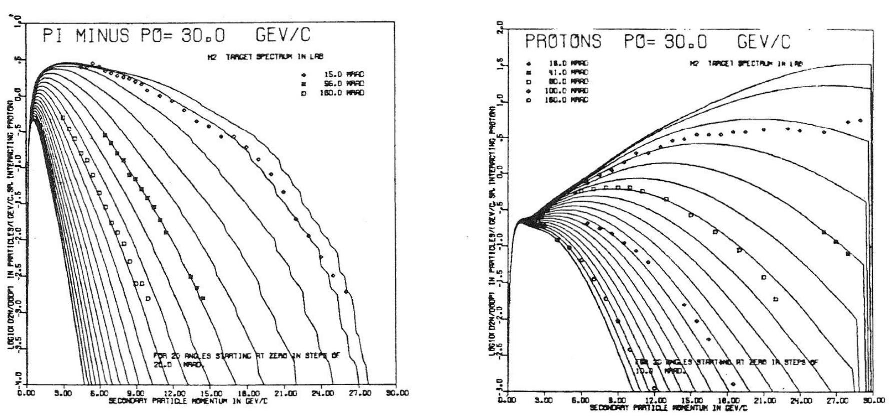


Fig. 25.2 Predicted and measured momentum spectra of π^- and p. Figures taken from the 'Atlas of Particle Spectra' [16]. *On left* the π^--angular spectra computed starting at zero (upper right) in steps of 20 mrad, data are for 15, 96, and 160 mrad. *On right* the p-angular spectra computed starting at zero (upper right) in steps of 10 mrad, data are for 18, 41, 60, 100, and 160 mrad

$$H\tau(p^2) = H\sum_{m_{\text{in}}}\delta_0(p^2 - m_{\text{in}}^2) + \sum_{n=2}^{\infty}\frac{1}{n!}\int\delta^4\left(p - \sum_{i=1}^{n}p_i\right)\prod_{i=1}^{n}H\tau(p_i^2)\mathrm{d}^4p_i\ , \quad (25.17)$$

where $\tau(m^2)\mathrm{d}m^2 \equiv \rho(m)\mathrm{d}m$, H is a universal constant, and m_{in} runs over the low mass stable hadrons (often only the pion). The philosophy was the same as before (with slight differences): $\tau(p^2)$ on the left-hand side of Eq. (25.17) was the density of states of the fireball with mass $m = \sqrt{p_\mu p^\mu}$ to be described, while the right-hand side expressed that it was either just one of the input particles or else composed of particles i having mass spectra $\tau(p_i^2)$.

Frautschi, Hamer, and Carlitz [24, 26–30] drew many important conclusions which would have been impossible to draw in the earlier canonical language. The new language used a vocabulary familiar to particle physicists (phase space), while the canonical one was repellent to them at that time. (Today, knowledge of thermodynamics is more widespread among high energy physicists.) Some of the new results were:

- Fireballs would predominantly decay into two fragments, one heavy and one light.
- By iterating their BE with realistic input, they found numerically $T_0 \approx 140$ MeV and $a = 2.9 \pm 0.1$ (which ruled out the until then adopted value $a = 5/2$).
- Each imposed conservation law implemented by fixing a quantum number, e.g., baryon number $\rho(B,m)$, in the mass spectrum, increases the value of a by 1/2.
- The microcanonical description gives a detailed account of what happens near T_0 (clustering, overshooting T_0, etc.).

W. Nahm [31] had meanwhile proved analytically that a must be exactly 3. This and the Frautschi–Hamer numerical result strengthened everyone's belief that T_0 must be a *limiting temperature*, since the energy density tends to infinity when $T \to T_0$. Thus, *there was no indication of a phase transition to a new state of matter* (alas, once more).

More checks with experiments were made ($\mathrm{p\overline{p}}$ annihilation, pair production, correlations, heavy particle production, ISR secondary spectra, Ericson fluctuations [32], and so on), which cannot be reviewed here. Instead I refer to the excellent review papers by E.M. Ilgenfritz, J. Kripfganz, and H.J. Möhring [33] and by A. Tounsi [34], in which the current technical improvements of the model (carried out in Leipzig, CERN, and Paris) are also described.

During these years, a group around L. Sertorio was busy relating the SBM to S-matrix theory [35–37] and clearing up many of the main open questions. This work is so extensive that I must refer to Sertorio's very complete and important review paper [38], which gives a full account of the more theoretical aspects up to 1978.

Other work included attempts, some successful, some less so, to equip the model with the following:

- Bose and Fermi statistics (present in the original canonical formulation [8], but extremely difficult to formulate in Frautschi's microcanonical one); Bose statistics was achieved [39–42], not Fermi.

- Internal symmetries [charge, isospin, SU(3), etc.] using group theoretical methods (based on early pioneering work by F. Cerulus [43]) [44–50].
- Angular momentum conservation [51, 52].

Much attention was given to the relation between SBM and dual resonance, string, and Regge pole models with the result that there was indeed a close resemblance, but not identity of the models [53–57]. Related to this question was a reconsideration of large-angle elastic scattering and Ericson fluctuations [32, 58–63].

Nucleon–antinucleon annihilation received new interest [64–68].

The description of particle production (momentum spectra, production rates, and so on) was improved [69–75] and extended to non-symmetrical collisions (πN, KN, etc.) [76] and even to p–nucleus collisions [77].

Exact analytical solutions of the BE

During the same years, our analytical understanding of the BE Eq. (25.17) advanced and solutions were constructed. The pioneer was J. Yellin, who had proposed Eq. (25.17) and solved it formally by Fourier transform (strictly speaking, not existing) and, via Eq. (25.22), by expansion in terms of phase space integrals [25]. This triggered a series of papers in which the BE was solved by Laplace transform (which does exist). The decay integral equation (sum over all possible decay modes) and its identity with the BE were discovered and a number of formal properties derived [45, 78–84]. Even a second quantized SBM was constructed, which led to the strange discovery that the creation and destruction operators of hadron clusters obey a non-associative (and, of course, non-commutative) algebra [85].

The bootstrap function

The analytical solution of the BE via Laplace transform mentioned above was so important for further development that I present it here. It goes as follows. Define the Laplace transforms

$$\Phi(\beta) := \int \mathrm{e}^{-\beta_\mu p^\mu} H\tau(p^2)\mathrm{d}^4 p\ , \tag{25.18}$$

$$\varphi(\beta) := \int \mathrm{e}^{-\beta_\mu p^\mu} H\delta_0(p^2 - m_{\mathrm{in}}^2)\mathrm{d}^4 p\ , \tag{25.19}$$

where $(\beta_\mu \beta^\mu)^{-1/2} = T$ is the temperature in the rest frame of the system. Laplace transforming Eq. (25.17) then gives

$$\Phi(\beta) := \varphi(\beta) + \mathrm{e}^{\Phi(\beta)} - \Phi(\beta) - 1\ . \tag{25.20}$$

Writing

$$G(\varphi) := \Phi(\beta)\ , \tag{25.21}$$

this yields the equation

$$\varphi = 2G - \mathrm{e}^{G} + 1\ , \tag{25.22}$$

which has been solved for the 'bootstrap function' $G(\varphi)$ by power expansion [25] and by an integral representation exploiting the analytical structure of Eq. (25.22) [86]. Most illuminating is the graphical solution of Eq. (25.22): draw the curve $x = 2y - \mathrm{e}^{y} + 1$ on left in Fig. 17.5 and interpret it as the graph of the inverse function, on right in Fig. 17.5.

It follows that, for *physical* systems,

$$\varphi \le \varphi_0 = \ln 4 - 1\ , \tag{25.23}$$

$$\Phi \le \Phi_0 = \ln 2\ . \tag{25.24}$$

In the simple case of one input mass, we have from Eq. (25.19)

$$\varphi(\beta) = 2\pi H m_{\mathrm{in}}^2 \frac{K_1(\beta m_{\mathrm{in}})}{\beta m_{\mathrm{in}}}\ ,$$

and the condition Eq. (25.23) reads

$$2\pi H m_{\mathrm{in}}^2 \frac{K_1(\beta_0 m_{\mathrm{in}})}{\beta_0 m_{\mathrm{in}}} = \ln 4 - 1\ , \tag{25.25}$$

which fixes a maximal temperature $T_0 = 1/\beta_0 \approx m_{\mathrm{in}}$. Thus, in this version of the model, T_0 is calculable (depending on H and m_{in}).

Furthermore, one recovers the earlier result of Nahm [31] that the singularity of $G(\varphi)$ at φ_0 is of the square-root type: $G(\varphi) = \Phi_0 - \mathrm{const.}\sqrt{\varphi_0 - \varphi} + \cdots$, from which it follows by inverse Laplace transformation of Eq. (25.18) that $\tau(m^2) \sim \mathrm{const.}\, m^{-3}\exp(m/T_0)$ [80], confirming once more that the energy density at T_0 is infinite and hence T_0 is a limiting temperature which might be attained or even exceeded only in transient states of superheating, but never in true equilibrium: *so still no phase transition to a new regime* (alas, for the fourth time).

It is important to observe that the same equation Eq. (25.22) is obtained *independently* of:

- the number of spacetime dimensions [83],
- the number of input particles (trivial),
- Abelian or non-Abelian symmetry constraints [50].

What changes is only the function φ, which may become very complicated, depending on various input masses and chemical potentials besides the temperature, while the relation between Φ and φ is always given by Eq. (25.22). Therefore, the 'bootstrap function' $G(\varphi)$ has a sort of invariant significance for SBM and it will also govern the discussion of the phase transition to which we will eventually come. [Note, however, that changing the BE by truncating it [80, 87, 88] or by making the volume n-dependent [89] will also change Eq. (25.22).] This invariance of the Laplace transformed BE suggests that it may appear in other contexts, and indeed it does: in renormalization theory, in nonlinear differential equations, and in combinatorics (references given in [86]). In this latter field, Eq. (25.22) was already discovered in the last century by E. Schröder [90]. He was solving a combinatorial problem which he explained in the following words (free translation):

> Given n elements, e.g., material points freely movable in space, one divides the set of n elements arbitrarily into subsets; some possibly containing only one element, some with two, three, etc., elements. Each set of more than one element is to be surrounded by a closed surface, which, together with its content, shall be considered as a new element on an equal footing with all the others. This new set of (obviously fewer than n) elements is submitted to the same procedure as long as one likes or until it ends because the next step would result in a single element. We wish to find the number of different complexions which can be generated by this procedure.

This is indeed SBM in its most abstract form. He terminates his paper with the prophetic words:

> Considerably more difficult would be the solution of the above problems in the case where, among the original elements, several are identical.

As if he had already known that, more than 100 years later, we would have difficulties with Bose and Fermi statistics!

The state of affairs up to 1978

The bootstrap equation (various versions) had been solved and the ensuing thermodynamics had been applied in particle physics, astrophysics, and cosmology. For particle physics applications, further assumptions about collective motions had been necessary [15], assumptions that had nothing to do with bootstrap thermodynamics, but contributed to the successes of the model. These and the confirmations coming from the dual models strengthened the belief that the model was basically right. The only challenge, large $p_\perp$, was interpreted as (at least partly) due to pre-equilibrium processes [15, 91] and/or angular momentum [52].

The picture was then as follows:

- Fireballs consist of fireballs (still pointlike, which is inconsistent, but bothered nobody).
- As a consequence, there is a limiting temperature T_0 which cannot be reached at finite energy density.

This picture seemed to provide a non-perturbative, analytically soluble model of the many-body aspects of strong interactions, running against a ‘phase transition’ whose other side would, however, be a forbidden land. It turned out later that removing the inconsistency of *pointlike* constituent fireballs would drastically change the situation and make the ‘phase transition’ a true transition with the other side no longer forbidden.

The state of affairs through 1978 is reviewed in the articles:

- Ref. [14] of 1973 for a heuristic introduction, ideology, and general results;
- Ref. [34] of 1973 and Ref. [33] of 1977 for a critical discussion and confrontation with experiment, and
- [38] of 1978 for more formal questions, the relation to the *S*-matrix, and so on.

Each of these needs the others for complementation.

25.4 The Phase Transition: Hadron Matter–Quark Matter

A phase transition has two sides. In our particular case, only lattice-QCD is so far able to handle both sides simultaneously; all other approaches (including ours) treat them separately and join them in hybrid models. That a quark-gluon matter phase might exist has been speculated since as early as 1969 [92] (maybe earlier[1]) and the first theoretical paper really devoted to it seems to be Carruthers' in 1973 [93]. From about 1975 on, so many papers appeared that I must refer to the articles collected in the Bielefeld proceedings of 1980 and 1982 [94, 95], where references are given.

In the present context, the approach from the hadron side is of interest. The obvious argument – who did invent it? – would be: if you compress hadron matter sufficiently, hadrons will overlap and cease to exist as such. You then have a quark-gluon soup. The ideas was used in many papers and got much support from the MIT bags [96], which are quark-gluon matter inside, hadrons outside. As many of the properties of bags strangely resemble the clusters (formerly called fireballs) of the SBM, one might expect that bootstrap thermodynamics would be equivalent to the thermodynamics of a bag gas. Indeed, an explicit model constructed by Baacke [97] made a phase transition very likely to happen near T_0. Two years earlier, Cabibbo and Parisi [98] had already proposed that the singularity found in SBM at T_0 should be related to the transition to the quark-gluon phase. These two arguments supported each other beautifully: SBM would provide for the *singularity necessary for a true phase transition*, while the bag gas model produced the crossing of the pressure curves of the plasma and the hadron gas near T_0.

Unfortunately, SBM was still in an underdeveloped stage in which two arguments spoke against Cabibbo and Parisi's interpretation:

- The energy density $\mathscr{E} \to \infty$ when $T \to T_0$.
- Quarks and gluons did not appear anywhere in SBM.

Therefore, how could a singularity, *which could not be reached* at finite energy density, indicate a *transition* to a phase whose *constituents did not appear* in the model? I felt strongly this way.

[1] I have reviewed Ref. [92] and find no mention of quark or parton phase, or for that matter, phase transition in this well-known publication. The very first mention of matter made of quarks that I know today about is by Ivanenko and Kurdgelaidze in 1965, see Ref. [8] in Chapter 11 on page 91. (JR)

Some people, however, felt – rightly – that these objections would be overcome by technical development without touching the essential features of SBM. Thus a number of papers, originating mostly in the fertile and critical soil of Bielefeld, investigated the various aspects of a phase transition, in the presence of an exponential spectrum: critical exponents, influence of fugacities, types of transition, and others [81, 99–106], to mention only a few references.

At this time, W. Nahm, during a workshop at Erice [107], severely criticized the shortcomings of the model and pointed out that they would forbid its application in astrophysics and to phase transitions, while particle physics applications might still be reasonable [108]. His criticism coincided, however, with a new formulation of SBM, just presented at the same workshop, in which most of the inconsistencies were removed [109], and to which we now turn.

Hadron volumes

The breakthrough came when J. Rafelski and myself tried to apply SBM to heavy ion collisions. There it was obvious that we could no longer deal with pointlike 'constituent clusters' of the extended clusters to be described. It took us a considerable time to equip the model with baryon conservation and proper hadron volumes. Our efforts, helped in a later stage by I. Montvay, resulted in the following treatment of the volume question:

- In the fully relativistic formulation we strived for, the volume had to be a four-vector parallel to the system it confined [110]:

$$V \longrightarrow V^{\mu} = A(m,\dots)p^{\mu} \; . \tag{25.26}$$

- If a cluster c consisted of clusters i, then

$$V_{\mathrm{c}}^{\mu} = K\sum_i V_i^{\mu} \; , \tag{25.27}$$

  where K is some constant.
- Hence, with Eq. (25.26),

$$A(m_{\mathrm{c}},\dots)p_{\mathrm{c}}^{\mu} = K\sum_i A(m_i,\dots)p_i^{\mu} \; . \tag{25.28}$$

- Since momentum conservation requires $p_{\mathrm{c}}^{\mu} = \sum p_i^{\mu}$, *it follows that the volume is proportional to the mass*:

$$K = 1 \text{ (dense packing)} \; , \quad A = \text{const. (independent of } m,\dots) \equiv (4\mathscr{B})^{-1} \; . \tag{25.29}$$

We identify $\mathscr{B}$ with the bag constant. These and other results were presented in Erice in the 1978 Workshop on Hadronic Matter at Extreme Energy Density [109].

However, this paper leaves one question unsolved: after treating the volume correctly on the level of the BE, the hadron gas should also take particle volumes into account (as already done by J. Baacke [97]). We later succeeded in doing so [111], by introducing the notion of the 'available volume', defined by subtracting (cluster) volumes from the 'external' (confining box) volume:

$$\Delta^{\mu} := V^{\mu}_{\text{ext}} - \sum_{\text{all particles } i} V^{\mu}_{i} \ . \tag{25.30}$$

Then, instead of keeping (as usual) the external volume fixed, we required Δ to be constant when summing over particle numbers and integrating over the mass spectrum, whereby V^{μ}_{ext} was pushed to infinity. By this trick, the partition function became *formally* that of *pointlike particles in the available volume* Δ, which we knew how to calculate explicitly. Thereafter, Δ had to be eliminated by using

$$\langle V_{\text{ext}} \rangle = \Delta + \frac{\langle E \rangle}{4\mathscr{B}} \ . \tag{25.31}$$

Then the energy density (and all other physical quantities) could be calculated. It turned out that

$$\frac{\langle E \rangle}{\langle V_{\text{ext}} \rangle} = \mathscr{E}(T,\mu) = \frac{\mathscr{E}_{\text{pt}}(T,\mu)}{1+\mathscr{E}_{\text{pt}}(T,\mu)/4\mathscr{B}} \ , \tag{25.32}$$

and similarly for other quantities. Here $\mathscr{E}_{\text{pt}}$ is the fictitious 'point particle energy density', which we can explicitly calculate and which diverges on the 'critical curve' [given by an equation like Eq. (25.25) with the baryon chemical potential μ incorporated]. Thus on the critical curve, *the energy density is now finite* [Eq. (25.32) with $\mathscr{E}_{\text{pt}}(T,\mu) \to \infty$]:

$$\mathscr{E}(T_{\text{crit}},\mu_{\text{crit}}) = 4\mathscr{B} = \text{bag energy density}\,. \tag{25.33}$$

The baryon number density is also finite along this curve, while the average cluster mass and volume tend to infinity, and the pressure to zero: *everything coalesces into one infinite cluster with the energy density of a bag*. Since inside highly excited (large) bags we have a quark-gluon gas, we should identify the single infinite cluster with quark-gluon matter, the transition to which is achieved just on the critical curve.

We have finally arrived at a sophisticated, relativistic van der Waals-type gas of strongly interacting particles: finite (mass proportional) cluster volumes represent repulsive forces, while the exponential mass spectrum resulting from bootstrap represents the attractive forces. The thus completed SBM leads to a phase transition on a critical curve (Fig. 25.3) qualitatively common to many models [95,106,112–116] to mention only a few of the many references. The results of QCD lattice calculations (see [95]) support our conclusions.

Yet, one might argue, how do you know that the singularity indicates a *phase transition to quark-gluon matter?* Nowhere in the SBM do quarks appear as input particles! Short of a proof, I can at least propose a chain of heuristic arguments:

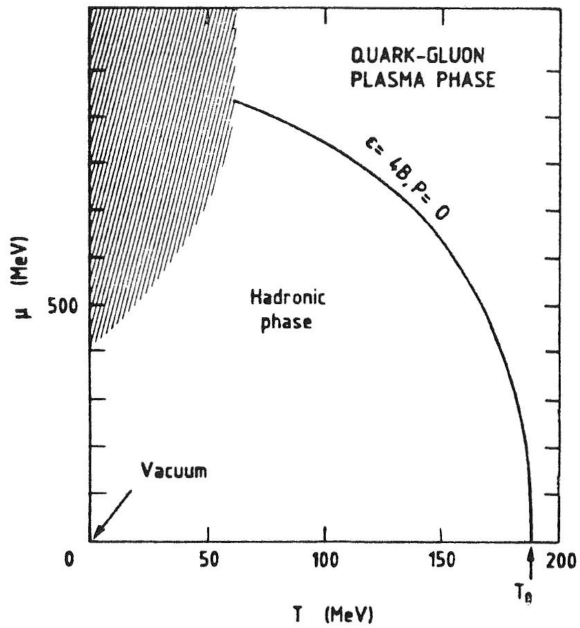


Fig. 25.3 The singular curve of the statistical bootstrap model. In the *shaded region*, the model is unreliable, because there the (otherwise negligible) effects of Bose–Einstein and Fermi–Dirac statistics become important.

- The identical properties of QCD bags and SBM clusters: both have the same mass–volume relation and the same mass spectrum up to the finest details with respect to conservation laws [117, 118]. These identities lead to the conclusion that QCD bags of interacting fields and SBM clusters are complementary descriptions of the same objects below the transition.
- If we are thus free to claim that, inside our clusters we do indeed have quark-gluon matter (though correlated by colour neutrality of constituent subclusters), then we do indeed *implicitly* have quarks and gluons in the model.
- The very nature of the bootstrap approach makes it rather irrelevant which are the input particles, provided the basic quantum numbers are represented. Instead of starting with quarks and gluons and imposing their interaction and confinement to colourless bound states, *we have accepted nature's own solution of the QCD bound state problem* and used the π and the nucleon (others could be added) as input. (After all, to describe a helium gas, you do not start with neutrons, protons, and electrons!)

Thus, without having a formal proof, I believe one can justify asserting that SBM predicts a phase transition (of which not yet all details are known) to a quark-gluon matter phase, as J. Rafelski and I proposed at the 1980 Bielefeld symposium[2] [119, 120].

A further criticism was brought up by V.V. Dixit and H. Satz: in the standard thermodynamical limit, a hard sphere gas, *even with an exponential mass spectrum*, cannot produce a singularity at any finite temperature [121]. Therefore, our method with the available volume violates the standard rules of taking the thermodynamic limit. The question was discussed at length [122] and it was shown that, since hadron matter at the regime near transition cannot be enclosed in boxes of fixed volume, the standard thermodynamical limit is not suitable. Using the 'grand canonical pressure ensemble' (without recourse to the available volume technique), the result of Dixit and Satz was reproduced, (see Chapter 24), our old results were also recovered, and this settled the question.

Here I must stop. Work goes on. For a review of present developments, applications, problems, and progress, see J. Rafelski and M. Danos [123, 124], also [125], and a rather daring and controversial paper of mine [91].

And if you ask me now why it took 27 years to arrive at the present (still problematic) state, let me answer with Shakespeare [126]:

> There are more things in Heaven and Earth,
> Horatio, than are dreamt of in your philosophy.

Acknowledgements

My sincere thanks and admiration go to all who have worked in this field and brought it into the present state, but in particular to those with whom I had, over the years, the pleasure of close contact, discussions, and collaboration: A. Auberson, R.D. Carlitz, F. Cerulus, M. Chaichian, J. Engels, B. Escoubès, E. Etim, G. Fast, R. Fiore, S. Frautschi, H. Grote, L.W. Jones, J. Kripfganz, J. Letessier, I. Montvay, W. Nahm, J. and G. Ranft, J. Rafelski, K. Redlich, H. Satz, K. Schilling, L. Sertorio, A. Tounsi, L. Turko, J. Vandermeulen, and U. Wambach.

References

1. H. Satz: Fortschr. Phys. **33**, 259 (1985)
2. E. Fermi: Progr. Theor. Phys. **5**, 570 (1950)
3. F. Cerulus, R. Hagedorn: Supp. Nuovo Cimento **9**, 646, 659 (1958)
4. G. Fast, R. Hagedorn: Nuovo Cimento **27**, 208 (1963); G. Fast, R. Hagedorn, L.W. Jones: Nuovo Cimento **27**, 856 (1963)
5. G. Auberson, B. Escoubès: Nuovo Cimento **36**, 628 (1965)
6. H. Satz: Nuovo Cimento **37**, 1407 (1965)

[2] Two months later a workshop at GSI was held in October 1980, and in this volume we reprint these two contributions in Chapters 26 and 27.

7. J. Vandermeulen: Bull. Soc. Roy. Sci., Lieège, Belgium **46**, 57 (1977)
8. R. Hagedorn: Suppl. Nuovo Cimento **3**, 147 (1965)
9. J. Orear: Phys. Lett. **13**, 190 (1964)
10. E. Beth, G.E. Uhlenbeck: Physica **4**, 915 (1937)
11. S.Z. Belenkij: Nucl. Phys. **2**, 259 (1956)
12. R. Hagedorn: Nucl. Phys. B **24**, 93 (1970)
13. R. Hagedorn: CERN Yellow Report 71-12
14. R. Hagedorn: In Cargèse Lectures in Physics, **6**, 643 (1973) (Gordon & Breach, New York)
15. R. Hagedorn, J. Ranft: Suppl. Nuovo Cimento **6**, 169 (1968)
16. H. Grote, R. Hagedorn, J. Ranft: CERN Report (Black) (1970)
17. R. Hagedorn: Suppl. Nuovo Cimento **6**, 311 (1968)
18. G. Matthiae: Nucl. Phys. B **7**, 142 (1968)
19. R. Hagedorn: Astron. & Astrophys. **5**, 184 (1970)
20. K. Huang, S. Weinberg: Phys. Rev. Lett. **25**, 895 (1970)
21. J.C. Wheeler: Astrophys. J. **169**, 105 (1971)
22. S. Fubini, D. Gordon, G. Veneziano: Phys. Lett. B **29**, 679 (1969)
23. S. Fubini, G. Veneziano: Nuovo Cimento A **64**, 811 (1969)
24. S. Frautschi: Phys. Rev. D **3**, 2821 (1971)
25. J. Yellin: Nucl. Phys. B **52**, 583 (1973)
26. C.J. Hamer, S. Frautschi: Phys. Rev. D **4**, 2125 (1971)
27. R.D. Carlitz: Phys. Rev. D **5**, 3231 (1971)
28. S. Frautschi: Nuovo Cimento A **12**, 133 (1972)
29. C.J. Hamer: Phys. Rev. D **10**, 1458 (1974)
30. C.J. Hamer: Nucl. Phys. B **95**, 173 (1975)
31. W. Nahm: Nucl. Phys. B **45**, 525 (1972)
32. S.C. Frautschi: In [107], p. 1
33. E.M. Ilgenfritz, J. Kripfganz, H.J. Möhring: Fortschr. Phys. **25**, 123 (1977)
34. A. Tounsi: Sixteenth Scottish Universities Summer School 1973, London 1974
35. A. Bassetto, L. Sertorio: Nuovo Cimento A **14**, 548 (1965)
36. P. Fré, L. Sertorio: Nuovo Cimento A **28**, 538 (1975)
37. P. Fré: Fortschr. Phys. **25**, 579 (1977)
38. L. Sertorio: Rivista Nuovo Cimento **2**, 1 (1979)
39. K. Fabricius, U. Wambach: Nucl. Phys. B **62**, 212 (1973)
40. J. Engels, K. Fabricius, K. Schilling: Phys. Lett. B **59**, 477 (1975)
41. J. Kripfganz, J. Ranft: Nucl. Phys. B **100**, 302 (1975)
42. M. Chaichian, M. Hayashi: Phys. Rev. D **15**, 402 (1977)
43. F. Cerulus: Nuovo Cimento **19**, 528 (1961)
44. E.M. Ilgenfritz, J. Kripfganz: Nucl. Phys. B **62**, 141 (1973)
45. C.J. Hamer: Phys. Rev. D **8**, 3558 (1973)
46. J. Engels, K. Fabricius, K. Schilling: Phys. Lett. B **53**, 65 (1974)
47. J. Engels, K. Fabricius, K. Schilling: Nuovo Cimento A **23**, 581 (1974)
48. F. Csikor, I. Farkas, Z. Katona, I. Montvay: Nucl. Phys. B **74**, 343 (1974)
49. J. Kripfganz, E.M. Ilgenfritz: Phys. Lett. B **48**, 329 (1974)
50. K. Redlich, L. Turko: Z. Phys. B **92**, 207 (1975)
51. C.B. Chiu, R.L. Heimann: Phys. Rev. D **4**, 3184 (1971)
52. R. Hagedorn, U. Wambach: Nucl. Phys. B **123**, 382 (1977)

53. M.I. Gorenstein, V.A. Miransky, V.P. Shelest, B.V. Struminsky, G.M. Zinovjev: Phys. Lett. B **43**, 73 (1973)
54. H. Satz: Phys. Lett. B **44**, 373 (1973)
55. J. Dethlefsen, H.B. Nielsen, H.C. Tze: Phys. Lett. B **48**, 48 (1974)
56. M.I. Gorenstein, V.A. Miransky, V.P. Shelest, G.M. Zinovjev, H. Satz: Nucl. Phys. B **76**, 453 (1974)
57. A. Strumia, G. Venturi: Lett. Nuovo Cimento **13**, 337 (1975)
58. F. Johns, B. Margolis, W.J. Meggs, R.K. Logan: Phys. Rev. Lett. **29**, 756 (1972)
59. G. Eilam, Y. Gell, B. Margolis, W.J. Meggs: Phys. Rev. B **8**, 2871 (1973)
60. B. Margolis, W.J. Meggs, R.K. Logan: Phys. Rev. D **8**, 199 (1973)
61. C.B. Chiu: Phys. Rev. D **13**, 2091 (1976)
62. S. Kogitz, R.K. Logan, S. Tanaka: Phys. Rev. D **14**, 1420 (1976)
63. R.K. Logan, M. Bates, S. Kogitz: Phys. Rev. D **25**, 2441 (1982)
64. C.J. Hamer: Nuovo Cimento A **12**, 162 (1972)
65. H.J. Möhring, J. Kripfganz, E.M. Ilgenfritz, J. Ranft: Nucl. Phys. B **85**, 221 (1975)
66. H.J. Möhring: Nucl. Phys. B **87**, 509 (1975)
67. J. Vandermeulen: Bull. Soc. Roy. Sci., Liège, Belgium **46**, 57 (1977)
68. H.J. Möhring: Acta Phys. Polon. B **9**, 125 (1978)
69. J. Ranft: Phys. Lett. B **31**, 529 (1970)
70. R. Hagedorn, J. Ranft: Nucl. Phys. B **48**, 157 (1972)
71. J. Letessier, A. Tounsi: Nuovo Cimento A **11**, 353 (1972)
72. J. Letessier, A. Tounsi: Nuovo Cimento A **13**, 557 (1973)
73. J. Letessier, A. Tounsi: Nuovo Cimento A **15**, 358 (1973)
74. J. Kripfganz, J. Ranft: Phys. Lett. B **55**, 301 (1975)
75. J. Ranft, G. Ranft: Nucl. Phys. B **92**, 207 (1975)
76. H.R. Gerhold, B. Buschbeck: Nuovo Cimento A **22**, 39 (1974)
77. J.P. Alard, A. Baldit, R. Brun, J.P. Costilhes, J. Dhermain, J. Fargaix, L. Fraysse, J. Pellet, G. Roche, J.C. Tamain, A. Cordaillat, A. Pasinetti: Nuovo Cimento A **30**, 320 (1975)
78. E.M. Ilgenfritz, J. Kripfganz: Nucl. Phys. B **56**, 241 (1973)
79. I. Montvay: Phys. Lett. B **42**, 466 (1972)
80. R. Hagedorn, I. Montvay: Nucl. Phys. B **59**, 45 (1973)
81. R. Fiore, R. Page, L. Sertorio: Nuovo Cimento A **37**, 45 (1977)
82. M. Feigenbaum, L. Sertorio: Nuovo Cimento A **43**, 31 (1978)
83. H. Satz: Phys. Rev. D **20**, 582 (1979)
84. R. Fiore: Z. Phys. C **4**, 281 (1980)
85. E. Etim, R. Hagedorn: Nucl. Phys. B **131**, 429 (1977)
86. R. Hagedorn, J. Rafelski: Comm. Math. Phys. **83**, 563 (1982)
87. C.B. Chiu, J. Johnson: Phys. Lett. B **42**, 475 (1972)
88. C.B. Chiu: Nucl. Phys. B **54**, 170 (1973)
89. I.Ya. Pomeranchuk: Dokl. Akad. Nauk. **78**, 889 (1951)
90. E. Schröder: Z. Math. Phys. **15**, 361 (1870). (For a modern representation of this work, see L. Comtet: *Advanced Combinatorics*, Reidel Pub. Co. 1970.)
91. R. Hagedorn: Riv.Nuovo Cim. **6**, 1 (1984)
92. J.D. Bjorken, E. Paschos: Phys. Rev. **185**, 1975 (1969)
93. P. Carruthers: Collective Phenomena **1**, 147 (1973)
94. H. Satz (Ed): *Statistical Mechanics of Quarks and Hadrons*, Bielefeld Symposium 1980 (North Holland 1980)

95. M. Jacob, H. Satz (Eds): *Quark Matter Formation and Heavy Ion Collisions*, Bielefeld Workshop 1982 (World Scientific 1982)
96. A. Chodos, R.L. Jaffe, K. Johnson, C.B. Thorn, V.F. Weisskopf: Phys. Rev. D **19**, 3471 (1974)
97. J. Baacke: Acta Phys. Polon. B **8**, 625 (1977)
98. N. Cabibbo, G. Parisi: Phys. Lett. B **59**, 67 (1975)
99. D.E. Miller: Phys. Rev. Lett. B **64**, 193 (1976)
100. I. Montvay, H. Satz: Nuovo Cimento A **39**, 425 (1977)
101. R. Fiore, R. Page, L. Sertorio: Nuovo Cimento A **44**, 531 (1978)
102. R. Fiore: Z. Phys. C **1**, 305 (1979)
103. J. Letessier, A. Tounsi, H. Satz: Nuovo Cimento A **45**, 580 (1978)
104. H. Satz: In [107], p. 269
105. F. Karsch, D.E. Miller: Phys. Lett. B **79**, 259 (1978)
106. T. Celik, H. Satz: Z. Phys. C **1**, 163 (1979)
107. N. Cabibbo, L. Sertorio (Eds): *Hadronic Matter at Extreme Energy Density*, Erice, Italy 1978 (Plenum Press 1980)
108. W. Nahm: In [107], p. 219
109. R. Hagedorn, I. Montvay, J. Rafelski: In [107], p. 49; In this volume we offer a more focused reort, see Chapter 23
110. B. Touschek: Nuovo Cimento B **58**, 295 (1968)
111. R. Hagedorn, J. Rafelski: Phys. Lett. B **97**, 136 (1980)
112. J.I. Kapusta: Nucl. Phys. B **148**, 461 (1979)
113. O.K. Kalashnikov, V.V. Klimov: Phys. Lett. B **88**, 328 (1979)
114. E.V. Shuriak: Physics Reports **61**, 71 (1980)
115. K. Kajantie, H. Miettinen: Z. Phys. C **9**, 341 (1981)
116. J. Kuti, B. Luckacs, J. Polonyi, K. Szlachanyi: Phys. Lett. B **95**, 75 (1980)
117. J.I. Kapusta: Nucl. Phys. B **196**, 1 (1982)
118. K. Redlich: Z. Phys. C **21**, 69 (1983)
119. R. Hagedorn, J. Rafelski: In [94], p. 237
120. J. Rafelski, R. Hagedorn: In [94], p. 253
121. V.V. Dixit, F. Karsch, H. Satz: Phys. Lett. B **101**, 412 (1981)
122. R. Hagedorn: Z. Phys. C **17**, 265 (1983)
123. J. Rafelski: In [125], p. 331
124. J. Rafelski, M. Danos: *Perspectives in High Energy Nuclear Collisions*, Inst. f. Theor. Phys. Frankfurt Univ. and US Nat. Bureau of Standards NBSIR-83-2725
125. M. Jacob, J. Tran Thanh Van (Eds): Physics Reports **88**, 321 (1982)
126. W. Shakespeare: Hamlet (1601)

Part III

Melting Hadrons, Boiling Quarks
Heavy Ion Path to Quark-Gluon Plasma

edited by Johann Rafelski

Contributions by:

Rolf Hagedorn, Johann Rafelski

In 1977 we embarked on the study of hot nuclear-hadron matter and its connection to deconfined hot quark-gluon matter, which we soon called QGP. By late 1978, in addition to us, several other researchers recognized that one can melt the confining vacuum structure and reach quark deconfinement in RHI collisions. By 1979, as awareness of paradigm-changing discovery opportunity spread, the leading particle and nuclear physics laboratories were evaluating their options.

The following reports characterize the events at key laboratories from the period Summer 1980 – Summer 1983:

a) The **GSI laboratory** at Darmstadt, Germany. GSI, a participant in the LBL, and the nascent CERN HI programs, was also preparing its own RHI accelerator proposal, see Chapter 13. In October 1980 a workshop was staged. Hagedorn, see Chapter 26, showed methods from the *pp* collisions adapting these to the new *AA* relativistic collision domain. I presented, see Chapter 27, specific results that near term experiments could address. These two contributions, published by the GSI home press, summarized our understanding of the collision dynamics, the approach to equilibrium, the properties of hadron gas, and of hot quark-gluon matter, the dynamical evolution of the dense fireball, and, propose strangeness and strange antibaryons as the signature of QGP. Few copies of the GSI report survived to the present.

b) At **CERN** RHI collisions were long seen as a possible research direction for the ISR collider. Chapter 28 describes the context and provides the QGP presentation made in January 1981 to the ISR 'soft hadron' community. Soon after constraints arising from the need to build the LEP (present day LHC tunnel) redirected the attention at CERN to the SPS. Chapter 29 presents the related decisive discussion at the CERN Science Policy Committee (SPC) meeting of June 1982. This discussion relied on a meeting held in May 1982 at Bielefeld. My Bielefeld 'Strangeness' report did not appear in the proceedings, due to several mishaps described in Chapter 30. Influential in shaping the experimental program, this report Chapter 31, was hard to find when the experiments began a decade later.

c) Another workshop defining the **LBL** future project was held June 1983. I presented strangeness signature of QGP, see Chapter 32. My LBL report was distributed in a LBL printed proceeding volume, and disappeared from view in consideration of **RHIC at BNL** becoming the US experimental facility.

In consideration of the overlap between CERN-Bielefeld Chapter 31 and LBL-Chapter 32 lectures I have omitted the duplicate material such that Chapter 31 is focused on strangeness production in QGP, and Chapter 32 on strangeness in hadronic gas. All the ideas presented in regard to strangeness QGP signature are reproduced verbatim in these sections.

Our (Hagedorn and Rafelski) work on RHI collisions presented here shows two primary insights: a) accessibility of quark deconfinement at relatively modest (low SPS-range) heavy ion energies, and b) the opportunity that strangeness signature of quark-gluon plasma holds for the discovery of both quark deconfinement, and a new phase of matter.

A general retrospective in Chapter 33 completes this book.

Chapter 26
How to Deal with Relativistic Heavy Ion Collisions – 1980

Rolf Hagedorn

Abstract A qualitative review is given of the theoretical problems and possibilities arising when one tries to understand what happens in relativistic heavy ion collisions. The striking similarity between these and *pp* collisions suggests the use of techniques similar to those used five to twelve years ago in *pp* collisions to disentangle collective motions from thermodynamics. A very heuristic and qualitative sketch of statistical bootstrap thermodynamics concludes an idealized picture in which a relativistic heavy ion collision appears as a superposition of moving 'fireballs' with equilibrium thermodynamics in the rest frames of these fireballs. The interesting problems arise where this theoretician's picture deviates from reality: non-equilibrium, more complicated motion (shock waves, turbulence, spin) and the collision history. Only if these problems have been solved or shown to be irrelevant can we safely identify signatures of unusual states of hadronic matter as, for example, a quark-gluon plasma or density isomers.

26.1 Introduction

During the last two years when I was working with Johann Rafelski on the Statistical Bootstrap Model [1] in order to adapt it to describe hot nuclear matter, I came more and more often across people concerned with relativistic heavy ion collisions, and also slowly became acquainted with the literature of this field – only to become more and more aware of how similar its problems are to those encountered in the beginning of particle physics. Of the many different theoretical models invented and applied in the development of hadron physics, there is one – the 'thermodynam-

---

Invited lecture at Quark Matter 1: *Workshop on Future Relativistic Heavy Ion Experiments* at the Gesellschaft für Schwerionenforschung (GSI), Darmstadt, Germany, 7-10 October 1980; circulated in the GSI81-6 Orange Report, R. Bock and R. Stock, editors; and as preprint CERN-TH-3014 dated 5 January, 1981 available at http://cds.cern.ch/record/134307.

(deceased) CERN-TH, 1211 Geneve 23, Switzerland

ical model' [2], which tries to describe just those aspects of high energy particle collisions which are most strikingly similar to the main ones of relativistic heavy ion collisions, namely, the many-body aspects with an intimate mixture of coherent collective and incoherent stochastic movements.

I think one can still claim that the thermodynamical model was successful when applied with care and precaution. The well-known 'large transverse momenta' do not invalidate this model; they belong to phenomena outside its range of validity, as I shall explain later. That there was success at all – one dared to apply statistical thermodynamics to two-body collisons of elmentary (sic) particles – was due to the many degrees of freedom in the final states and, without doubt, also to the colliding 'elementary' particles being much less elementary than one thought 30 years ago. The analogy with relativistic heavy ion collisions becomes obvious when the 'elementary' particles are considered as bags [3] filled with quarks and gluons. If I anticipate here that the present form of the statistical bootstrap model has good reasons to claim that, in collisions with a few GeV per nucleon, the individual bags will melt into a single bag, then the analogy between a *pp* collision and a relativistic heavy ion collision is perfect; remaining differences in the theoretical treatment of these collisions are quantitative, but not principal.

It is therefore not surprising that several ideas of the thermodynamical model have been independently rediscovered by people concerned with relativistic heavy ion collisions. With all this in mind, I have the courage to dig deep into the past and uncover a few forgotten things which may still be useful for today's relativistic heavy ion collisions. The rather explicit list of references should compensate for the extremely qualitative style of this talk.

Notation and abbreviations: We use energy units MeV and GeV and set

$$\hbar = c = k = 1 \, .$$

We use the abbreviations the relativistic heavy ion collision for relativistic heavy ion collisions and SBM for statistical bootstrap model.

26.2 Collective Motions

To my knowledge, Weisskopf [4] was the first to apply thermodynamics to the emission of particles from excited nuclei. The situation was favourable to such an approach: the excitation energy was low and the compound nucleus was long lived enough to reach an equilibrium state.

One would think that this could no longer be true in elementary collisions or the relativistic heavy ion collision. Nevertheless, when the first pion producing *pp* collisions were analyzed, Koppe [5] realized that they could be interpreted as pion evaporation from some hot object of elementary dimensions. To honour Koppe for this pioneering work, this model was called the 'Fermi statistical model' [6].

The hot dense object became known as a 'fireball'. Very soon it was discovered that a single fireball could not explain the momentum distribution of emitted particles; it was impossible to find, for any given event, a Lorentz frame in which the momentum distribution was isotropic. Indeed, this should not have been expected, since even if a single fireball had formed it would, in general, have a very high spin. Moreover, phase space calculations show that the actual anisotropy – a forward/backward jet in the centre-of-momentum frame – cannot be accounted for by assuming a single, high spin fireball [7]. This is easily understood: the initial state has very definite phase relations between its individual partial waves; a single fireball, even if considered as a statistical sum over spins, cannot reproduce these phase relations.

It was then found, with the help of ingeniously chosen variables [8], that two fireballs moving with large opposite velocities in the CM frame were a much better approximation of reality. Adding a third fireball, at rest in CM, substantially improved the picture [9]. Of course, the two oppositely moving fireballs need not have the same mass nor the same speed, and the third could also have some velocity in the CM. Therefore, one should rather introduce mass and velocity distributions, but then why have just three fireballs? Why not sometimes one, sometimes two, and sometimes three or even more? Thus, one should also introduce a distribution for the number of fireballs. It seems that, in this way, one obtains so much freedom that one can fit everything. This is not so if some simple model assumptions are made which are based on observation and which are very restrictive. This was done in the thermodynamical model [2], which I shall briefly describe. It was designed to predict inclusive momentum distributions and branching ratios of particles produced in high energy *pp* collisions, but it was later easily adapted to πp, Kp [10], and *pA* (even heavy nuclei) collisions [11]. Some of the simplifying assumptions may be grossly wrong if extended to the relativistic heavy ion collision. I shall come back to this.

The simplifying postulates were [2, 12]:

Postulate 1. In high energy collisions of hadrons, collective motions have only components in the direction of the collision axis. It is possible to find a continuum of comoving Lorentz frames (local rest frames) such that a comoving observer will, in his neighbourhood, see only thermal motion. Turbulence is absent.

Postulate 2. All the kinetic energy of the incoming particles, which disappears by decelerating hadronic matter, is adiabatically and locally converted into excitation energy (heat).

Postulate 1 is illustrated in Figs. 26.1, and 26.2 The first figure images are taken from my CERN lectures in 1971 [12], while the second are from recent theoretical articles on the relativistic heavy ion collision [13, 14]. Figure 26.1 is a picture of the distribution at the moment of impact, while Figs. 26.2 show a time development, on left in a model [13] and on right in a simplified hydrodynamic calculation [14], the two results are obtained under different assumptions about the equation of state from hydrodynamical calculations for a symmetric system. Some transverse collective motion is present in the latter case

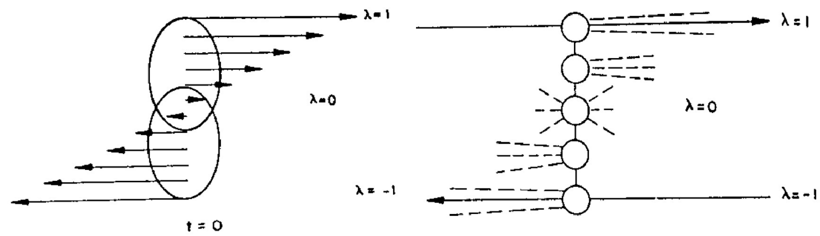


Fig. 26.1 Velocity distribution in a collision at the moment of impact, interpreted as continuous (*on left*) or as a probability distribution of fireballs (*on right*), from Ref. [12]

Useful variables

We use β = velocity and $\gamma = (1-\beta^2)^{-1/2}$ = Lorentz factor. A momentum four-vector is then $p = m(\gamma, \beta\gamma)$. A Lorentz transformation along a given direction is fully determined by β or γ. It brings a particle from rest to velocity β.

A very useful variable is the 'rapidity' η defined by

$$\gamma = \cosh\eta \ , \qquad \beta\gamma = \sinh\eta \ , \qquad \beta = \tanh\eta \ . \tag{26.1}$$

Hence η is the angle if a Lorentz transformation is represented as a rotation in Minkowski space. As the product of two rotations about the same axis is the rotation with the sum of the two angles, it follows for the product of two parallel and rotation free Lorentz transformations

$$L(\eta_2) \circ L(\eta_1) = L(\eta_1 + \eta_2) \ . \tag{26.2}$$

In applying Postulate 1, we shall have to ascribe at any moment to a fireball a velocity along the collision axis and, according to Postulate 2, an internal excitation equal to the kinetic energy which has disappeared by decelerating it from the initial to its present velocity. The excitation energy of a fireball must therefore be a function of the initial as well as of the actual velocity. Therefore, a suitable velocity variable should contain both the actual and the initial velocities. Giving the initial one a subscript zero, reasonable choices for velocity variables are then

$$\lambda := \frac{\sinh\eta}{\sinh\eta_0} \ , \quad \text{or } \lambda := \frac{\eta}{\eta_0} \ , \quad \text{or } \lambda := \frac{\gamma - 1}{\gamma_0 - 1}\,\mathrm{sign}(\beta) \ . \tag{26.3}$$

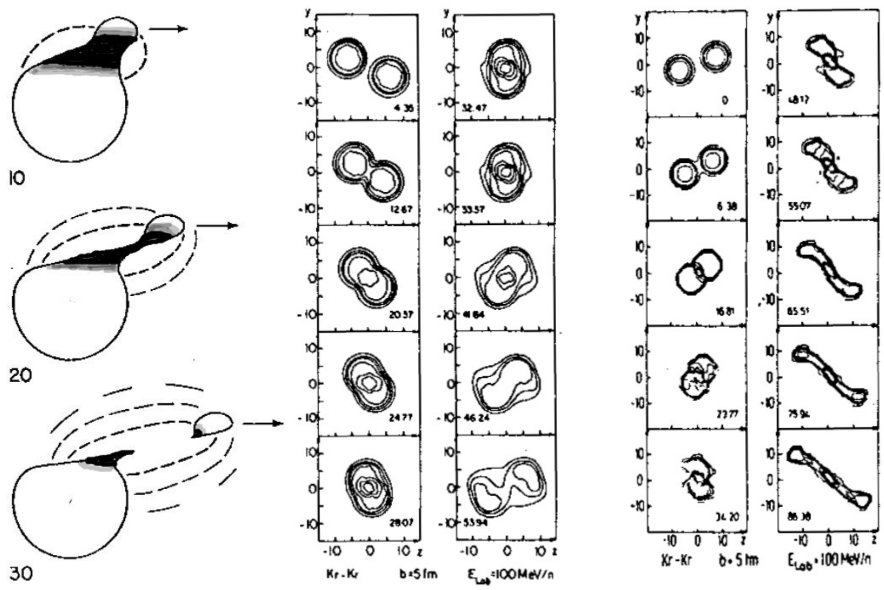


Fig. 26.2 *On left*: Shape and velocity distribution in a collision of 250 MeV/nucleon Neon on Uranium, seen at different times from top to bottom 10, 20, 30 fm/s after the moment of impact (qualitative figure follows Fig 1 in Ref. [13]); *On right:* Temporal development of density distribution in a low energy collision from a hydrodynamical calculation (the two sets of figures belong to different hydrodynamical assumptions) from Ref. [14]

Note that $\lambda := \beta/\beta_0$ is not suitable since β and β_0 are almost always near to one and thus such a λ would have no 'resolution power' for analyzing a relativistic velocity spectrum. The three other possible choices above all have good resolution power and they share the property $-1 \leq \lambda \leq 1$. The first choice makes λ almost equal to Feynman's variable x [15], while the second choice does not seem to have been used, and the third is still used today in the thermodynamical model [2, 16, 17]. It has the advantage of being physically obvious, since $(\gamma - 1)/(\gamma_0 - 1)$ is the ratio of the actual kinetic energy density to the initial one of a decelerated volume element. However, it has the disadvantage of not being an analytic function as the other two are. Today I would prefer the first choice, but in this talk I leave the choice open. It could be any of the three or even some other one.

Momentum distributions

What is the situation now? We assume that there is a velocity distribution of hadronic matter which will depend on time t, space point $\mathbf{x}$, impact parameter $\mathbf{b}$, and 'velocity' λ :

$$\text{longitudinal velocity distribution} \quad u(\lambda, \mathbf{x}, t, \mathbf{b}) \,. \tag{26.4}$$

If properly normalized, $u(\lambda, \mathbf{x}, t, \mathbf{b})\mathrm{d}^3x\,\mathrm{d}^3b$ is the probability that the piece of matter contained in d^3x has, at time t, the velocity λ when the impact parameter of the collision was in $\{\mathbf{b}, \mathrm{d}^3b\}$.

According to Postulate 1, such a piece of matter is, for a comoving observer, hot matter at rest in equilibrium, having a certain local temperature T. From very general arguments about black body radiation, it then follows that, in this volume element, the momentum distribution of particles with mass m will be

$$\mathrm{d}^3x\, f_m(\mathbf{p}, T)\mathrm{d}^3p = \frac{\mathrm{d}^3x}{(2\pi)^3} \frac{\mathrm{d}^3p}{\exp\frac{\sqrt{p^2+m^2}}{T} \pm 1} \,. \tag{26.5}$$

For $m = 0$, this is just the Planck formula which initiated quantum physics.

The local temperature T can now be calculated from Postulate 2 if an equation of state is known (see below). Consider a piece of incoming hadron matter. Before the collision, it has the rest energy density ε_0 of cold hadron matter. Now follow it as a comoving observer until it has decelerated from its initial Lorentz factor γ_0 to the actual Lorentz factor γ at time t. For the comoving observer, it is still at rest, but now has rest energy density ε because the initial kinetic energy has become excitation energy. Postulate 2 asserts that

$$\varepsilon\gamma = \varepsilon_0\gamma_0 \,. \tag{26.6}$$

Assuming the equation of state is known, we furthermore have

$$\varepsilon = \varepsilon(T) \,.$$

This can be inverted to give $T(\varepsilon)$, and since we have $\varepsilon = \varepsilon_0\gamma_0/\gamma$, it follows that $T = T(\lambda, \gamma_0)$.

Now put everything together to obtain the momentum distribution of particles of mass m in any fixed Lorentz frame. To be definite, we may choose the CM frame of the collision:

$$\Big[W_m(\mathbf{p})\mathrm{d}^3p\Big]_{\mathrm{CM}} = \int_0^{2R} \mathrm{d}^3b \int \mathrm{d}t\, \mathrm{d}^3x \int \mathrm{d}\lambda\, u(\lambda, \mathbf{x}, t, \mathbf{b}) L(\lambda) \Big[f_m\big(T(\lambda, \gamma_0), \mathbf{p}'\big)\mathrm{d}^3p'\Big] \,. \tag{26.7}$$

This formula does the following: for fixed $\lambda, \mathbf{x}, t, \mathbf{b}$, the momentum distribution $f_m(T, \mathbf{p}')\mathrm{d}^3p'$ in the local λ rest frame is Lorentz transformed by $L(\lambda)$ to the CM frame and then the integrations sum up all these local contributions to yield $W_m(\mathbf{p})_{\mathrm{CM}}$.

We now observe that neither the local spectrum $f_m(T, \mathbf{p}')$ nor the Lorentz transformation depend on $\mathbf{x}, t, \mathbf{b}$. Therefore, we can immediately integrate over these variables and obtain a new weight function

$$VF(\lambda, \gamma_0) := \int \mathrm{d}^3b \int \mathrm{d}t\, \mathrm{d}^3x\, u(\lambda, \mathbf{x}, t, \mathbf{b}) \,, \tag{26.8}$$

where V is the total interaction volume and Eq. (26.7) reduces to

$$W_m(\mathbf{p})\mathrm{d}^3p = \int_{-1}^{1} F(\lambda,\gamma_0)L(\lambda)\Big[V f_m\big(T(\lambda,\gamma_0),\mathbf{p}'\big)\mathrm{d}^3p'\Big] , \tag{26.9}$$

where $F(\lambda,\gamma_0)$ now picks up all contributions to a given λ summed over the entire spacetime history and all impact parameters. This formula can be written in a manifestly covariant way.

In Eq. (26.9), everything is known except the weight function $F(\lambda,\gamma_0)$. As a probability distribution, it must obey

$$\int_{-1}^{0} F(\lambda,\gamma_0)\mathrm{d}\lambda = \int_{0}^{1} F(\lambda,\gamma_0)\mathrm{d}\lambda = 1 . \tag{26.10}$$

We normalize it independently over each half interval in order to allow target and projectile to have different mass. Now $F(\lambda,\gamma_0)$ is normalized and defined over the γ_0 independent interval $\{-1,0;0,1\}$. Hence, if γ_0 varies, F can only change shape. One would like to choose the definition of λ in such a way that $F(\lambda,\gamma_0)$ becomes a scaling function:

$$\lim_{\gamma_0\to\infty} F(\lambda,\gamma_0) = F(\lambda) . \tag{26.11}$$

It has turned out that the choice made in the thermodynamical model [2], that is, $\lambda = (\gamma-1)/(\gamma_0-1)$, which was made with that aim, almost led to the desired behaviour of Eq. (26.11). From 10 GeV to 1000 GeV (ISR), $F(\lambda,\gamma_0)$ did not detectably depend on γ_0 if fitted to experiments [18]. However, if any, then only one choice of λ can lead to Eq. (26.11), because if $\lambda_f = f(\gamma,\gamma_0)$ does so, then any other choice $\lambda_g = g(\gamma,\gamma_0)$ will not, unless a function of λ_f above.

Determination of the weight function $F(\lambda,\gamma_0)$

One can have two attitudes:

1. Try to calculate $F(\lambda,\gamma_0)$ from some model:[1]
    - from Regge poles [19],
    - from form factors [20],
    - from relativistic kinematics [21,22],
    - from hydrodynamics [23],
    - from Monte Carlo cascade calculations [24],
    - from the Boltzmann collision equation [25],

[1] Not all the listed references set out to calculate $F(\lambda,\gamma_0)$, but the models yield information which can be interpreted in terms of such a function.

- from any other models (I apologize to the authors not quoted due to my ignorance).

2. Find it by parametrization and fit to experiments. This was done by several authors [2, 17]. When fitted to pion production at *one* primary energy, the same $F(\lambda)$ gave good predictions for other different energies (up to ISR), for other secondaries (π, N, Y, K, $\overline{\mathrm{N}}$, $\overline{\mathrm{Y}}$, d, $\overline{\mathrm{d}}$, He^3, $\overline{\mathrm{He}}^3$, etc.) and for other projectiles and/or targets [10, 11]. As $F(\lambda)$ was parametrized with only two parameters (remaining the same and constant for all these processes), one could say that $F(\lambda)$ was nearly (i.e., within the precision of the fit and the comparison of predictions with the data [11]) a universal function, although at ISR energies, the behaviour at λ near zero suggested a violation of the desired form of scaling [18]. The various $F(\lambda)$ calculated from models differ among each other and from the empirical one, but never dramatically, except for a possible singularity at $\lambda = 0$.

Violations of the postulates 1 and 2

Our postulates worked rather well in particle physics, but they may fail in the relativistic heavy ion collision as follows:

a) Transverse collective motions

The function $F(\lambda)$ is designed to represent only longitudinal collective motions. In principle, there should also be a function $G(\lambda_\perp)$ for transverse collective motions, or better still, a function $F(\lambda)$ with λ representing three-dimensional collective motions. While in *pp* collisions, this was not necessary; hydrodynamic calculations [23] for the relativistic heavy ion collision indicate the existence of non-negligible transverse components of hydrodynamic flow and of shock waves [26]. Figure 26.3 shows the result of a non-relativistic hydrodynamical calculation [27]. Clearly, all such calculations greatly depend on the assumptions made for the equations of state, viscosity, compressibility, and so on. As a child, I was much impressed when I discovered that solid cold tar was like a liquid if one had patience (a stone would sink into a tar barrel within a couple of days), but it would shatter like glass if hit hard. Thus transverse motion may depend on the collision energy.

Theoretical work in this domain can greatly profit from experiment; we will always have a superposition of collective and heat motion. Heat motion is (as we shall see) limited to typical values as, for example, at ISR energies [28]:

$$\begin{aligned} \langle P_\perp \rangle_{\text{proton}} &\approx 500\ \text{MeV}/c \quad \Longrightarrow \quad \beta_\perp^{(\text{p})} \approx .47\ , \\ \langle P_\perp \rangle_\pi &\approx 350\ \text{MeV}/c \quad \Longrightarrow \quad \beta_\perp^{(\pi)} \approx .93\ . \end{aligned} \tag{26.12}$$

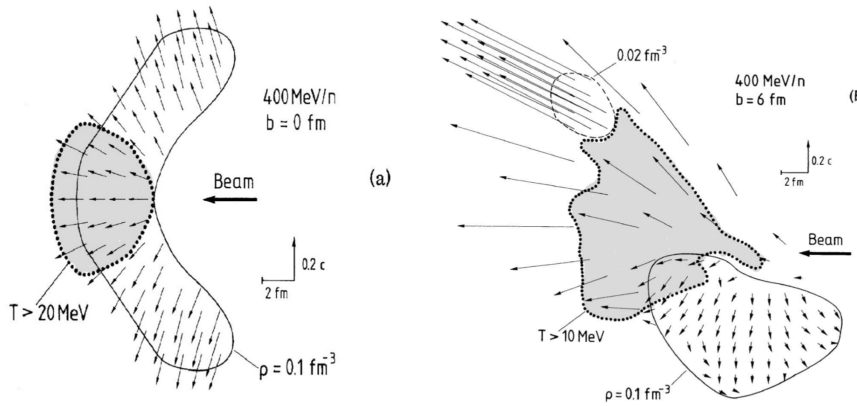


Fig. 26.3 Temperature, density, and velocity distributions in 400 MeV/nucleon Ne–U collisions, results from hydrodynamical calculations of Ref. [27]. Note the significant transverse velocity component.

Hence, the chance to observe transverse collective motion despite the thermal noise grows with the mass. For $m \gg T$, i.e., $m \gg m_\pi$, we have [2, 12]

$$\langle p_\perp(m,T)\rangle \approx \sqrt{\frac{\pi m T}{2}}\,. \tag{26.13}$$

Hence, with

$$\beta_\perp^2 = \frac{\gamma_\perp^2 - 1}{\gamma_\perp^2} = \frac{p_\perp^2}{m^2 + p_\perp^2}\,, \tag{26.14}$$

$$\langle \beta_\perp \rangle \approx \frac{\langle p_\perp \rangle}{m} \approx \sqrt{\frac{\pi T}{2m}} \qquad (m \gg T)\,, \tag{26.15}$$

the typical temperature for high collision energies is $T = 160$ MeV. This gives, with Eq. (26.15), for protons $\langle \beta \rangle \approx 0.52$, which shows that Eq. (26.15) already gives a good estimate for $m = m_{\mathrm{N}}$ and is rather good for heavier masses. Since even for very large collision energies $T \lesssim T_0 \approx 160$ MeV, we have for all energies above a few GeV/nucleon,

$$\langle \beta_\perp \rangle_A \approx .52/\sqrt{A}\,, \tag{26.16}$$

where A is the nucleon number of the emitted fragment. For heavy fragments, the thermal transverse velocity $\beta_\perp$ is therefore small and may become smaller than the transverse collective velocity. Therefore, the transverse momentum of heavy fragments should be studied carefully because it allows one to determine the collective transverse motion and to compare it to hydrodynamic calculations. Turbulence might also be detected by such measurements.

b) Violation of postulate 2

Postulate 2 is also certainly violated and here it seems to be difficult to say how this could be experimentally detected because there are many violating mechanisms:

- It is not true that an incoming volume element will simply be decelerated. It will also be deformed and its matter content will be mixed by mutual interpenetration with that of the collision partner. Nevertheless, it seems that the velocity weight function $F(\lambda, \gamma_0)$ is able to absorb part of this type of violation and that the fast rise of the local temperature towards some limiting value does the rest to dissimulate it.
- While heating up matter, some particle emission already takes place. Thus, heating is not quite adiabatic. However, emission is damped by $\exp(-m/T)$ so that only near the highest temperature reached in the heating process will particle emission be significant.
- Equilibrium might not be established even locally. We will come back to this point.
- Heat conduction might take place [29]. This will be negligible if break-up and particle emission is faster than heat transfer.
- Reabsorption of emitted particles violates not so much Postulate 2 as the assumption made in Eq. (26.9) that emission in the local rest frame is isotropic. Figure 26.4a shows how this can generate an asymmetry in the angular distribution ('hot spot') [29]. Figure 26.4b shows that this asymmetry would also disappear if longitudinal break-up into many fireballs is fast.

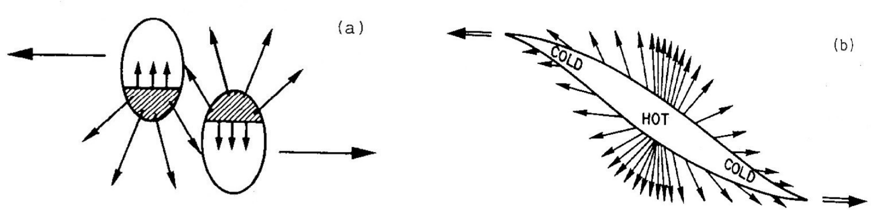


Fig. 26.4 Two extreme possibilities for the situation after a collision. (**a**) Two 'hot spots' leading to an asymmetry in the lab distribution of produced particles. (**b**) A fast stretching continuum will not lead to an asymmetry.

26.3 Statistical Bootstrap Thermodynamics

The whole philosophy and all technical details of SBM are described in the literature [1, 12, 30]. Here I shall be very qualitative.

The partition function

Consider a microscopic system confined to a volume V and embedded in a heat bath of temperature T. It will have an energy level spectrum $S = \{E_0, E_1, E_2, \ldots, E_n, \ldots\}$. As an example, think of an ideal gas of one sort of particles of mass m. Then the probability of finding the system in the energy level E_n is proportional to $\exp(-E_n/T)$. Normalised to one, we have

$$W_m = \frac{\exp(-E_n/T)}{\sum_{i=1}^{\infty} \exp(-E_i/T)} \ . \tag{26.17}$$

The expectation value of its energy is

$$\begin{aligned}\langle E \rangle &= \sum_{n=0}^{\infty} E_n W_n = \frac{\sum E_n \exp(-E_n/T)}{\sum \exp(-E_n/T)} \\ &= -\frac{\mathrm{d}}{\mathrm{d}(1/T)} \ln \left[\sum_{n=1}^{\infty} \exp(-E_n/T) \right] .\end{aligned} \tag{26.18}$$

The expression in square brackets is the partition function

$$Z(T,V) := \sum_{n=1}^{\infty} \exp(-E_n/T) =: \int_0^{\infty} \sigma(E,V) \mathrm{e}^{-E/T} \mathrm{d}E \ . \tag{26.19}$$

The density of states $\sigma(E,V)$ is defined by this identity; $\sigma(E,V)\mathrm{d}E$ is the number of energy levels in the interval $\{E, \mathrm{d}E\}$. Equation (26.19) states that the partition function is the Laplace transform of the density of states.

From $Z(T,V,\ldots)$ all interesting thermodynamic quantities can be derived by logarithmic differentiation, as in Eq. (26.18). Apart from T and V, the partition function may depend on further variables like chemical potentials (one for each conservation law), external fields, etc.

Interaction

We learn from chemists that, if there are atoms of sorts A and B which can undergo exothermic chemical reactions liberating the heat Q, viz.,

$$A + B \rightleftarrows (AB) + Q \ , \tag{26.20}$$

where (AB) is a molecule consisting of atoms A and B, then one introduces just *three* different particles A, B, and (AB) with masses m_A, m_B, and m_{AB}, with

$$m_{AB} := m_A + m_B - E_{\text{bind}} \ , \qquad Q \equiv E_{\text{bind}} \ . \tag{26.21}$$

If then no other sorts of particles and no other reactions occur, this is sufficient to calculate the chemical equilibrium rates $A : B : (AB)$ and the equations of state. One simply considers a *three component ideal gas*:

$$Z(T,V,\dots) = Z_A(T,V,\dots)Z_B(T,V,\dots)Z_{(AB)}(T,V,\dots) \tag{26.22}$$

and calculates everything from

$$\ln Z = \ln Z_A + \ln Z_B + \ln Z_{(AB)} \,. \tag{26.23}$$

We need not know any details except E_{bind} about the interaction between A and B, nor of the internal structures of A, B, and (AB). The values of m_A, m_B, and m_{AB} (which contains E_{bind}) are sufficient to represent the interaction for all questions about the equilibrium state (not, for example, for the question of how fast equilibrium is reached). This method can be pushed further. We could also include molecules $(A_l B_k)$ with $l,k = 1,2,\dots$ or add further elementary objects (atoms) $C,D,\dots$ and consider molecules $(A_i B_j C_k D_l \dots)$ as well as their excited states.

Back to particle physics. Here we include *all* possible reactions and *all* bound states, excited states, and resonances of the elementary input particles. The latter are chosen by convenience. One could start with quarks and gluons, but equally well with pions and nucleons (or with all these at the same time). Let us consider pions and nucleons (one could add strange, charmed, and other particles).

Figure 26.5 shows a mass distribution of π and its resonances, A and its resonances, and bound states (nuclei) of (πN) and its resonances. We know from the chemists that we need all these masses and that we have only to consider a mixture of ideal gases, one for each particle mass (labelled $i = 1,\dots,\infty$):

$$\ln Z(T,V,\dots) = \sum_i \ln Z_{m_i}(T,V,\dots) =: \int \mathrm{d}m\, \rho(m) \ln Z_m(T,V,\dots) \,, \tag{26.24}$$

where $\rho(m)\mathrm{d}m$ is the number of different sorts of particles in $\{m, \mathrm{d}m\}$, while $Z_m(T,V,\dots)$ is the *ideal gas* partition function for an unrestricted number of par-

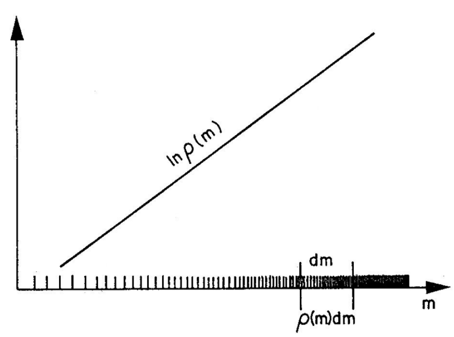


Fig. 26.5 The mass spectrum $\rho(m)$ of hadrons (schematically). Each line represents one particle and their density grows exponentially with the mass.

ticles of mass m, and dots indicate further variables (chemical potentials). The number of particles has to be unrestricted because:

- their total number is unrestricted due to particle creation and annihilation,
- the number of each sort changes via 'chemical' reactions, e.g.,

$$\mathrm{N} + \mathrm{N} \longrightarrow \mathrm{N} + \mathrm{N}^* + 2\pi\,.$$

For the partition function of an ideal gas of an arbitrary number of particles, we find in any textbook

$$\ln Z_m(T,V) = V f(m,T) \mathrm{e}^{-m/T}\,, \qquad f(m,T) \underset{m \gg T}{\longrightarrow} \left(\frac{mT}{2\pi}\right)^{3/2}\,. \tag{26.25}$$

Note that the factor $\mathrm{e}^{-m/T}$ is missing in most textbooks, since in non-relativistic statistical mechanics it is an irrelevant normalizing factor. In the relativistic situation it is *the* important part as it governs the equilibrium between particle creation and annihilation. Hence,

$$\ln Z(T,V) = V \int_0^\infty f(m,T) \rho(m)\, \mathrm{e}^{-m/T}\, \mathrm{d}m \tag{26.26}$$

is the partition function for the strongly interacting π–N gas, i.e., for the simplest strongly interacting hadron gas. What now is $\rho(m)$? We have the π, N, all nuclei with their excited states, π resonances, A resonances, πN states and their resonances, etc. Only a finite number of them is known, but there are many more still unknown. The finite number of known states is, in general, sufficient to calculate some interesting quantities. This has been done for a long time – recently and in the context of nucleosynthesis in the early universe as well as for the relativistic heavy ion collision in some pioneering papers [32, 33]. In particular, the two papers by A.Z. Mekjan [33] are an excellent introduction to many fundamental concepts and open questions – most recommended reading!

What a finite number of states, included in the integral of Eq. (26.26) for $\ln Z$, cannot do, is to generate a singularity of the partition function, in other words, generate a phase transition. As one sees from Eq. (26.26), $\ln Z(T,V)$ is analytic in the entire right half of the complex T plane if $O[\rho(m)] = m^\alpha$, $\alpha < \infty$. If $\rho(m)$ grows exponentially, $\rho(m) \sim C m^\alpha \exp(m/T_0)$, then the integral of Eq. (26.26) does not exist for $\mathrm{Re}(T) > T_0$ and $\ln Z(T,V)$ has a singularity at T_0, as first observed by Yu.B. Rumer [34], years before the SBM was proposed.

The bootstrap hypothesis

An incomplete $\rho(m)$, however, useful for computing low temperature properties of the system, will fail to exhibit critical phenomena. We therefore need the complete

$\rho(m)$, $0 \leq m < \infty$. Indeed, only the complete spectrum can represent the full interaction; it is equivalent to the eigenphase representation of the S matrix [30].

We obtain the full mass spectrum from the 'bootstrap' hypothesis. The idea goes like this. From Eqs. (26.19) and (26.26), we have

$$\begin{aligned} Z(T,V) &= \int_0^\infty \sigma(E,V)\mathrm{e}^{-E/T}\mathrm{d}E \,, \\ Z(T,V) &= \exp\left[V\int_0^\infty f(m,T)\rho(m)\mathrm{e}^{-m/T}\mathrm{d}m\right] \,. \end{aligned} \tag{26.27}$$

The *same* function $Z(T,V)$ is expressed in different ways, once by the density of states of the *whole* system and once by the mass spectrum of its *constituents*.

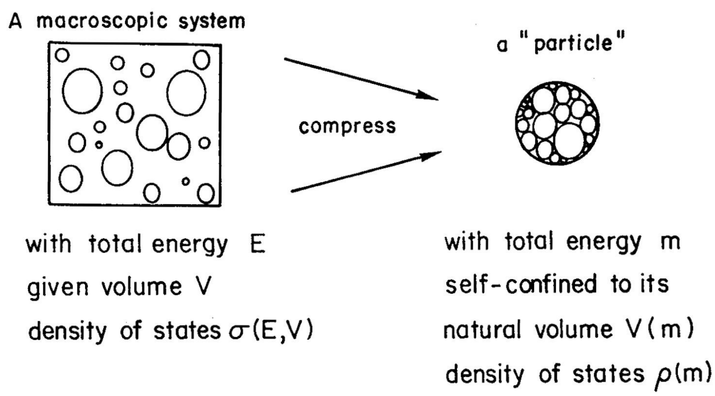


Fig. 26.6 One step in the argument leading to the statistical bootstrap.

We must clearly understand the physical meaning of $\sigma(E,V)$ and of $\rho(m)$ (see Fig. 26.6):

- $\sigma(E,V)\mathrm{d}E$ is the number of states between E and $E+\mathrm{d}E$ of an interacting system enclosed in any externally given volume V.
- $\rho(m)\mathrm{d}m$ is the number of states (i.e., different particles) between m and $m+\mathrm{d}m$ of an interacting system confined to its 'natural volume', i.e., to the volume resulting from the forces keeping these masses together as bound states or resonances.

Thus $\sigma(E,V)$ refers, in general, to some macroscopic system, while $\rho(m)$ refers to particles. Here the reader should hold on for a moment and imagine that we could compress the macroscopic system to that small volume which would be the natural volume $V(E)$ belonging to the energy E. What would happen? It would itself become a 'particle' – just one among the infinite number counted by the mass spectrum. Thus $\sigma\big(E,V(E)\big)$ would have to be equal to the mass spectrum at $m=E$, namely, $\sigma\big(m, V(m)\big) = \rho(m)$. This argument is so important that I will repeat it in another formulation:

> The interaction reigning in the macroscopic system enclosed in V is identical to the one creating the various bound states and resonances, keeping them together awhile and squeezing them into their natural volumes. On the other hand, we have claimed that the existence of all these – just exactly these! – bound states and resonances with all possible 'chemical' reactions between them, does represent – even generate – this interaction. Hence,
>
> **the interaction** is generated by reactions between bound states and resonances, which themselves are generated by the interaction, which is generated by reactions between bound states and resonances, and so on ad infinitum.

This circular reasoning is a special example of the more general 'bootstrap philosophy' proposed by G.F. Chew [35].

Coming back to the above gedankenexperiment, if we could compress the system with energy E and volume V to its natural volume $V(E)$, it would not be distinguishable from a resonance or a bound state with mass $m = E$ and volume $V(m)$. However, it then also follows that such a particle is also 'composed of' other particles just as it was before compressing and because it is subject to the same interaction (Fig. 26.6). Then of course $\sigma\big(E, V(E)\big)$ is the number of states between E and $E + \mathrm{d}E$ of a system confined to its natural volume; this is just the definition of $\rho(m)$ at $E = m$. Therefore, the function $\rho(m)$ is at the same time:

- the density of states of a composite system confined to its natural volume,
- and the mass spectrum of the constituents of such a system,

and ρ generates the interaction which generates ρ. This double role of ρ can be illustrated by a highly simplified 'bootstrap equation' in which everything except the double role has been omitted:

$$\rho(m) \sim \rho(m_1)\rho(m_2)\ldots\rho(m_N)\,, \quad \text{with} \sum_{i=1}^{N} m_i = m \text{ for any } N\,. \tag{26.28}$$

Such a type of equation has only exponential solutions. Actually, the arguments are much more subtle and the equation for $\rho(m)$ is not as simple as Eq. (26.28), but the conclusion remains the same: ρ is of the exponential type

$$\rho(m) = g(m)\mathrm{e}^{m/T_0}\,, \qquad O[g(m)] = m^{\alpha}\,, \quad \alpha < \infty\,, \tag{26.29}$$

where $g(m)$ is not exponential. It is not easy to determine $g(m)$, but its asymptotic behaviour for $m \to \infty$ is well known. In fact, $g(m) \to C/m^3$. The reader will find more information in [1, 30, 31]. Here I mention only two things. The constants C and T_0 can be guessed from a simplified model involving only pions [36]: $T_0 \approx m_\pi$. Such a spectrum fits well the lower part of the known spectrum of hadrons where we are sure to have found all resonances [1] and T_0 gives about the right slope.

Furthermore, the same T_0 accounts quantitatively for the well-known limited mean transverse momenta of particles produced in high energy collisions because the partition function will become singular at T_0 (as explained above) and $T_0 \approx m_\pi$ should be in some sense a limiting temperature or the critical temperature of a phase transition (boiling point of hadron matter). We shall come back to this.

Thus the bootstrap hypothesis allows one to predict the (averaged) hadronic mass spectrum and relates it to one of the most prominent features of high energy particle production – limited mean transverse momenta. It might be expected that it can also be applied to the relativistic heavy ion collision.

The singularity of the partition function: baryon conservation

We have to conserve the baryon number in the relativistic heavy ion collision. So far we have ignored this, but now it will be built in. In order to do so, we must study the singularity of the partition function. We insert the exponential mass spectrum of Eq. (26.29) into Eq. (26.26) for $\ln Z$, combine the two non-exponential functions $f(m,T)$ of Eq. (26.26) and $g(m)$ of Eq. (26.29) into a new non-exponential function $h(m,T)$, and obtain

$$\ln Z(T,V) = V \int_0^\infty h(m,T)\, \mathrm{e}^{m/T_0} \mathrm{e}^{-m/T}\, \mathrm{d}m \;, \tag{26.30}$$

where it is obvious that this integral exists for $T < T_0$ and has a singularity at $T = T_0$, the nature of which depends on $h(m,T)$ and does not interest us at the moment.

Now we split $\ln Z$ into two parts, $\ln Z_\pi$ and $\ln Z_\mathrm{N}$, where the first one only contains pions and pionic resonances, while the second contains all baryonic states:

$$\ln Z = \ln Z_\pi + \ln Z_\mathrm{N} \;, \tag{26.31}$$

where

$$\ln Z_{\pi,\mathrm{N}} = V \int_0^\infty h_{\pi,\mathrm{N}}(m,T)\, \mathrm{e}^{m/T_0} \mathrm{e}^{-m/T}\, \mathrm{d}m \;. \tag{26.32}$$

Here we claim that the asymptotic part e^{m/T_0} of the mass spectrum is the same for pions and baryons. Qualitatively, this can be understood by considering all hadrons with a given baryon number b and a very large mass such that $m \gg b m_\mathrm{N}$. For such large masses, the presence of a few baryons is irrelevant as most of the mass is due to excitation of non-baryonic degrees of freedom. Hence, for any fixed baryon number b, the asymptotic part of the mass spectrum must be the same and equal to the pionic one. This conclusion can be proved rigorously [37].

Consider now the baryonic partition function

$$\ln Z_\mathrm{N} = V \int_0^\infty h_\mathrm{N}(m,T)\, \mathrm{e}^{m/T_0} \mathrm{e}^{-m/T}\, \mathrm{d}m \;. \tag{26.33}$$

The factor $e^{-m/T}$ is proportional to the probability of creating a mass m at temperature T. This factor is extremely small for small T and so would be the number of baryons. If we wish $\ln Z_N$ to exhibit a given number of brought-in baryons, we must counteract the small factor $e^{-m/T}$. This can be done by artificially lowering m by subtracting some suitable Δm from it. This Δm should account for the actual average baryon number $\langle b \rangle$ we wish to impose and it should lie between 0 and m ($\Delta m = 0$ is no correction, $\Delta m = m$ is just excluded as too much). Thus we put

$$\Delta m = m \frac{\mu}{\mu_N} , \tag{26.34}$$

where μ is some parameter to be adjusted to yield the wanted $\langle b \rangle$. Then replacing $e^{-m/T}$ by $e^{-(m-\Delta m)/T}$, we obtain a new baryonic partition function

$$\ln Z_N(T, V, \mu) = V \int_0^\infty h_N(m, T) \exp \frac{m}{T_0} \exp \left[-\frac{m}{T} \left(1 - \frac{\mu}{m_N} \right) \right] dm . \tag{26.35}$$

This partition function, in which μ can be chosen to give any desired expectation value $\langle b \rangle$ of the baryon number, has no longer an isolated singularity at T_0, but a singular curve in the (T, μ) plane. Indeed, it will be singular where the total exponent vanishes:

$$T_{\text{crit}} = T_0(\mu) := T_0(1 - \mu/m_N) . \tag{26.36}$$

This is the broken straight line in Fig. 26.7.

The above arguments are oversimplified in order to make the idea clear. In reality, one proceeds differently and $\ln Z_N$ has a slightly different form, the critical curve in the (T, μ) plane is not trivial to calculate [31] and it looks like the curve in Fig. 26.7 instead of being a straight line. The partition function exists below that curve.

The new parameter μ introduced here (in a sloppy way) is called the chemical potential related to baryon number conservation. There is an extra chemical potential for every conserved quantity. Ours is the relativistic chemical potential. In non-relativistic statistical mechanics, it is defined as $\mu_{NR} = \mu - m_N$. This is consis-

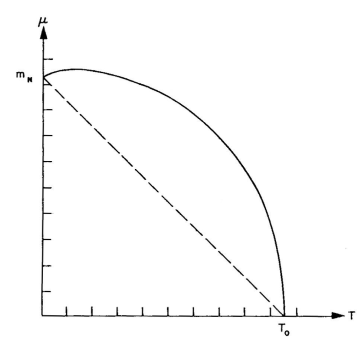


Fig. 26.7 The critical curve in the (T, μ) plane. The *broken line* is obtained from the handwaving 'derivation' in the text, while the *full curve* results from the model of Ref. [31].

tent with omitting the factor $e^{-m/T}$ in non-relativistic situations. Suppose we deal with a nucleon gas non-relativistically, then for the one-particle case

$$E - \mu = \frac{p^2}{2m_\mathrm{N}} + m_\mathrm{N} - (\mu_\mathrm{NR} + m_\mathrm{N}) = \frac{p^2}{2m_\mathrm{N}} - \mu_\mathrm{NR}$$

and

$$e^{-(E-\mu)/T} = \exp\left(-\frac{p^2}{2m_\mathrm{N}T} + \frac{\mu_\mathrm{NR}}{T}\right) .$$

Thus the factor $e^{-m_\mathrm{N}/T}$ has disappeared.

With the knowledge of $\rho(m)$ and the introduction of the chemical potential for baryon number conservation, $Z(T,V,\mu)$ has become calculable and is ready for application to the problem of highly excited hadronic (nuclear) matter. I stress that here I have only presented the general ideas. The complete analytical solution is technically more involved, but known in every detail [31].

The partition function for real (extended) particles

All we have done so far suffers from a most unrealistic tacit assumption, namely, that our particles are pointlike. For dilute gases, this is known to be a good approximation. We, however, consider dense matter. Indeed, when applying the bootstrap argument, the system considered has the density of a composite particle, i.e., roughly nuclear density. From nuclear physics, we know that describing a nucleus as a gas of pointlike nucleons is a bad approximation. We also know that the volume of a nucleus is proportional to the number of its nucleons, i.e., to its total mass. In a relativistic situation, it is not possible to distinguish between mass due to the rest masses of constituents and mass due to kinetic energy. Hence, relativistically, the natural volume $V(m)$ of a particle must be proportional to the mass m, viz.,

$$V(m) = \frac{m}{4\mathscr{B}} , \tag{26.37}$$

where $\mathscr{B}$ is a fundamental constant with the dimension of energy density. Relation Eq. (26.37) is borne out not only by low energy nuclear physics, but also by bag models [3][2] and by the statistical bootstrap model, as can be seen as follows:

- First write Eq. (26.37), which is valid in the particle's rest frame, in covariant form

$$V^\mu(m) = \frac{p^\mu}{4\mathscr{B}} , \tag{26.38}$$

  by which the four-volume V^μ is defined. In the rest frame, this reduces to (26.37) and therefore is the unique generalization of (26.37) and of the corresponding low energy nuclear property.

[2] We take $\mathscr{B}$ to be the 'bag constant'. Then $4\mathscr{B}$ is the energy density of a bag.

- Consider a particle as a densely packed assembly of any number of other particles with masses m_i:

$$V^{\mu}(m) = \sum_i V^{\mu}(m_i) \,, \quad \text{for any set } \{m_1, m_2, \ldots\} \,. \tag{26.39}$$

- Bootstrap tells us that the m_i have the same composite internal structure as the composite m. Hence, the $V^{\mu}(m_i)$ must obey Eq. (26.38) with the same $\mathscr{B}$.
- Therefore,

$$V^{\mu}(m) = \frac{p^{\mu}}{4\mathscr{B}} = \frac{1}{4\mathscr{B}} \sum_{i=1}^{N} p_i^{\mu} \,, \tag{26.40}$$

which is an identity, since $p^{\mu} = \sum_i p_i^{\mu}$ for any partition. This proves Eq. (26.37) to be true in statistical bootstrap.

Coming back to the partition function of a macroscopic system, we now introduce the notion of the available volume [31]

$$\Delta := V - \sum_{i=1}^{N} V(m_i) \,, \tag{26.41}$$

where V is the externally given volume enclosing the system in a heat bath [for such a system, Eq. (26.38) is of course not true; it only holds for each of its constituents] and Δ is what remains after taking the proper volumes of all constituents away, just as in the van der Waals gas:

Δ is the volume in which the particles move *as if* they were pointlike, while in reality they have finite proper volumes and move in V.

There are some differences with the van der Waals gas, however:

1. The proper volumes $V(m_i)$ are not equal.
2. The proper volumes will have to be written covariantly

$$V^{\mu}(m_i) = \frac{p_i^{\mu}}{4\mathscr{B}} \,, \qquad p_i^{\mu} p_{i\mu} = m_i^2 \,.$$

3. The usual factor four multiplying $\sum V_i$ is missing. It arises only for particles that are rigid spheres of equal radius. Our particles are deformable and of different sizes, in which case the factor is one.
4. The second van der Waals correction, which simulates attractive forces by subtracting a density dependent term from the pressure, is not necessary here since bootstrap takes care of attractive forces (and to all orders in the virial expansion).

The above statement (vdW) implies that one obtains the partition function of real extended particles enclosed in V by calculating the partition function of pointlike particles enclosed in the volume Δ:

$$Z_{\text{real}}(T,V,\mu) \equiv Z_{\text{pt}}(T,\Delta,\mu) \ . \tag{26.42}$$

Consider a particular microstate of the system where the particles have momenta p_i^μ ($i = 1,\ldots,N$). In that case, Eq. (26.41) reads

$$\Delta^\mu = V^\mu - \sum_{i=1}^{N} \frac{p_i^\mu}{4\mathscr{B}} = V^\mu - \frac{p^\mu}{4\mathscr{B}} \ , \tag{26.43}$$

with p^μ being the momentary total four-momentum of the system (p^μ fluctuates due to the heat bath). How then can we insert Δ in the partition function? The difficulties seem great since Z_{pt} is equal to e^{Z_1}, the one-particle function [see Eq. (26.27)], and yet we shall introduce a quantity which depends on all momenta and even fluctuates. We solve the problem by a tour de force. We choose Δ to be our independent volume-like parameter. Then

$$V^\mu = \Delta^\mu + \frac{p^\mu}{4\mathscr{B}} \tag{26.44}$$

for any state contributing to $\ln Z$. Thus now V is no longer fixed. It has, however, an expectation value. In the rest frame of the heat bath

$$\langle V(E,\Delta)\rangle = \Delta + \frac{\langle E\rangle}{4\mathscr{B}} \ . \tag{26.45}$$

Properties of the real hadron gas

From Eqs. (26.42) and (26.45), we can calculate all the usual thermodynamic variables. As an example, we calculate $\langle E\rangle/\langle V\rangle$, the energy density. Equation (26.18) says that

$$\langle E\rangle = T^2 \frac{\partial}{\partial T} \ln Z_{\text{real}} = T^2 \frac{\partial}{\partial T} \ln Z_{\text{pt}}(T,\Delta,\mu) \ , \tag{26.46}$$

and since $\ln Z_{\text{pt}}(T,\Delta,\mu)$ is proportional to Δ [see, e.g., Eq. (26.35)],

$$\langle E\rangle = \Delta \left[\frac{1}{\Delta} T^2 \frac{\partial}{\partial T} \ln Z_{\text{pt}}(T,\Delta,\mu) \right] \ . \tag{26.47}$$

The expression in square brackets is the (Δ independent) energy density of a gas of pointlike particles, viz., $\varepsilon_{\text{pt}}(T,\mu)$. Hence,

$$\langle E\rangle = \Delta \times \varepsilon_{\text{pt}} = \left[\langle V(E,\Delta)\rangle - \frac{\langle E\rangle}{4\mathscr{B}} \right] \varepsilon_{\text{pt}}(T,\mu) \ . \tag{26.48}$$

Here ε_{pt} does not depend on Δ. Furthermore, Δ can be chosen so that any given value $\langle V(E,\Delta)\rangle$ is assumed ($\geq \langle E\rangle/4\mathscr{B}$). Hence we can now consider $\langle V\rangle$ as a variable which can be prescribed and we can thus solve Eq. (26.38) without regard for the implicit dependence of $\langle V\rangle$ on $\langle E\rangle$:

$$\langle E\rangle = \langle V\rangle \frac{\varepsilon_{\rm pt}(T,\mu)}{1+\varepsilon_{\rm pt}(T,\mu)/4\mathscr{B}} \tag{26.49}$$

and

$$\frac{\langle E\rangle}{\langle V\rangle} = \varepsilon_{\rm real}(T,\mu) = \frac{\varepsilon_{\rm pt}(T,\mu)}{1+\varepsilon_{\rm pt}(T,\mu)/4\mathscr{B}}\ . \tag{26.50}$$

Furthermore, from Eq. (26.48), $\langle E\rangle = \Delta\times\varepsilon_{\rm pt}$, we find with Eq. (26.50),

$$\langle V(T,\Delta,\mu)\rangle = \Delta\left[1+\frac{\varepsilon_{\rm pt}(T,\mu)}{4\mathscr{B}}\right]\ , \qquad \Delta = \langle V\rangle\left[1-\frac{\varepsilon_{\rm real}(T,\mu)}{4\mathscr{B}}\right]\ . \tag{26.51}$$

It turns out that all 'real' intensive quantities like pressure, baryon number density, etc., are related to the 'point' intensive quantities as $\varepsilon_{\rm real}$ is related to $\varepsilon_{\rm pt}$:

$$\begin{aligned}
\varepsilon_{\rm real}(T,\mu) &= \frac{\Delta}{\langle V\rangle}\varepsilon_{\rm pt}(T,\mu)\ ,\\
P_{\rm real}(T,\mu) &= \frac{\Delta}{\langle V\rangle}P_{\rm pt}(T,\mu) \qquad \text{(pressure)}\ ,\\
\nu_{\rm real}(T,\mu) &:= \frac{\langle b\rangle}{\langle V\rangle} = \frac{\Delta}{\langle V\rangle}\nu_{\rm pt}(T,\mu)\ .
\end{aligned} \tag{26.52}$$

Behaviour near the critical curve

Inspection of the exact partition function [31] reveals that $\varepsilon_{\rm pt}(T,\mu)\to\infty$ when the system approaches the critical curve. While, for given T,μ, one can choose Δ to yield any given $\langle V\rangle \geq \langle E\rangle/4\mathscr{B}$ [see Eq. (26.45), Eq. (26.51) says how], the ratio $\Delta/\langle V\rangle$ is a definite function of T and μ, tending to zero when the system approaches the critical curve:

$$\lim_{\rm (crit)} \varepsilon_{\rm real}(T,\mu) = 4\mathscr{B}\ , \tag{26.53}$$

that is, on the critical curve, the whole system assumes the density $4\mathscr{B}$ of its constituents [remember Eq. (26.37)] and therefore has become just one giant 'particle'. Closer inspection [31] yields for the pressure and the baryon number density

$$\lim_{\rm (crit)} P_{\rm real}(T,\mu) = 0\ , \qquad \lim_{\rm (crit)} \nu_{\rm real}(T,\mu) = \nu_{\rm crit}(T,\mu) \neq 0,\infty\ . \tag{26.54}$$

As the critical curve is reached at finite energy density, nothing prevents it being reached in actual particle collisions and nothing prevents it even being passed over, provided the collision was energetic enough. Considering hadrons as quark-gluon bags, the hadronic gas becomes then on the critical curve a giant quark-gluon bag, and it should be described as an interacting quark-gluon gas on the other side. This, at least, is our (J. Rafelski and R. Hagedorn) present interpretation [31] (Fig. 26.8).

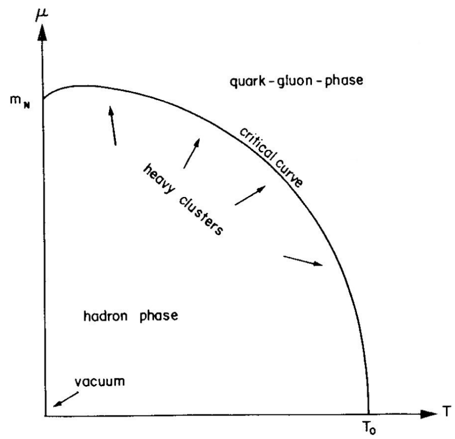


Fig. 26.8 Physical interpretation of different regions of the (T, μ) plane as proposed in Ref. [31].

This thermodynamics, combined with a good description of the collective motions, should then give a model for the relativistic heavy ion collision:

$$\text{Complete description} = \int \{\text{collective motion}\} \otimes \left\{ \begin{array}{c} \text{local bootstrap} \\ \text{thermodynamics} \end{array} \right\}. \quad (26.55)$$

While the local bootstrap thermodynamics is known, we still know little about the collective motions, which thermselves depend again on the local thermodynamics. There remains a great deal of work to be done!

26.4 Is there Equilibrium in the Relativistic Heavy Ion Collision?

Nobody expects global equilibirum (except perhaps in selected ‘central’ collisions), but there are good reasons to doubt even that there is local equilibrium, because the duration of a collision, the lifetime of resonances, and the time needed to create a particle are all of the same order of magnitude. In particle collisions one can agree that thermodynamic equilibrium does not require a number of collisions of existing particles, but that the quantum mechanical probability distributions governing the creation of particles are such that the new-born particles seem to come from an equilibrium state [12]. This might be different in the relativistic heavy ion collision where many particles are already present before the collision and have to undergo collisions individually and/or coherently.

The argument for equilibrium seems to be valid in particle physics because the thermodynamical model which rests heavily on it was on the whole successful. With very few free parameters chosen once and for all, it covered collisions of different sorts of particles with lab energies between 10 and 1000 GeV, describing rather well the features of particle production for all sorts of particles and with production rates ranging over 12 orders of magnitude. While many models are quantitatively superior in restricted areas, the thermodynamical model was (and still is) the only one covering the whole reasonably well. There are also some failures:

1. Two-particle correlations are not well described by the simple thermodynamical model [38].
2. The large $p_\perp$ processes observed at ISR energies [39] are not predicted by the model.

The second failure is particularly interesting because it concerns only about one per thousand of all produced particles – the rest behave as the model predicts. Figure 26.9 shows what happens qualitatively. The $p_\perp$ distribution takes off the straight line predicted by the model at $p_\perp \gtrsim 1.5$ GeV/c and stays higher up depending on the height of the collision energy. Why does this not kill the model? Because the straight parts of all curves, coinciding below ~ 1.5 GeV/c have the same energy-independent shape corresponding to a temperature ~ 165 MeV, just as the model says. If the shape of this part had decreased steadily with rising energy, the model would have been in serious trouble. The large transverse momenta can be understood as being due to pre-hadronization processes taking place in the quark-gluon phase and emitting some energetic quark or gluon *before* crossing the critical curve. All the particles belonging to the straight line below 1.5 GeV/c would then be emitted from the hadronic phase *after* crossing the critical curve.

While in *pp* collisions at ISR energies there are thus definite traces of pre-equilibrium processes happening in the quark-gluon phase, such indications are as yet missing in proton–nucleus and nucleus–nucleus collisions, presumably because the collision energies are not yet high enough. In any case, models along the lines described in the previous sections have been applied to *pA* and *AA* collisions. I will mention just a few (apologizing to any authors not mentioned because of my ignorance):

- The ‘Black Book’ on particle spectra [11] calculates the *pA* collisions between 20 and 70 GeV primary lab energy with *A* being Be, Al, Cu, and Pb. A single collective velocity function $F(\lambda)$ covers all of these energies and targets in satisfactory agreement with data (where available). Production of heavy fragments is not calculated.
- J.P. Alard [40] pioneered the calculation of nucleon and heavy fragment emission (He^4, Be, Li, ...) using a special $F(\lambda)$ and introducing chemical potentials.
- A. Mekjian [41] calculated the relativistic heavy ion collision with $F(\lambda) \sim \delta(\lambda - \lambda_0)$, where λ_0 was adjusted to represent a single moving fireball with thermodynamics restricted to the lower, explicitly known part of the spectrum.
- J. Gosset, J.I. Kapusta, and G.D. Westfall [42] obtained good results using a velocity distribution (‘fire streak’) derived from kinematical considerations by W.D. Myers [13] and with thermodynamics using the lower part of the spectrum [32, 41].

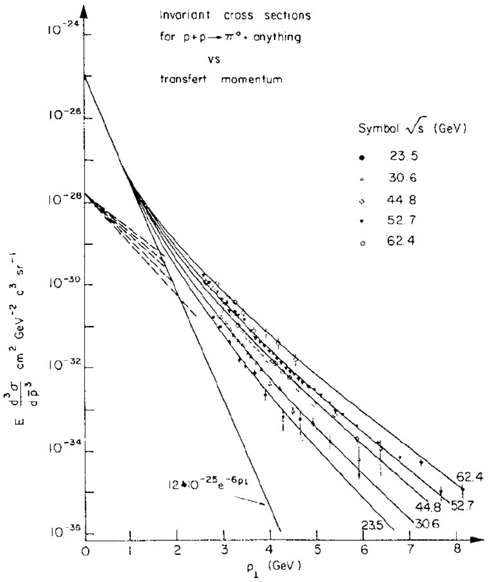


Fig. 26.9 Transverse momentum distribution of π^0 at ISR energies [39]. The curves (hand drawn by the present author) can be well approximated by a superposition of three exponentials. At $p_\perp \lesssim 1.5$ GeVc, the SBM prediction holds with $T \approx 165$ MeV, while at $p_\perp \gtrsim 5$ GeV/c, a temperature growing $\approx E_{\mathrm{c.m.}}^{2.7}$ (*broken straight lines*) suggests a plasma of gluons and not quite massless quarks.

- R. Malfliet [25] derived a collective velocity distribution [in the sense of $F(\lambda)$, but taking into account its temporal evolution] from the relativistic Boltzmann equation. Supplied with low-spectrum thermodynamics and nucleon–nucleon cross-sections, the model yielded particle spectra in good agreement with the relativistic heavy ion collision data.

All these attempts are based on the assumption of

$$\{\text{local equilibrium}\} \otimes \{\text{collective motion}\}$$

and they are all more or less successful. This is surprising.

The way to equilibrium

It is necessary to understand why there can be local equilibrium at least approximately. The problem has been considered in the literature [43–45]. Clearly, the approach to equilibrium takes time. After a sufficient time has elapsed, any system will come to equilibrium. What is less obvious is that the equilibrium state reached will, everything else being equal, depend on the volume available [12, 45]. Here I shall outline the ideas without going into detail. There are two kinds of equilibrium: kinetic and chemical:

- *Kinetic equilibrium* means equipartition of the total kinetic energy among the particles then present. This is a fast process which needs only very few collisions per particle. We assume this kind of equilibrium to be established instantaneously and locally, which means that a local temperature can be defined meaningfully. This temperature can still vary in space and time.
- *Chemical equilibrium* is equilibrium between the numbers of different species of particles. Being in equilibrium means for a given species that its rate of creation balances its death rate. To arrive at that state may take a short or a long time depending on cross-sections, lifetimes, densities, and so on.

Consider a simplified example which exhibits the main point: a quasi-ideal pion–nucleon gas in which a third kind of particle with a conserved charge Q can be created. Let A and $\overline{\mathrm{A}}$ denote this particle and its antiparticle and a the density of either A or $\overline{\mathrm{A}}$, whence $a = N_{\mathrm{A}}/V$. Further, let n be the number density of pions plus nucleons, viz., $n = (N_{\pi} + N_{\mathrm{N}})/V$. Then

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \left.\frac{\mathrm{d}a}{\mathrm{d}t}\right|_{\text{creation}} - \left.\frac{\mathrm{d}a}{\mathrm{d}t}\right|_{\text{annihilation}} , \tag{26.56}$$

$$\left.\frac{\mathrm{d}a}{\mathrm{d}t}\right|_{\text{creation}} = \sigma_{\mathrm{c}} v_{\mathrm{T}} n^2 W_{\text{pair}} . \tag{26.57}$$

Here σ_{c} is the inelastic cross-section (assumed energy independent and equal for all collisions πN, $\pi\pi$, AA), v_{T} the mean thermal velocity, and n the density of pions plus nucleons. We have assumed that A has a sufficiently large mass so that its contribution to n is negligible: $n \gg a$ at all times. W_{pair} is the pair creation probability per collision. Also,

$$\left.\frac{\mathrm{d}a}{\mathrm{d}t}\right|_{\text{annihilation}} = \sigma_{\mathrm{A}} v_{\mathrm{TA}} a^2 , \tag{26.58}$$

where σ_{A} is the annihilation cross-section of A and v_{TA} is the mean thermal velocity of the A particles. Hence,

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \sigma_{\mathrm{c}} v_{\mathrm{T}} n^2 W_{\text{pair}} - \sigma_{\mathrm{A}} v_{\mathrm{TA}} a^2 =: \alpha - \beta a^2 , \quad \alpha := \sigma_{\mathrm{c}} v_{\mathrm{T}} n^2 W_{\text{pair}} , \quad \beta := \sigma_{\mathrm{A}} v_{\mathrm{TA}} , \tag{26.59}$$

where α, β are constants fixed by particle properties, densities, and temperature. Obviously equilibrium is reached when

$$\frac{\mathrm{d}a}{\mathrm{d}t} = 0 , \qquad a(\infty) = \sqrt{\frac{\alpha}{\beta}} . \tag{26.60}$$

The system approaches this value from below or from above depending on the initial value $a(0)$, as shown in Fig. 26.10. From the thermodynamical model, we know (first paper of Ref. [2] and CERN lectures Ref. [12]) that

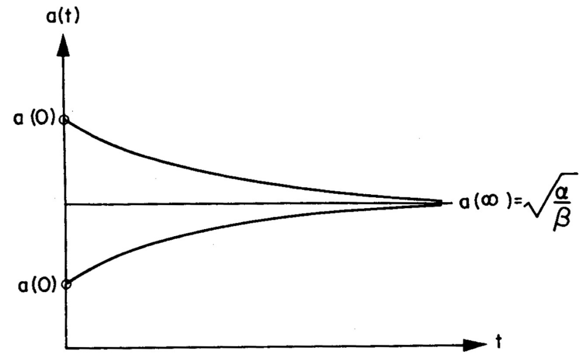


Fig. 26.10 The approach to equilibrium (qualitative) predicted by our simple model for two different initial values $a(0)$.

$$W_{\text{pair}} \sim e^{-2m_\text{A}/T} \;, \tag{26.61}$$

so that

$$a(\infty) = \text{Const.} \times \mathrm{e}^{-m_\text{A}/T} \;. \tag{26.62}$$

Consider now $a(0)$. This initial value is determined by the process which creates the πN gas in which the creation of A$\overline{\text{A}}$ pairs takes place. Let this process be an the relativistic heavy ion collision. Then in the first instant, *AA* collisions at high (not thermal) velocities take place and, in these collisions, pions as well as A$\overline{\text{A}}$ pairs are created. Then kinetic equilibrium between all these is rapidly reached and Eqs. (26.56)–(26.58) can be applied, with $a(0)$ being the density of A resulting from the pairs created in the first impact, viz.,

$$a(0) \sim \mathrm{e}^{-2m_\text{A}/T} \;, \tag{26.63}$$

the proportionality constant depending on the details of the collision (energy, number of nucleons, etc.).

Above, we have made the assumption that the density a is so small that it can be neglected against n. This implies that after creation the particles A and $\overline{\text{A}}$ part from each to large distances and that A($\overline{\text{A}}$) has to wait for annihilation until colliding with some other particle $\overline{\text{A}}$(A). This, however, is only possible if the total volume available to the whole system is so large that the pairs created in the first instance can escape to distances which are large compared to the range of the annihilation interaction. If the volume were so small as to keep them always within annihilation distance, the number of pairs would always remain proportional to $\exp(-2m_\text{A}/T)$ as in *pp* collisions [2, 12]. Between such a small and a very large volume, many intermediate situations may occur in the relativistic heavy ion collision. Therefore, it is important to know the volume dependence of the equilibrium distribution $a(\infty)$, which may vary between $\sim \exp(-2m_\text{A}/T)$ and $\sim \exp(-m_\text{A}/T)$ [45]. (Our above equation is valid only for large volumes, as $n \gg a$ was assumed.)

The time needed to reach equilibrium depends on the values of α and β, which determine the slopes of the curves in Fig. 26.10. Equilibrium is then reached when $|a(t) - a(\infty)|^2$ is of the order of the natural fluctuations in equilibrium, i.e.,

$$\left|a(t) - a(\infty)\right|^2 \underset{\text{equilbrium}}{\approx} \langle a^2 \rangle - \langle a \rangle^2 \, . \tag{26.64}$$

From this the equilibrium time can be determined [44].

Expansion and cooling

Expansion and cooling after onset of a relativistic heavy ion collision are not equilibrium processes, but the only way presently known of describing them theoretically seems to be via a sequence of quasi-equilibrium states. Two approaches have been discussed in the literature [41,43]: expansion with constant energy and with constant entropy. We believe [31] that constant entropy is wrong in early stage, while particle creation still takes place, no external work is done, and the process is irreversible. On the other hand, total energy is conserved, but it is insufficient to characterize the process, since we can calculate from thermodynamics only energy densities and do not know the dynamical expanding volume. On the other hand, baryon number is also conserved and must be taken into account. Again, we can only calculate the baryon number density and do not know the volume. The ratio of the two conserved quantities $\langle E \rangle$ and $\langle b \rangle$ is then also conserved and the unknown expanding volume drops out:

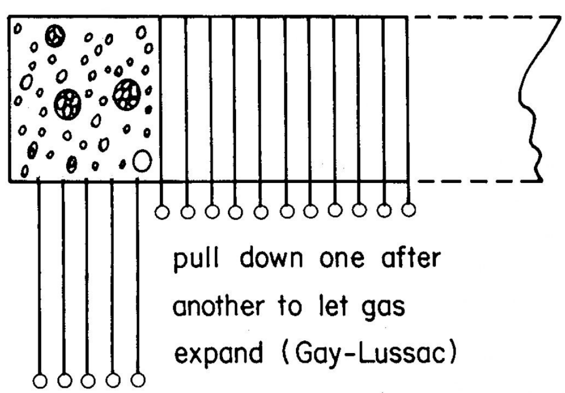


Fig. 26.11 Freely expanding hadron matter seen as a sequence of equilibrium states. Energy and baryon number are conserved, entropy increases.

$$\frac{\langle E \rangle}{\langle b \rangle} = \frac{\varepsilon}{v} = \frac{E}{b}\bigg|_{\text{initial}} = \text{Const.} \tag{26.65}$$

We therefore advocate the calculation of cooling curves given by (26.65) as a succession of quasi-equilibrium states [31]. We may visualize this as in Fig. 26.11 where we imagine the separating walls pulled down one after another, each time waiting until a new equilibrium is established. At some stage, pulling down more walls will not change any more the momentum distribution and particle ratios. We have then reached an ideal gas situation: the equilibrium has been frozen [41, 43]. This state will be the one recorded on our particle detectors, but particles being emitted during the whole process will superimpose on it [31]. Clearly, the process pictured in Fig. 26.11 is only an approximation of reality, since equilibrium is not reached after each decay step. But it will be at first a much better approximation than that obtained from the assumption of constant entropy.

26.5 Conclusions

Despite a great number of well worked out partial theoretical models, we do not yet know enough to build a theory which describes coherently the whole of the relativistic heavy ion collision. Under these circumstances, even the detailed models cannot be adequately tested since we have not yet learned to disentangle the dozen or so different mechanisms mixed up in an the relativistic heavy ion collision. There exists, however, a fully worked out analogue computer programme based on the one and only true, complete theory: the relativistic heavy ion collision experiment. As we do not know the programme, but only its output for a given input, learning the theory from it is far from trivial. We should try to force it to give answers to the following unsettled questions:

- Is there a unique $F(\lambda)$?
- How important are transverse collective motions and turbulence?
- Must hydrodynamics be used?
- Is bootstrap thermodynamics right?
- How fast is equilibrium reached locally?
- Which is the best thermometer?
- How do fireballs cool and expand?
- Is there a phase transition to a quark-gluon plasma?
- Do ISR jets (large $p_\perp$) indicate such a phase transition?

These questions result from theoretical prejudice. Given that these prejudices might be reasonable to start with, the following experiments will be interesting:

1. Measure total multiplicities or relative ratios of secondaries: π, N, K, Λ, Σ, d, t, He^3, He^4, Be, Li, ..., and of as many of their antiparticles as feasible.
2. Measure for all of these the mean transverse momentum $\langle p_\perp \rangle$ as a function of the primary energy per nucleon E/A and of the secondary's mass m.

For these first type types of measurement, try to make things as simple as possible: projectile $\equiv$ target and trigger for central collisions, in order to approach the theoretician's dream object – a single fireball.

3. Measure inclusive momentum distributions $W(\mathbf{p})\mathrm{d}^3p$ as a function of E/A and m. Try to fit with some $F(\lambda) \otimes \{$local bootstrap thermodynamics$\}$:
   - Can one find an energy independent $F(\lambda)$?
   - Is it the same for all targets and projectiles?
   - Is it independent of the emitted secondary?
   - Does one need transverse collective motion?

For these measurements *all* events must be taken; triggering for central collisions or selecting events according to any specific criteria would distort the picture. One might start, however, by colliding equal nuclei and later make projectile $\neq$ target. Pay special attention to transverse momenta of heavy fragments, the heavier the better [see Eq. (26.16)].

4. Look for asymmetries in individual events:
   - Azimuthal, i.e., non-isotropic in the angle about the collision axis. Such an asymmetry should arise from angular momentum conservation [46] and be large in peripheral, small in central collisions.
   - While the first type of azimuthal asymmetry would still maintain symmetry with respect to reflection on the collision axis, even that might be destroyed by the 'hot spot' mechanism [29] leading to a right–left asymmetry.
5. Look for 'abnormal' things such as:
   - sideward jets,
   - unusually large $p_\perp$,
   - unexpected (from equilibrium thermodynamics) particle ratios, in particular involving anti-(Λ,Σ).

This last group of observations would have to be interpreted as evidence supporting the theoretical picture of a phase transition to a quark-gluon plasma. Any process originating in that phase and surviving the return to the hadron phase would leave traces of the sort mentioned.

Acknowledgements A great deal of what has been said here is due to a long and fruitful collaboration with J. Rafelski, without whom I might never have become interested in the relativistic heavy ion collision.

References

1. (A) Some of the more basic papers about SBM, its main ideas and difficulties:
   R. Hagedorn: Suppl. Nuovo Cimento **3**, 147 (1964);

R. Hagedorn: Nuovo Cimento **6**, 311 (1968);
G. Matthiae: Nucl. Phys. B **7**, 142 (1968);
R. Hagedorn: Nucl. Phys. B **24**, 93 (1970);
K. Huang, S. Weinberg: Phys. Rev. Lett. **25**, 855 (1970);
S. Frautschi: Phys. Rev. D **3**, 2821 (1971);
C.J. Hamer, S. Frautschi: Phys. Rev. D **4**, 2126 (1971);
W. Nahm: Nucl. Phys. B **45**, 525 (1972);
R.D. Carlitz: Phys. Rev. D **5**, 3231 (1972);
I. Montvay: Phys. Lett. B **42**, 466 (1972);
J. Yellin: Nucl. Phys. B **52**, 583 (1973);
C. Hamer: Phys. Rev. D **8**, 3558 (1973);
M.I. Gorenstein, *et. al*: Phys. Lett. B **43**, 73 (1973);
H. Satz: Phys. Lett. B **44**, 373 (1973);
E.M. Ilgenfritz, J. Kripfganz: Nucl. Phys. B **62**, 141 (1973);
K. Fabricius, V. Wambach: Nucl. Phys. B **62**, 212 (1973);
J. Engels, K. Fabricius, K. Schilling: Nuovo Cimento A **23**, 581 (1974);
J. Kripfganz, E.M. Ilgenfritz: Phys. Lett. B **48**, 329 (1974);
J. Engels, K. Fabricius, K. Schilling: Phys. Lett. B **53**, 65 (1974);
F. Csikor, I. Farkas, Z. Katona, I. Montvay: Nucl. Phys. B **74**, 343 (1974);
M.I. Gorenstein, *et. al*: Nucl. Phys. B **76**, 453 (1974);
C.J. Hamer: Phys. Rev. D **10**, 1458 (1974);
C.J. Hamer: Nucl. Phys. B **95**, 173 (1975);
J. Engels, K. Fabricius, K. Schilling: Phys. Lett. B **59**, 477 (1975);
S. Frautschi: Nucl. Phys. B **91**, 125 (1975);
P. Fré, L. Sertorio: Nuovo Cimento A **28**, 538 (1975);
J. Kripfganz: Nucl. Phys. B **100**, 302 (1975);
N. Cabibbo, G. Parisi: Phys. Lett. B **59**, 67 (1975);
D.E. Miller: Phys. Lett. B **64**, 193 (1976);
E. Etim, R. Hagedorn: Nucl. Phys. B **131**, 429 (1977);
M. Feigenbaum, L. Sertorio: Nuovo Cimento A **43**, 31 (1978);
J. Letessier, A. Tounsi, H. Satz: Nuovo Cimento A **45**, 590 (1978);
H. Satz: Phys. Rev. D **20**, 582 (1979);
R. Fiore: Zeit. f. Physik C **1**, 305 (1979);
K. Redlich, L. Turko: Zeit. f. Physik C **5**, 201 (1980).

(B) For reviews and lectures see [12] and [30].

(C) See also [25, 31, 36, 45].

2. R. Hagedorn, J. Ranft: Suppl. Nuovo Cimento **6**, 169 (1968);
R. Hagedorn: Nuovo Cimento A **56**, 1027 (1968); see also [17].
3. A. Chodos, *et. al*: Phys. Rev. D **9**, 3471 (1974);
K. Johnson: Acta Phys. Pol. B **6**, 865 (1975);
P. Hasenfratz, J. Kuti: Phys. Rep. C **40**, 75 (1978)
4. V. Weisskopf: Phys. Rev. **52**, 295 (1937);
R. Hagedorn, W. Macke: in *Kosmische Strahlung*, ed. by W. Heisenberg, Springer, Berlin (1953)
5. H. Koppe: Zeit. f. Naturforschung A **3**, 251 (1948); Phys. Rev. **76**, 688 (1949)
6. E. Fermi: Prog. Theor. Phys. **5**, 570 (1950)
7. T. Ericson: Nuovo Cimento **21**, 605 (1961);
F. Cerulus: Nuovo Cimento **22**, 958 (1961);
H. Satz: Fortschr. d. Physik **11**, 445 (1963);
H. Joos, H. Satz: Nuovo Cimento **34**, 619 (1964);
K. Zalewski: Acta Phys. Pol. **28**, 933 (1965);
A. Kotanski, K. Zalewski: Nuovo Cimento **40**, 134 (1965)

8. C. Castagnoli,*et. al*: Nuovo Cimento **10**, 1539 (1953);
G. Cocconi: Phys. Rev. **111**, 1699 (1958)
9. A. Jabs: Zeit. f. Physik **222**, 1 (1969) and references therein
10. H.R. Gerhold, B. Buschbeck: Nuovo Cimento A **22**, 39 (1974);
K. Hänssgen: Application of the thermodynamical model to πN and KN collisions and charged and neutral strange particle production, Leipzig preprint KMU-HEP-7605 (1976)
11. (A) Prediction and comparison of momentum spectra with experimental data for *pp* and *pA* collisions:
H. Grote, R. Hagedorn, J. Ranft: Atlas of Particle Spectra, CERN black report (1970).
(B) A variety of other predictions and/or confrontations with experiment:
S. Frautschi, C.J. Hamer: Phys. Rev. D **4**, 2125 (1971);
F. Johns, B. Margolis, W.J. Meggs, R.K. Logan: Phys. Rev. Lett. **29**, 756 (1972);
C.J. Hamer: Nuovo Cimento A **12**, 162 (1972);
S. Frautschi: Nuovo Cimento A **12**, 133 (1972);
B. Margolis, W.J. Meggs, R.K. Logan: Phys. Rev. D **8**, 199 (1973);
G. Eilam, Y. Gell, B. Margolis, W.J. Meggs: Phys. Rev D **8**, 2871 (1973);
F. Csikor, I. Farkas, I. Montvay: Nucl. Phys. B **79**, 92 (1974);
H.J. Möhring, J. Kripfganz, E.M. Ilgenfritz, J. Ranft: Nucl. Phys. B **85**, 221 (1975);
H.J. Möhring: Acta Phys. Pol. B **9**, 125 (1978);
H.J. Möhring: Nucl. Phys. B **87**, 509 (1975);
C.B. Chiu: Phys. Rev. D **13**, 2091 (1976);
S. Kogitz, R.K. Logan, S. Tanaka: Phys. Rev. D **14**, 1420 (1976);
J. Vandermeulen: Bull. Roy. Sci. de Liège **46**, 57 (1977);
J. Vandermeulen: Contribution à l'étude phénoménologique de l'annihilation nucléon antinucléon, mémoire présenté Université de Liège, Faculté des Sciences (1976);
S. Frautschi: Ericson fluctuations and the new Argonne data on πN scattering. In *Hadron Matter at Extreme Energy Density*, Ettore Majorana Internat. Sci. Series (1980), Phys. Sci. ed. by N. Cabibbo and L. Sertorio;
R.K. Logan, M. Bates, S. Kogitz: Statistical bootstrap duality II, University Toronto preprint (1980);
see also [11, 12, 17]
12. R. Hagedorn: Cargèse Lecture in Phys. **6**, 643 (1973) (Gordon & Breach, New York3);
R. Hagedorn: Thermodynamics of strong interactions. CERN yellow report 71/12;
E.M. Ilgenfritz, J. Kripfganz, H.J. Möhring: Fortschr. d. Phys. **25**, 123 (1977);
A. Tounsi: Fourteenth Scottish Summer School, 1973, publ. in *Phenomenology at High Energies*, London 1974
13. W.D. Myers: Nucl. Phys. A **296**, 177 (1978)
14. H. Stöcker et al.: GSI preprint 80-9 Darmstadt, Germany (1980)
15. R.P. Feynman: Phys. Rev. Lett. **23**, 1415 (1969)
16. J. Ranft: 'SPUKJ' Programme W129 of the CERN programme library report CERN-LabII-RA-75-2; and Leipzig University report TUL 37-7 (1975)
17. J. Letessier, A. Tounsi: Nuovo Cimento A **11**, 353 (1972); ibid. **13**, 557 (1973); ibid. **15**, 358 (1973);
J. Kripfganz, J. Ranft: Phys. Lett. B **55**, 301 (1975);
J. Ranft, G. Ranft: Nucl. Phys. B **92**, 207 (1975);
see also [2]
18. J. Hagedorn, J. Ranft: Nucl. Phys. B **48**, 157 (1972): see also E. Celeghini, L. Lusanna: [22]
19. G. Ranft, J. Ranft: Phys. Lett. B **32**, 207 (1970)
20. F. Cerulus: A liquid drop model for high energy elastic and inelastic scattering. Proceedings of the colloquium on high multiplicity hadronic interactions, Ecole Polytech. Paris III Session (1970)
21. J. Hüfner, J. Knoll: Nucl. Phys. A **290**, 460 (1977)

22. E. Celeghini, L. Lusanna: Nucl. Phys. B **69**, 58 (1974); see also [13,20]
23. J. Hofmann, W. Scheid, W. Greiner: Nuovo Cimento A **33**, 343 (1976): J. Hofmann, B. Müller, W. Greiner: Phys. Lett. B bf 82, 195 (1979)
24. J. Cugnon, T. Mizutani, J. Vandermeulen: University Liège preprint (1980);
J. Cugnon, T. Mizutani, J. Vandermeulen: Equilibration in rel. nucl. coll.; a Monte Carlo calculation. Caltech preprint MAP-10 (1980);
J. Cugnon: Non-equilibrium aspects in rel. nucl. coll. Caltech preprint MAP-15 (1980);
J. Cugnon, S.E. Koonin: Pion spectra and the geometry of nucl. coll. Caltech preprint MAP-17 (1980)
25. R. Malfliet: Phys. Rev. Lett. **44**, 864 (1980);ibid: Non-equilibrium dynamics in high energy nucleus–nucleus collisions I and II. Groningen preprint KVI-270 (1980)
26. H.G. Baumgardt et al.: Zeit. f. Physik A **273**, 359 (1975); see also [23]
27. H. Stöcker, J.A. Maruhn, W. Greiner: Phys. Rev. Lett. **44**, 725 (1980)
28. G. Giacomelli, M. Jacob: Phys. Rep. **55**, 1 (1979)
29. R.M. Weiner: Phys. Rev. Lett. **32**, 630 (1974);
R. Weiner, M. Weström: Nucl. Phys. A **286**, 282 (1977);
R. Weiner, M. Weström: Phys. Rev. **34**, 1523 (1975);
R.M. Weiner: Phys. Rev. D **13**, 1363 (1976)
30. L. Sertorio: Rivista del Nuovo Cimento **2**, 1 (1979) (A review stressing fundamental aspects)
31. R. Hagedorn, J. Rafelski: Phys. Lett. B **97**, 136 (1980); See also: preprints CERN-TH-2947 and CERN-TH-2969 "From hadron gas to quark matter I & II", in the *Proceedings of the International Symposium on Statistical Mechanics of Quarks and Hadrons*, Bielefeld, Germany, August 1980, ed. by H. Satz, (North Holland Publishing Company); See also following Chapter 27; See also [1]
32. J.I. Kapusta: Phys. Rev. C **16**, 1493 (1977)
33. A.Z. Mekjian: Phys. Rev. C **17**, 1051 (1978); ibid: Nucl. Phys. A **312**, 491 (1978)
34. Yu.B. Rumer: J. Expl. Theor. Phys. **38**, 1899 (1960)
35. G.F. Chew, S. Mandelstam: Phys. Rev. **119**, 467 (1960);
F. Zachariasen: Phys. Rev. Lett. **7**, 112 (1961);
G.F. Chew, S. Frautschi: Phys. Rev. Lett. **7**, 394 (1961);
G.F. Chew, M. Gell-Mann, A.H. Rosenfeld: Scientific American **210**, 74 (1964);
G.F. Chew: Science **161**, 762 (1968); ibid: Physics Today **23**, 23 (1970)
36. R. Hagedorn, I. Montvay: Nucl. Phys. B **59**, 45 (1973);
E.M. Ilgenfritz, J. Kripfganz: Nucl. Phys. B **56**, 241 (1973)
37. R. Hagedorn, J. Rafelski: Manuscript in preparation for Physics Reports – This manuscript was never completed, however, the material we developed is now presented in this volume drawn from different conference reports
38. G. Ranft, J. Ranft: Nucl. Phys. B **83**, 217 (1973);
J. Ranft: Fortschr. d. Physik **23**, 467 (1975)
39. F.W. Büsser et al. (CERN–Columbia–Rockefeller Collaboration): Phys. Lett. B **46**, 471 (1973)
40. J.P. Alard et al.: Nuovo Cimento A **30**, 320 (1975)
41. A.Z. Mekjian: Nucl. Phys. A **312**, 491 (1978)
42. J. Gosset, J.I. Kapusta, G.D. Westfall: Phys. Rev. C **18**, 844 (1978)
43. A.Z. Mekjian: Phys. Rev. C **17**, 1051 (1978)
44. I. Montvay, J. Zimanyi: Nucl. Phys. A **316**, 490 (1979)
45. J. Rafelski, M. Danos: Phys. Lett. B **97**, 279 (1980)
46. C.B. Chiu, R.L. Heimann: Phys. Rev. D **4**, 3184 (1971);
R. Hagedorn, U. Wambach: Nucl. Phys. B **123**, 382 (1977)

Chapter 27
Extreme States of Nuclear Matter – 1980

Johann Rafelski

Abstract The theory of hot nuclear fireballs consisting of all possible finite-size hadronic constituents in chemical and thermal equilibrium is presented. As a complement of this hadronic gas phase characterized by maximal temperature and energy density, the quark bag description of the hadronic fireball is considered. Preliminary calculations of temperatures and mean transverse momenta of particles emitted in high multiplicity relativistic nuclear collisions together with some considerations on the observability of quark matter are offered.

27.1 Overview

I wish to describe, as derived from known traits of strong interactions, the likely thermodynamic properties of hadronic matter in two different phases: the hadronic gas consisting of strongly interacting but individual baryons and mesons, and the dissolved phase of a relatively weakly interacting quark-gluon plasma. The equations of state of the hadronic gas can be used to derive the particle temperatures and mean transverse momenta in relativistic heavy ion collisons, while those of the quark-gluon plasma are more difficult to observe experimentally. They may lead to recognizable effects for strange particle yields. Clearly, the ultimate aim is to understand the behaviour of hadronic matter in the region of the phase transition from gas to plasma and to find characteristic features which will allow its experimental observation. More work is still needed to reach this goal. This report is an account of my long and fruitful collaboration with R. Hagedorn [1].

Invited lecture at Quark Matter 1: *Workshop on Future Relativistic Heavy Ion Experiments* at the Gesellschaft für Schwerionenforschung (GSI), Darmstadt, Germany, 7-10 October 1980; circulated in the GSI81-6 Orange Report, pp. 282-324, R. Bock and R. Stock, editors; Reprinted in the 'Bormio 1981' Winter School proceedings. Presented here in complete original format omitting SBM pictures shown in previous chapters.

Department of Physics, The University of Arizona, Tucson, AZ 85721, USA

The theoretical techniques required for the description of the two phases are quite different: in the case of hadronic gas, a strongly attractive interaction has to be accounted for, which leads to the formation of the numerous hadronic resonances – which are in fact bound states of several (anti) quarks. if this is really the case, then our intuition demands that at sufficiently high particle (baryon) density the individuality of such a bound state will be lost. In relativistic physics in particular, meson production at high temperatures might already lead to such a transition at moderate baryon density. As is currently believed, the quark–quark interaction is of moderate strength, allowing a perturbative treatment of the quark-gluon plasma as relativistic Fermi and Bose gases. As this is a very well studied technique to be found in several reviews [2], we shall present the relevant results for the relativistic Fermi gas and restrict the discussion to the interesting phenomenological consequences. Thus the theoretical part of this report will be devoted mainly to the strongly interacting phase of hadronic gas. We will also describe some experimental consequences for relativistic nuclear collisions such as particle temperatures, i.e., mean transverse momenta and entropy.

As we will deal with relativistic particles throughout this work, a suitable generalization of standard thermodynamics is necessary, and we follow the way described by Touschek [3]. Not only is it the most elegant, but it is also by simple physical arguments the only *physical* generalization of the concepts of thermodynamics to relativistic particle kinematics. Our notation is such that $\hbar = c = k = 1$. The inverse temperature β and volume V are generalized to become four-vectors:

$$\begin{aligned} E &\longrightarrow p^\mu = (p^0, \mathbf{p}) = m u^\mu \;, \quad u_\mu u^\mu = 1 \;, \\ \frac{1}{T} &\longrightarrow \beta^\mu = (\beta^0, \beta) = \frac{1}{T} v^\mu \;, \quad v_\mu v^\mu = 1 \;, \\ V &\longrightarrow V^\mu = (V^0, \mathbf{V}) = V w^\mu \;, \quad w_\mu w^\mu = 1 \;, \end{aligned} \tag{27.1}$$

where u^μ, v^μ, and w^μ are the four-velocities of the total mass, the thermometer, and the volume, respectively. Usually, $\langle u^\mu \rangle = v^\mu = w^\mu$.

We will often work in the frame in which all velocities have a timelike component only. In that case we shall often drop the Lorentz index μ, as we shall do for the arguments $V = V_\mu$, $\beta = \beta_\mu$ of different functions.

The attentive reader may already be wondering how the approach outlined here can be reconciled with the concept of quark confinement. We will now therefore explain why the occurrence of the high temperature phase of hadronic matter – the quark-gluon plasma – is still consistent with our incapability to liberate quarks in high energy collisions. It is thus important to realize that the currently accepted theory of hadronic structure and interactions, quantum chromodynamics [4], supplemented with its phenomenological extension, the MIT bag model [5], allows the formation of large space domains filled with (almost) free quarks. Such a state is expected to be unstable and to decay again into individual hadrons, following its free expansion. The mechanism of quark confinement requires that all quarks recombine to form hadrons again. Thus the quark-gluon plasma may be only a transitory form of hadronic matter formed under special conditions and therefore quite difficult to detect experimentally.

We will recall now the relevant postulates and results that characterize the current understanding of strong interactions in quantum chromodynamics (QCD). The most important postulate is that the proper vacuum state in QCD is not the (trivial) perturbative state that we (naively) imagine to exist everywhere and which is little changed when the interactions are turned on/off. In QCD, the true vacuum state is believed to a have a complicated structure which originates in the glue ('photon') sector of the theory. The perturbative vacuum is an excited state with an energy density $\mathscr{B}$ above the true vacuum. It is to be found inside hadrons where perturbative quanta of the theory, in particular quarks, can therefore exist. The occurrence of the true vacuum state is intimately connected to the glue–glue interaction. Unlike QED, these massless quanta of QCD, also carry a charge – color – that is responsible for the quark–quark interaction.

In the above discussion, the confinement of quarks is a natural feature of the hypothetical structure of the true vacuum. If it is, for example, a color superconductor, then an isolated charge cannot occur. Another way to look at this is to realize that a single colored object would, according to Gauss' theorem, have an electric field that can only end on other color charges. In the region penetrated by this field, the true vacuum is displaced, thus effectively raising the mass of a quasi-isolated quark by the amount $\mathscr{B}V_{\text{field}}$.

Another feature of the true vacuum is that it exercises a pressure on the surface of the region of the perturbative vacuum to which quarks are confined. Indeed, this is just the idea of the original MIT bag model [6]. The Fermi pressure of almost massless light quarks is in equilibrium with the vacuum pressure $\mathscr{B}$. When many quarks are combined to form a giant quark bag, then their properties inside can be obtained using standard methods of many-body theory [2]. In particular, this also allows the inclusion of the effect of internal excitation through a finite temperature and through a change in the chemical composition.

A further effect that must be taken into consideration is the quark–quark interaction. We shall use here the first order contribution in the QCD running coupling constant $\alpha_s(q^2) = g^2/4\pi$. However, as $\alpha_s(q^2)$ increases when the average momentum exchanged between quarks decreases, this approach will have only limited validity at relatively low densities and/or temperatures. The collective screening effects in the plasma are of comparable order of magnitude and should reduce the importance of perturbative contributions as they seem to reduce the strength of the quark–quark interaction.

From this general description of the hadronic plasma, it is immediately apparent that, at a certain value of temperature and baryon number density, the plasma must disintegrate into individual hadrons. Clearly, to treat this process and the ensuing further nucleonisation by perturbative QCD methods is impossible. It is necessary to find a semi-phenomenological method for the treatment of the thermodynamic system consisting of a gas of quark bags. The hadronic gas phase is characterized by those reactions between individual hadrons that lead to the formation of new particles (quark bags) only. Thus one may view [7–9] the hadronic gas phase as being an assembly of many different hadronic resonances, their number in the interval $(m^2, m^2 + \mathrm{d}m^2)$ being given by the mass spectrum $\tau(m^2, b)\mathrm{d}m^2$. Here the baryon number b is the only discrete quantum number to be considered at present. All bag–bag interaction is contained in the mutual transmutations from one state to another. Thus the gas phase has the characteristic of an infinite component ideal gas phase of extended objects. The quark bags having a finite size force us to formulate the theory of an extended, though otherwise ideal multicomponent gas.

It is a straightforward exercise, carried through in the beginning of the next section, to reduce the grand partition function Z to an expression in terms of the mass spectrum $\tau(m^2, b)$. In principle, an experimental form of $\tau(m^2, b)$ could then be used as an input. However, the more natural way is to introduce the statistical bootstrap model [7], which will provide us with a theoretical τ that is consistent with assumptions and approximations made in determining Z.

In the statistical bootstrap, the essential step consists in the realization that a composite state of many quark bags is in itself an 'elementary' bag [1, 10]. This leads directly to a nonlinear integral equation for τ. The ideas of the statistical bootstrap have found a very successful application in the description of hadronic reactions [11] over the past decade. The present work is an extension [1, 9, 12] and application [1, 13] of this method to the case of a system containing any number of finite size hadronic clusters with their baryon numbers adding up to some fixed number. Among the most successful predictions of the statistical bootstrap, we record here the derivation of the limiting hadronic temperature and the exponential growth of the mass spectrum.

We see that the theoretical description of the two hadronic phases – the individual hadron gas and the quark-gluon plasma – is consistent with observations and with the present knowledge of elementary particles. What remains is the study of the possible phase transition between those phases as well as its observation. Unfortunately, we can argue that in the study of temperatures and mean transverse momenta of pions and nucleons produced in nuclear collisions, practically all information about the hot and dense phase of the collision is lost, as most of the emitted particles originate in the cooler and more dilute hadronic *gas* phase of matter. In order to obtain reliable information on quark matter, we must presumably perform more specific experiments. We will briefly point out that the presence of numerous $\bar{s}$ quarks in the quark plasma suggest, as a characteristic experiment, the observation $\overline{\Lambda}$ hyperons.

We close this report by showing that, in nuclear collisions, unlike pp reactions, we can use equilibrium thermodynamics in a large volume to compute the yield of strange and anti-strange particles. The latter, e.g., $\overline{\Lambda}$, might be significantly different from what one expects in pp collisions and give a hint about the properties of the quark-gluon phase.

27.2 Thermodynamics of the Gas Phase and the SBM

Given the grand partition function $Z(\beta, V, \lambda)$ of a many-body system, all thermodynamic quantities can be determined by differentiation of $\ln Z$ with respect to its arguments. Here, λ is the fugacity introduced to conserve a discrete quantum number, here the baryon number. The conservation of strangeness can be carried through in a similar fashion leading then to a further argument λ_s of Z. Whenever necessary, we will consider Z to be implicitly dependent on λ_s.

The grand partition function is a Laplace transform of the level density $\sigma(p, V, b)$, where p_μ is the four-momentum and b the baryon number of the many-body system enclosed in the volume V:

$$Z(\beta, V, \lambda) = \sum_{b=-\infty}^{\infty} \lambda^b \int \sigma(p, V, b) \mathrm{e}^{-\beta_\mu p^\mu} \mathrm{d}^4 p \ . \tag{27.2}$$

We recognize the usual relations for the thermodynamic expectation values of the baryon number,

$$\langle b \rangle = \lambda \frac{\partial}{\partial \lambda} \ln Z(\beta, V, \lambda) \ , \tag{27.3}$$

and the energy–momentum four-vector,

$$\langle p_\mu \rangle = -\frac{\partial}{\partial \beta_\mu} \ln Z(\beta, V, \lambda) \ , \tag{27.4}$$

which follow from the definition in Eq. (27.2).

The theoretical problem is to determine $\sigma(p, V, b)$ in terms of known quantities. Let us suppose that the physical states of the hadronic gas phase can be considered as being built up from an arbitrary number of massive objects, henceforth called clusters, characterized by a mass spectrum $\tau(m^2, b)$, where $\tau(m^2, b)\mathrm{d}m^2$ is the number of different elementary objects (existing in nature) in the mass interval $(m^2, m^2 + \mathrm{d}m^2)$ and having the baryon number b. As particle creation must be permitted, the number N of constituents is arbitrary, but constrained by four-momentum conservation and baryon conservation. Neglecting quantum statistics (it can be shown that, for $T \gtrsim 40$ MeV, Boltzmann statistics is sufficient), we have

$$\sigma(p, V, b) = \sum_{N=0}^{\infty} \frac{1}{N!} \int \delta^4 \left(p - \sum_{i=1}^{N} p_i \right) \sum_{\{b_i\}} \delta_k \left(b - \sum_{i=1}^{N} b_i \right) \prod_{i=1}^{N} \frac{2\Delta_\mu p_i^\mu}{(2\pi)^3} \tau(p_i^2, b_i) \mathrm{d}^4 p_i \ . \tag{27.5}$$

The sum over all allowed partitions of b into different b_i is included and Δ is the volume available for the motion of the constituents, which differs from V if the different clusters carry their proper volume V_{ci}:

$$\Delta^\mu = V^\mu - \sum_{i=1}^{N} V_{\mathrm{c}i}^\mu \ . \tag{27.6}$$

The phase space volume used in Eq. (27.5) is best explained by considering what happens for one particle of mass m_0 in the rest frame of Δ_μ and β_μ :

$$\int \mathrm{d}^4 p_i \frac{2\Delta_\mu p_i^\mu}{(2\pi)^3}\,\mathrm{e}^{-\beta\cdot p}\,\delta_0(p_i^2-m^2) = \Delta_0 \int \frac{\mathrm{d}^3 p_i}{(2\pi)^3}\,\mathrm{e}^{-\sqrt{p^2+m^2}\beta_0} = \Delta_0 \frac{Tm^2}{2\pi^2} K_2(m/T)\ . \tag{27.7}$$

The density of states in Eq. (27.5) implies that the creation and absorption of particles in kinetic and chemical equilibrium is limited only by four-momentum and baryon number conservation. These processes represent the strong hadronic interactions which are dominated by particle productions. $\tau(m^2,b)$ contains all participating elementary particles and their resonances. Some remaining interaction is here neglected or, as we do not use the complete experimental τ, it may be considered as being taken care of by a suitable choice of τ. The short range repulsive forces are taken into account by the introduction of the proper volume V of hadronic clusters.

One more remark concerning the available volume Δ is in order here. If V were considered to be given and an independent thermodynamic quantity, then in Eq. (27.5), a further built-in restriction limits the sum over N to a certain $N_{\max}$, such that the available volume Δ in Eq. (27.6) remains positive. However, this more conventional assumption of V as the independent variable would significantly obscure our mathematical formalism. It is important to realize that we are *free* to select the available volume Δ as the independent thermodynamic variable and to consider V as a thermodynamic expectation value to be computed from Eq. (27.6):

$$V^\mu \;\longrightarrow\; \langle V^\mu\rangle = \Delta^\mu + \big\langle V_\mathrm{c}^\mu(\beta,\Delta,\lambda)\big\rangle\ . \tag{27.8}$$

Here $\langle V_\mathrm{c}^\mu\rangle$ is the average sum of proper volumes of all hadronic clusters contained in the system considered. As already discussed, the standard quark bag leads to the proportionality between the cluster volume and hadron mass. Similar arguments within the bootstrap model [9, 10], as for example discussed in the preceding Chapter 26 by R. Hagedorn, also lead to

$$\langle V_\mathrm{c}^\mu\rangle = \frac{\big\langle p^\mu(\beta,\Delta,\lambda)\big\rangle}{4\mathscr{B}}\ , \tag{27.9}$$

where $4\mathscr{B}$ is the (at this point arbitrary) energy density of isolated hadrons in the quark bag model [5].

Since our hadrons are under pressure from neighbours in hadronic matter, we have in principle to take instead of $4\mathscr{B}$ the energy density of a quark bag exposed to a pressure P [see Eq. (27.57) below]

$$\varepsilon_\mathrm{bag} = 4\mathscr{B} + 3P\ .$$

Combining Eqs. (27.8)–(27.10), we find, with $\varepsilon(\beta,\Delta,\lambda) = \langle p^\mu\rangle/\langle V^\mu\rangle = \langle E\rangle/\langle V\rangle$, that

$$\frac{\Delta}{\langle V(\beta,\Delta,\lambda)\rangle} = 1 - \frac{\varepsilon(\beta,\Delta,\lambda)}{4\mathscr{B}+3P(\beta,\Delta,\lambda)}\ . \tag{27.10}$$

As we shall see, the pressure P in the hadronic matter never rises above $\simeq 0.4\mathscr{B}$, see Fig. 27.3a below, and arguments following Eq. (27.63). Consequently, the inclusion of P above – the compression of free hadrons by the hadronic matter by about 10% – may be omitted for now from further discussion. However, we note that both ε and P will be computed as $\ln Z$ becomes available, whence Eq. (27.10) is an implicit equation for $\Delta/\langle V\rangle$.

It is important to record that the expression in Eq. (27.10) can approach zero only when the energy density of the hadronic gas approaches that of matter consisting of one big quark bag: $\varepsilon \to 4\mathscr{B}$, $P \to 0$. Thus the density of states in Eq. (27.5), together with the choice of Δ as a thermodynamic variable, is a consistent physical choice only up to this point. Beyond we assume that a description in terms of interacting quarks and gluons is the proper physical description. Bearing all these remarks in mind, we now consider the available volume Δ as a thermodynamic variable which by definition is positive. Inspecting Eq. (27.5) again, we recognize that the level density of the extended objects in volume $\langle V\rangle$ can be interpreted for the time being as the level density of point particles in a fictitious volume Δ :

$$\sigma(p,V,b) = \sigma_{\mathrm{pt}}(p,\Delta,b) \;, \tag{27.11}$$

whence this is also true for the grand canonical partition function in Eq. (27.2):

$$Z(\beta,V,\lambda) = Z_{\mathrm{pt}}(\beta,\Delta,\lambda) \;. \tag{27.12}$$

Combining Eqs. (27.2) and (27.5), we also find the important relation

$$\ln Z_{\mathrm{pt}}(\beta,\Delta,\lambda) = \sum_{b=-\infty}^{\infty} \lambda^b \int \frac{2\Delta_\mu p^\mu}{(2\pi)^3} \tau(p^2,b) \mathrm{e}^{-\beta_\mu p^\mu} \mathrm{d}^4 p \;. \tag{27.13}$$

This result can only be derived when the sum over N in Eq. (27.5) extends to infinity, thus as long as $\Delta/\langle V\rangle$ in Eq. (27.10) remains positive.

In order to continue with our description of hadronic matter, we must now determine a suitable mass spectrum τ to be inserted into Eq. (27.5). For this we now introduce the statistical bootstrap model. The basic idea is rather old, but has undergone some development more recently making it clearer, more consistent, and perhaps more convincing. The details may be found in [9] and the references therein. Here a simplified naive presentation is given. We note, however, that our present interpretation is non-trivially different from that in [9].

The basic postulate of statistical bootstrap is that the mass spectrum $\tau(m^2,b)$ containing all the ‘particles’, i.e., elementary, bound states, and resonances (clusters), is generated by the *same* interactions which we see at work if we consider our thermodynamical system. Therefore, if we were to compress this system until it reaches its natural volume $V_{\mathrm{c}}(m,b)$, then it would itself be almost a cluster appearing in the mass spectrum $\tau(m^2,b)$. Since $\sigma(p,\Delta,b)$ and $\tau(p^2,b)$ are both densities of states (with respect to the different parameters d^4p and $\mathrm{d}m^2$), we postulate that

$$\sigma(p,\Delta,b)\Big|_{\langle V\rangle \underset{\Delta\to 0}{\longrightarrow} V_\mathrm{c}(m,b)} \;\widehat{=}\; \text{const.} \times \tau(p^2,b)\;, \tag{27.14}$$

where $\widehat{=}$ means 'corresponds to' (in some way to be specified). As $\sigma(p,\Delta,b)$ is [see Eq. (27.5)] the sum over N of N-fold convolutions of τ, the above 'bootstrap postulate' will yield a highly nonlinear integral equation for τ.

The bootstrap postulate (27.14) requires that τ should obey the equation resulting from replacing σ in Eq. (27.5) by some expression containing τ linearly and by taking into account the volume condition expressed in Eqs. (27.8) and (27.9).

We cannot simply put $V = V_\mathrm{c}$ and $\Delta = 0$, because now, when each cluster carries its own dynamically determined volume, Δ loses its original meaning and must be redefined more precisely. Therefore, in Eq. (27.5), we tentatively replace

$$\begin{aligned}
\sigma(p,V_\mathrm{c},b) \;&\longrightarrow\; \frac{2V_\mathrm{c}(m,b)\cdot p}{(2\pi)^3}\tau(p^2,b) = \frac{2m^2}{(2\pi)^3 4\mathscr{B}}\tau(p^2,b)\;,\\
\frac{2\Delta\cdot p_i}{(2\pi)^3}\tau(p_i^2,b_i) \;&\longrightarrow\; \frac{2V_\mathrm{c}(m_i,b_i)\cdot p_i}{(2\pi)^3}\tau(p_i^2,b_i) = \frac{2m_i^2}{(2\pi)^3 4\mathscr{B}}\tau(p_i^2,b_i)\;.
\end{aligned} \tag{27.15}$$

Next we argue that the explicit factors m^2 and m_i^2 arise from the dynamics and therefore must be absorbed into $\tau(p_i^2,b_i)$ as dimensionless factors[1] m_i^2/m_0^2. Thus,

$$\begin{aligned}
\sigma(p,V_\mathrm{c},b) \;&\longrightarrow\; \frac{2m_0^2}{(2\pi)^3 4\mathscr{B}}\tau(p^2,b) = H\tau(p^2,b)\;,\\
\frac{2\Delta\cdot p_i}{(2\pi)^3}\tau(p_i^2,b_i) \;&\longrightarrow\; \frac{2m_0^2}{(2\pi)^3 4\mathscr{B}}\tau(p_i^2,b_i) = H\tau(p_i^2,b_i)\;,
\end{aligned} \tag{27.16}$$

with

$$H = \frac{2m_0^2}{(2\pi)^3 4\mathscr{B}}\;,$$

where either H or m_0 may be taken as a new free parameter of the model, to be fixed later. (If m_0 is taken, then it should be of the order of the 'elementary masses' appearing in the system, e.g., somwhere between m_π and M_N in a model using pions and nucleons as elementary input.) Finally, if clusters consist of clusters which consist of clusters, and so on, this should end at some 'elementary' particles (where what we consider as elementary is fixed by convention). Inserting Eq. (27.16) into Eq. (27.5), the bootstrap equation (BE) then reads

$$\begin{aligned}
H\tau(p^2,b) &= Hg_b\delta_0(p^2-\overline{m}_b^2) \\
&\quad + \sum_{N=2}^{\infty}\frac{1}{N!}\int\delta^4\left(p-\sum_{i=1}^{N}p_i\right)\sum_{\{b_i\}}\delta_k\left(b-\sum_{i=1}^{N}b_i\right)\prod_{i=1}^{N}H\tau(p_i^2,b_i)\mathrm{d}^4p_i\;.
\end{aligned} \tag{27.17}$$

[1] Here is the essential difference with [9], where another choice was made.

Clearly, the bootstrap equation (27.17) has not been derived. We have made it more or less plausible and state it as a postulate. For more motivation, see [9]. In other words, the bootstrap equation means that the cluster with mass $\sqrt{p^2}$ and baryon number b is either elementary (mass $\overline{m}_b$, spin isospin multiplicity g_b), or it is composed of any number $N \geq 2$ of subclusters having the same internal composite structure described by this equation. The bar over $\overline{m}_b$ indicates that one has to take the mass which the 'elementary particle' will have effectively when present in a large cluster, e.g., in nuclear matter, $\overline{m} = m - \langle E_{\text{bind}} \rangle$, and $\overline{m}_{\text{N}} \approx 925$ MeV. That this must be so becomes obvious if one imagines Eq. (27.17) solved by iteration (the iteration solution exists and is the physical solution). Then $H\tau(p^2, b)$ becomes in the end a complicated function of p^2, b, all $\overline{m}_b$, and all g_b. In other words, in the end a single cluster consists of the 'elementary particles'. As these are all bound into the cluster, their mass $\overline{m}$ should be the effective mass, not the free mass m. This way we may include a small correction for the long-range attractive meson exchange by choosing $\overline{m}_{\text{N}} = m - 15$ MeV.

Let us make a brief excursion to the bag model at this point. There the mass of a hadron is computed from the assumption of an *isolated* particle (= bag) with its size and mass being determined from the equilibrium between the vacuum pressure $\mathscr{B}$ and the internal Fermi pressure of the (valence) quarks. In a hadron gas, this is not true as a finite pressure is exerted on hadrons in matter. After a short calculation, we find the pressure dependence of the bag model hadronic mass:

$$M(P) = M(0)\frac{1 + 3P/4\mathscr{B}}{(1 + P/\mathscr{B})^{3/4}} = M(0)\left[1 + \frac{3}{32}\left(\frac{P}{\mathscr{B}}\right)^2 + \cdots\right] . \tag{27.18}$$

We have already noted that the pressure never exceeds $0.4\mathscr{B}$ in the hadronic gas phase, see Fig. 27.3a below, and arguments following Eq. (27.63). Hence we see that the increase in mass of constituents (quark bags) in the hadronic gas never exceeds 1.5% and is at most comparable with the 15 MeV binding in $\overline{m}$. In general, P is about $0.1B$ and the pressure effect may be neglected.

Thus we can consider the 'input' first term in Eq. (27.17) as being fixed by pions, nucleons, and whenever necessary by the usual strange members of meson and baryon multiplets. Furthermore, we note that the bootstrap equation (27.17) makes use of practically all the same approximations as our description of the level density in Eq. (27.5). Thus the solution of Eq. (27.17) is particularly suitable for our use.

We solve the BE by the same double Laplace transformation which we used before Eq. (27.2). We define

$$\varphi(\beta, \lambda) := \int \mathrm{e}^{-\beta_\mu p^\mu} \sum_{b=-\infty}^{\infty} \lambda^b H g_b \delta_0(p^2 - \overline{m}_b^2) \mathrm{d}^4 p = 2\pi H T \sum_{b=-\infty}^{\infty} \lambda^b g_b \overline{m}_b K_1(\overline{m}_b/T) \, ,$$

$$\Phi(\beta, \lambda) := \int \mathrm{e}^{-\beta_\mu p^\mu} \sum_{b=-\infty}^{\infty} \lambda^b H\tau(p^2, b) \mathrm{d}^4 p \, . \tag{27.19}$$

Once the set of input particles $\{\overline{m}_b, g_b\}$ is given, $\varphi(\beta,\lambda)$ is a known function, while $\Phi(\beta,\lambda)$ is unknown. Applying the double Laplace transformation to the BE, we obtain

$$\Phi(\beta,\lambda) = \varphi(\beta,\lambda) + \exp\Phi(\beta,\lambda) - \Phi(\beta,\lambda) - 1 \,. \tag{27.20}$$

This implicit equation for Φ in terms of φ can be solved without regard for the actual β,λ dependence. Writing

$$G(\varphi) := \Phi(\beta,\lambda)\,, \qquad \varphi = 2G - \mathrm{e}^G + 1\,, \tag{27.21}$$

we can draw the curve $\varphi(G)$, see Fig. 17.5a, and then invert it graphically, see Fig. 17.5b to obtain $G(\varphi) = \Phi(\beta,\lambda)$. $G(\varphi)$ has a square root singularity at $\varphi = \varphi_0 = \ln(4/\mathrm{e}) = 0.3863$. Beyond this value, $G(\varphi)$ becomes complex. There are further branches; for example Fig. 17.5b the *dashed line* represents the unphysical real solution branch.

Apart from this graphical solution, other forms of solution are known:

$$G(\varphi) = \sum_{n=1}^{\infty} s_n \varphi^n = \sum_{n=0}^{\infty} w_n (\varphi_0 - \varphi)^{n/2} = \text{integral representation}\,. \tag{27.22}$$

The expansion in terms of $(\varphi_0 - \varphi)^{n/2}$ has been used in our numerical work (12 terms yield a solution within computer accuracy) and the integral representation will be published elsewhere [15]. Henceforth, we consider $\Phi(\beta,\lambda) = G(\varphi)$ to be a known function of $\varphi(\beta,\lambda)$. Consequently, $\tau(m^2,b)$ is also in principle known. From the singularity at $\varphi = \varphi_0$, it follows [1] that $\tau(m^2,b)$ grows, for $m \gg m_{\mathrm{N}} b$, exponentially $\sim m^{-3}\exp(m/T_0)$. In some weaker form, this has been known for a long time [7, 16, 17].

27.3 The Hot Hadronic Gas

The definition of $\Phi(\beta,\lambda)$ in Eq. (27.19) in terms of the mass spectrum allows us to write a very simple expression for $\ln Z$ in the gas phase (passing now to the rest frame of the gas):

$$\ln Z(\beta,V,\lambda) = \ln Z_{\mathrm{pt}}(\beta,\Delta,\lambda) = -\frac{2\Delta}{(2\pi)^3 H}\frac{\partial}{\partial\beta}\Phi(\beta,\lambda)\,. \tag{27.23}$$

We recall that Eqs. (27.10) and (27.20) define (implicitly) the quantities Δ and Φ in terms of the physical variables V, β, and λ.

Let us now introduce the energy density $\varepsilon_{\mathrm{pt}}$ of the hypothetical pointlike particles as

$$\varepsilon_{\mathrm{pt}}(\beta,\lambda) = \frac{1}{\Delta}\left[-\frac{\partial}{\partial\beta}\ln Z_{\mathrm{pt}}(\beta,\Delta,\lambda)\right] = \frac{2}{(2\pi)^3 H}\frac{\partial^2}{\partial\beta^2}\Phi(\beta,\lambda)\,, \tag{27.24}$$

which will turn out to be quite helpful as it is independent of Δ. The proper energy density is

$$\varepsilon(\beta,\lambda) = \frac{1}{\langle V\rangle}\left(-\frac{\partial}{\partial\beta}\ln Z\right) = \frac{\Delta}{\langle V\rangle}\varepsilon_{\text{pt}} , \tag{27.25}$$

while the pressure follows from

$$P(\beta,\lambda)\langle V\rangle = T\ln Z(\beta,V,\lambda) = T\ln Z_{\text{pt}}(\beta,\Delta,\lambda) , \tag{27.26}$$

$$P(\beta,\lambda) = \frac{\Delta}{\langle V\rangle}\left[-\frac{2T}{(2\pi)^3 H}\frac{\partial}{\partial\beta}\Phi(\beta,\lambda)\right] =: \frac{\Delta}{\langle V\rangle}P_{\text{pt}} . \tag{27.27}$$

Similarly, for the baryon number density, we find

$$\nu(\beta,\lambda) = \frac{\langle b\rangle}{\langle V\rangle} =: \frac{\Delta}{\langle V\rangle}\nu_{\text{pt}}(\beta,\lambda) , \tag{27.28}$$

with

$$\nu_{\text{pt}}(\beta,\lambda) = \frac{1}{\Delta}\lambda\frac{\partial}{\partial\lambda}\ln Z_{\text{pt}} = -\frac{2}{(2\pi)^3 H}\lambda\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\beta}\Phi(\beta,\lambda) . \tag{27.29}$$

From Eqs. (27.24)–(27.24), the crucial role played by the factor $\Delta/\langle V\rangle$ becomes apparent. We note that it is quite straightforward to insert Eqs. (27.25) and (27.26) into Eq. (27.10) and solve the resulting quadratic equation to obtain $\Delta/\langle V\rangle$ as an explicit function of ε_{pt} and P_{pt}. First we record the limit $P \ll B$:

$$\frac{\Delta}{\langle V\rangle} = 1 - \frac{\varepsilon(\beta,\lambda)}{4\mathscr{B}} = \left[1 + \frac{\varepsilon_{\text{pt}}(\beta,\lambda)}{4\mathscr{B}}\right]^{-1} , \tag{27.30}$$

while the correct expression is

$$\frac{\Delta}{\langle V\rangle} = \frac{1}{2} - \frac{\varepsilon_{\text{pt}}}{6P_{\text{pt}}} - \frac{2\mathscr{B}}{3P_{\text{pt}}} + \sqrt{\frac{4\mathscr{B}}{3P_{\text{pt}}} + \left(\frac{1}{2} - \frac{\varepsilon_{\text{pt}}}{6P_{\text{pt}}} - \frac{2\mathscr{B}}{3P_{\text{pt}}}\right)^2} . \tag{27.31}$$

The last of the important thermodynamic quantities is the entropy S. By differentiating Eq. (27.26), we find

$$\frac{\partial}{\partial\beta}\ln Z = \frac{\partial}{\partial\beta}\beta P\langle V\rangle = P\langle V\rangle - T\frac{\partial}{\partial T}(P\langle V\rangle) . \tag{27.32}$$

Considering Z as a function of the chemical potential, viz.,

$$Z(\beta,V,\lambda) = Z(\beta,V,\mathrm{e}^{\mu\beta}) = \tilde{Z}(\beta,V,\mu) = \tilde{Z}_{\text{pt}}(\beta,\Delta,\mu) , \tag{27.33}$$

we find

$$\left.\frac{\partial}{\partial\beta}\ln Z\right|_{\mu,\Delta} = \frac{\partial}{\partial\beta}\ln\tilde{Z}_{\text{pt}}(\beta,\Delta,\mu) = -E + \mu\langle b\rangle , \tag{27.34}$$

with E being the total energy. From Eqs. (27.32) and (27.34), we find the 'first law' of thermodynamics to be

$$E = -P\langle V\rangle + T\frac{\partial}{\partial T}(P\langle V\rangle) + \mu\langle b\rangle \; . \tag{27.35}$$

Now quite generally,

$$E = -P\langle V\rangle + TS + \mu\langle b\rangle \; , \tag{27.36}$$

so that

$$S = \frac{\partial}{\partial T}\Big[P(\beta,\Delta,\mu)\langle V(\beta,\Delta,\mu)\rangle\Big]\Big|_{\mu,\Delta} \; . \tag{27.37}$$

Equations (27.26) and (27.34) now allow us to write

$$S = \frac{\partial}{\partial T}(P\langle V\rangle) = \ln \tilde{Z}_{\mathrm{pt}}(T,\Delta,\mu) + \frac{E-\mu b}{T} \; . \tag{27.38}$$

The entropy density in terms of the already defined quantities is therefore

$$\mathscr{S} = \frac{S}{\langle V\rangle} = \frac{P+\varepsilon-\mu\nu}{T} \; . \tag{27.39}$$

We shall now take a brief look at the quantities P, ε, ν, $\Delta/\langle V\rangle$. They can be written in terms of $\partial\Phi(\beta,\lambda)/\partial\beta$ and its derivatives. We note that [see Eq. (27.21)]

$$\frac{\partial}{\partial\beta}\Phi(\beta,\lambda) = \frac{\partial G(\varphi)}{\partial\varphi}\frac{\partial\varphi}{\partial\beta} \; , \tag{27.40}$$

and that $\partial G/\partial\varphi \sim (\varphi_0-\varphi)^{-1/2}$ near to $\varphi=\varphi_0=\ln(4/\mathrm{e})$ (see Fig. 17.5b). Hence at $\varphi=\varphi_0$, we find a singularity in the point particle quantities $\varepsilon_{\mathrm{pt}}$, ν_{pt}, and P_{pt}. This implies that all hadrons have coalesced into one large cluster. Indeed, from Eqs. (27.25), (27.27), (27.28), and (27.30), we find

$$\begin{aligned} \varepsilon &\longrightarrow 4\mathscr{B} \; , \\ P &\longrightarrow 0 \; , \\ \Delta/\langle V\rangle &\longrightarrow 0 \; . \end{aligned} \tag{27.41}$$

We can easily verify that this is correct by establishing the average number of clusters present in the hadronic gas. This is done by introducing an artificial fugacity ξ^N in Eq. (27.5) in the sum over N, where N is the number of clusters. Denoting by $Z(\xi)$ the associated grand canonical partition functions in Eq. (27.23), we find

$$\langle N\rangle = \xi\frac{\partial}{\partial\xi}\ln Z^{\xi}_{\mathrm{pt}}(\beta,\Delta,\lambda;\xi)\Big|_{\xi=1} = -\frac{2\Delta}{(2\pi)^3 H}\frac{\partial}{\partial\beta}\Phi(\beta,\lambda) \; , \tag{27.42}$$

which leads to the useful relation

$$P\langle V\rangle = \langle N\rangle T \; . \tag{27.43}$$

Thus as $P\langle V\rangle \to 0$, so must $\langle N\rangle$, the number of clusters, for finite T. We record the astonishing fact that the hadron gas phase obeys an 'ideal' gas equation, although of course $\langle N\rangle$ is not constant as for a real ideal gas but a function of the thermodynamic variables.

The boundary given by

$$\varphi(\beta,\lambda) = \varphi_0 = \ln(4/\mathrm{e}) \tag{27.44}$$

thus defines a critical curve in the β,λ plane. Its position depends, of course, on the actually given form of $\varphi(\beta,\lambda)$, i.e., on the set of 'input' particles $\{\overline{m}_b, g_b\}$ assumed and the value of the constant H in Eq. (27.16). In the case of three elementary pions π^+, π^0, and π^- and four elementary nucleons (spin $\otimes$ isospin) and four antinucleons, we have from Eq. (27.19)

$$\varphi(\beta,\lambda) = 2\pi HT\left[3m_\pi K_1(m_\pi/T) + 4\left(\lambda + \frac{1}{\lambda}\right)\overline{m}_\mathrm{N} K_1(\overline{m}_\mathrm{N}/T)\right] , \tag{27.45}$$

and the condition (27.44), written in terms of T and $\mu = T\ln\lambda$, yields the curve shown in Fig. 25.3 i.e., the 'critical curve', corresponding to $\varphi(T,\mu) = \varphi_0$ in the μ, T plane. Beyond it, the usual hadronic world ceases to exist. In the shaded region our theory is not valid, because we neglected Bose-Einstein and Fermi-Dirac statistics. For $\mu = 0$, the curve ends at $T = T_0$, where T_0, the 'limiting temperature of hadronic matter', is the same as that appearing in the mass spectrum [7, 9, 16, 17] $\tau(m^2, b) \sim m^{-3}\exp(m/T_0)$ (for $b \gg bm_\mathrm{N}$).

The value of the constant H in Eq. (27.16) has been chosen [13] to yield $T_0 =$ 190 MeV. This apparently large value of T_0 seemed necessary to yield a maximal average decay temperature of the order of 145 MeV, as required by [18]. (However, a new value of the bag constant then induces a change [1] to a lower value of $T_0 =$ 180 MeV.) Here we use

$$\begin{aligned} H &= 0.724\ \mathrm{GeV}^{-2} , \qquad T_0 = 0.19\ \mathrm{GeV} ,\\ m_0 &= 0.398\ \mathrm{GeV} \quad [\text{when } \mathscr{B} = (145\ \mathrm{MeV})^4] , \end{aligned} \tag{27.46}$$

where the value of m_0 lies as expected between m_π and m_N $[(m_\pi m_\mathrm{N})^{1/2} = 0.36\ \mathrm{GeV}]$.

The critical curve limits the hadron gas phase. By approaching it, all hadrons dissolve into a giant cluster, which is not in our opinion a hadron solid [14]. We would prefer to identify it with a quark-gluon plasma. Indeed, as the energy density along the critical curve is constant ($= 4\mathscr{B}$), the critical curve can be attained and, if the energy density becomes $> 4\mathscr{B}$, we enter into a region which cannot be described without making assumptions about the inner structure and dynamics of the 'elementary particles' $\{\overline{m}_b, g_b\}$ – here pions and nucleons – entering into the input function $\varphi(\beta,\lambda)$. Considering pions and nucleons as quark-gluon bags leads naturally to this interpretation.

27.4 The Quark–Gluon Phase

We now turn to the discussion of the region of the strongly interacting matter in which the energy density would be equal to or higher than $4\mathscr{B}$. As a basic postulate, we will assume that it consists of – relatively weakly – interacting quarks. To begin with, only u and d flavours will be considered as they can easily be copiously produced at $T \gtrsim 50$ MeV. Again the aim is to derive the grand partition function Z. This is a standard exercise. For the massless quark Fermi gas up to first order in the interaction [1, 2, 12], the result is

$$\ln Z_q(\beta,\lambda) = \frac{8V}{6\pi^2}\beta^{-3}\left[\left(1-\frac{2\alpha_s}{\pi}\right)\left(\frac{1}{4}\ln^4\lambda_q + \frac{\pi^2}{2}\ln^2\lambda_q\right) + \left(1-\frac{50}{21}\frac{\alpha_s}{\pi}\right)\frac{7\pi^4}{60}\right], \tag{27.47}$$

valid in the limit $m_q < T\ln\lambda_q$.

Here $g = (2s+1)(2I+1)C = 12$ counts the number of the components of the quark gas, and λ_q is the fugacity related to the quark number. As each quark has baryon number 1/3, we find

$$\lambda_q^3 = \lambda = \mathrm{e}^{\mu/T}\ , \tag{27.48}$$

where as before λ allows for conservation of the baryon number. Consequently,

$$3\mu_q = \mu\ . \tag{27.49}$$

The glue contribution is

$$\ln Z_{\mathrm{g}}(\beta,\lambda) = V\frac{8\pi^2}{45}\beta^{-3}\left(1-\frac{15}{4}\frac{\alpha_s}{\pi}\right)\ . \tag{27.50}$$

We notice the two relevant differences with the photon gas:

- The occurence of the factor eight associated with the number of gluons.
- The glue–glue interaction as gluons carry color charge.

Finally, let us introduce the vacuum term, which accounts for the fact that the perturbative vacuum is an excited state of the 'true' vacuum which has been renormalized to have a vanishing thermodynamic potential, $\Omega = -\beta^{-1}\ln Z$. Hence in the perturbative vacuum,

$$\ln Z_{\mathrm{vac}} = -\beta\mathscr{B}V\ . \tag{27.51}$$

This leads to the required positive energy density $\mathscr{B}$ within the volume occupied by the colored quarks and gluons and to a negative pressure on the surface of this region. At this stage, this term is entirely phenomenological, as discussed above. The equations of state for the quark-gluon plasma are easily obtained by differentiating

$$\ln Z = \ln Z_q + \ln Z_{\mathrm{g}} + \ln Z_{\mathrm{vac}} \tag{27.52}$$

with respect to β, λ, and V. The baryon number density, energy, and pressure are respectively:

$$\nu = \frac{1}{V}\lambda\frac{\partial}{\partial\lambda}\ln Z = \frac{2T^3}{\pi^2}\left(1-\frac{2\alpha_s}{\pi}\right)\left(\frac{1}{3^4}\ln^3\lambda + \frac{\pi^2}{9}\ln\lambda\right)\,, \tag{27.53}$$

$$\begin{aligned}\varepsilon &= -\frac{1}{V}\frac{\partial}{\partial\beta}\ln Z\\ &= \frac{6}{\pi^2}T^4\left[\left(1-\frac{2\alpha_s}{\pi}\right)\left(\frac{1}{4\cdot 3^4}\ln^4\lambda + \frac{\pi^2}{2\cdot 3^2}\ln^2\lambda\right) + \left(1-\frac{50}{21}\frac{\alpha_s}{\pi}\right)\frac{7\pi^4}{60}\right]\\ &\quad + \frac{8\pi^2}{15}T^4\left(1-\frac{15}{4}\frac{\alpha_s}{\pi}\right) + B\,,\end{aligned} \tag{27.54}$$

$$\begin{aligned}P &= T\frac{\partial}{\partial V}\ln Z\\ &= \frac{2T^4}{\pi^2}\left[\left(1-\frac{2\alpha_s}{\pi}\right)\left(\frac{1}{4\cdot 3^4}\ln^4\lambda + \frac{\pi^2}{2\cdot 3^2}\ln^2\lambda\right) + \left(1-\frac{50}{21}\frac{\alpha_s}{\pi}\right)\frac{7\pi^4}{60}\right]\\ &\quad + \frac{8\pi^2}{45}T^4\left(1-\frac{15}{4}\frac{\alpha_s}{\pi}\right) - \mathscr{B}\,.\end{aligned} \tag{27.55}$$

Let us first note that, for $T \ll \mu$ and $P = 0$, the baryon chemical potential tends to

$$\mu_{\mathrm{B}} = 3\mu_q \longrightarrow 3\mathscr{B}^{1/4}\left[\frac{2\pi^2}{(1-2\alpha_s/\pi)}\right]^{1/4} = 1010\,\mathrm{MeV}\,,\ \alpha_s = 1/2\,,\ \mathscr{B}^{1/4} = 145\,\mathrm{MeV}\,, \tag{27.56}$$

which assures us that interacting cold quark matter is an excited state of nuclear matter. We have assumed that, except for T, there is no relevant dimensional parameter, e.g., quark mass m_q or the quantity Λ which enters into the running coupling constant $\alpha_s(q^2)$. Therefore the relativistic relation between the energy density and pressure, viz., $\varepsilon - \mathscr{B} = 3(P + \mathscr{B})$, is preserved, which leads to

$$P = \frac{1}{3}(\varepsilon - 4\mathscr{B})\,, \tag{27.57}$$

a relation we have used occasionally before [see Eq. (27.10)].

From Eq. (27.57), it follows that, when the pressure vanishes, the energy density is $4\mathscr{B}$, independent of the values of μ and T which fix the line $P = 0$. This behaviour is consistent with the hadronic gas phase. This may be used as a reason to choose the parameters of both phases in such a way that the two lines $P = 0$ coincide. We will return to this point again below. For $P > 0$, we have $\varepsilon > 4\mathscr{B}$. Recall that, in the hadronic gas, we had $0 < \varepsilon < 4\mathscr{B}$. Thus, above the critical curve of the μ, T plane, we have the quark-gluon plasma exposed to an external force.

In order to obtain an idea of the form of the $P = 0$ critical curve in the μ, T plane for the quark-gluon plasma, we rewrite Eq. (27.55) using Eqs. (27.48) and (27.49) for $P = 0$:

$$\mathscr{B} = \frac{1-2\alpha_s/\pi}{162\pi^2}\left[\mu^2+(3\pi T)^2\right]^2 + \frac{T^4\pi^2}{45}\left[12\left(1-\frac{5}{3}\frac{\alpha_s}{\pi}\right)+8\left(1-\frac{15}{4}\frac{\alpha_s}{\pi}\right)\right]. \tag{27.58}$$

Here, the last term is the glue pressure contribution. (If the true vacuum structure is determined by the glue–glue interaction, then this term could be modified significantly.) We find that the greatest lower bound on temperature T_q at $\mu = 0$ is about

$$T_q \sim \mathscr{B}^{1/4} \approx 145\text{–}190\ \text{MeV}. \tag{27.59}$$

This result can be considered to be correct to within 20%. Its order of magnitude is as expected. Taking Eq. (27.58) as it is, we find for $\alpha_s = 1/2$, $T_q = 0.88\mathscr{B}^{1/4}$. Omitting the gluon contribution to the pressure, we find $T_q = 0.9\mathscr{B}^{1/4}$. It is quite likely that, with the proper treatment of the glue field and the plasma corrections, and with larger $\mathscr{B}^{1/4} \sim 190$ MeV, the desired value of $T_q = T_0$ corresponding to the statistical bootstrap choice will follow. Furthermore, allowing some reasonable T,μ dependence of α_s, we can then easily obtain an agreement between the critical curves.

However, it is not necessary for the two critical curves to coincide, even though this would be preferable. As the quark plasma is the phase into which individual hadrons dissolve, it is sufficient if the quark plasma pressure vanishes within the boundary set for non-vanishing positive pressure of the hadronic gas. It is quite satisfactory for the theoretical development that this is the case. In Fig. 27.1a, a qualitative picture of the two $P = 0$ lines is shown in the μ, T plane. Along the dotted straight line at constant temperature, we show in Fig. 27.1b the pressure as a function of the volume (a P,V diagram). The volume is obtained by inverting the baryon density at constant fixed baryon number:

$$V = \frac{\langle b\rangle}{\nu}. \tag{27.60}$$

The behaviour of $P\,(V,T = \text{const.})$ for the hadronic gas phase is as described before in the statistical bootstrap model. For large volumes, we see that P falls with rising V. However, when hadrons get close to each other so that they form larger and larger lumps, the pressure drops rapidly to zero. The hadronic gas becomes a state of few composite clusters (internally already consisting of the quark plasma). The second branch of the $P\,(V,T = \text{const.})$ line meets the first one at a certain volume $V = V_\mathrm{m}$.

The phase transition occurs for $T = $ const. in Fig. 27.1b at a vapour pressure P_v obtained from the conventional Maxwell construction: the shaded regions in Fig. 27.1b are equal. Between the volumes V_1 and V_2, matter coexists in the two phases with the relative fractions being determined by the magnitude of the actual volume. This leads to the occurrence of a third region, viz., the coexistence region of matter, in addition to the pure quark and hadron domains. For $V < V_1$, corresponding to $\nu > \nu_1 \sim 1/V_1$, all matter has gone into the quark plasma phase.

The dotted line in Fig. 27.1b encloses (qualitatively) the domain in which the coexistence between the two phases of hadronic matter seems possible. We further note that, at low temperatures $T \leq 50$ MeV, the plasma and hadronic gas critical

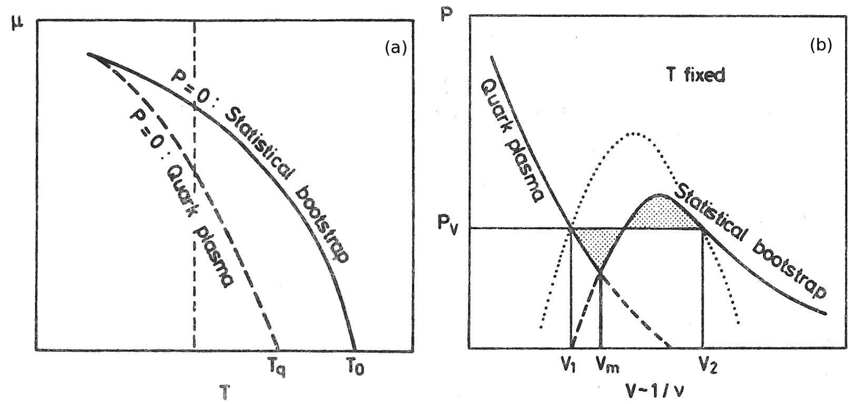


Fig. 27.1 (**a**) The critical curves ($P = 0$) of the two models in the T, μ plane (qualitatively). The region below the *full line* is described by the statistical bootstrap model and the region above the *broken line* by the quark-gluon plasma. The critical curves can be made to coincide. (**b**) P, V diagram (qualitative) of the phase transition (hadron gas to quark-gluon plasma) along the *broken line* $T =$ const. of (**a**). The coexistence region is found from the usual Maxwell construction (the *shaded areas* being equal).

curves meet each other in Fig. 27.1a. This is just the domain where, at present, our description of the hadronic gas fails, while the quark-gluon plasma also begins to suffer from infrared difficulties. Both approaches have a very limited validity in this domain.

The qualitative discussion presented above can be easily supplemented with quantitative results. But first we turn our attention to the modifications forced onto this simple picture by the experimental circumstances in high energy nuclear collisions.

27.5 Nuclear Collisions and Inclusive Particle Spectra

We assume that in relativistic collisions triggered to small impact parameters by high multiplicities and absence of projectile fragments [19], a hot central fireball of hadronic matter can be produced. We are aware of the whole problematic connected with such an idealization. A proper treatment should include collective motions and distribution of collective velocities, local temperatures, and so on [20], as explained in the preceding Chapter 26 by R. Hagedorn [10]. Triggering for high multiplicities hopefully eliminates some of the complications. In nearly symmetric collisions (projectile and target nuclei are similar), we can argue that the numbers of participants in the centre of mass of the fireball originating in the projectile or target are the same. Therefore, it is irrelevant how many nucleons do form the fireball – and the above symmetry argument leads, in a straightforward way, to a formula for the centre of mass energy per participating nucleon:

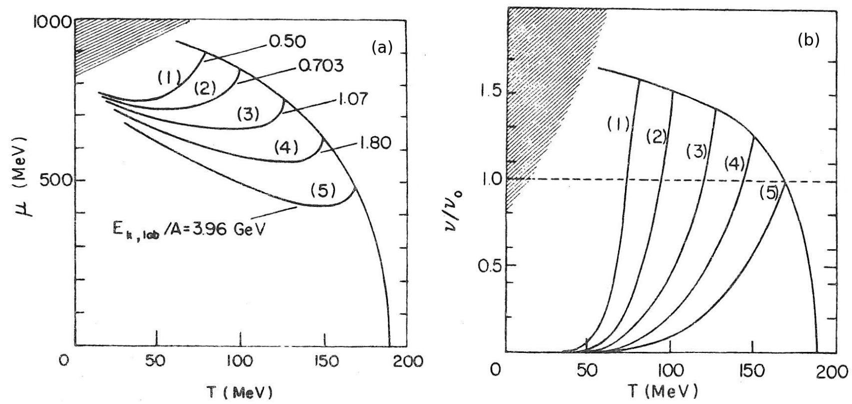


Fig. 27.2 (**a**) The critical curve of hadron matter (bootstrap), together with some 'cooling curves' in the T, μ plane. While the system cools down along these lines, it emits particles. When all particles have become free, it comes to rest on some point on these curves ('freeze out'). In the *shaded region*, our approach may be invalid. (**b**) The critical curve of hadron matter (bootstrap), together with some 'cooling curves' [same energy as in (**a**)] in the variables T and ν/ν_0 = ratio of baryon number density to normal nuclear baryon number density. In the *shaded region*, our approach may be invalid.

$$U := \frac{E_{\text{c.m.}}}{A} = m_{\text{N}} \sqrt{1 + \frac{E_{\text{k,lab}}/A}{2m_{\text{N}}}} \,, \tag{27.61}$$

where $E_{\text{k,lab}}/A$ is the projectile kinetic energy per nucleon in the laboratory frame. While the fireball changes its baryon density and chemical composition ($\pi + \text{p} \leftrightarrow \Delta$, etc.) during its lifetime through a change in temperature and chemical potential, the conservation of energy and baryon number assures us that U in Eq. (27.61) remains constant, assuming that the influence on U of pre-equilibrium emission of hadrons from the fireball is negligible. As U is the total energy per baryon available, we can, supposing that kinetic and chemical equilibrium have been reached, set it equal to the ratio of thermodynamic expectation values of the total energy and baryon number:

$$U = \frac{\langle E \rangle}{\langle b \rangle} = \frac{\text{E}(\beta, \lambda)}{\nu(\beta, \lambda)} \,. \tag{27.62}$$

Thus we see that, through Eq. (27.62), the experimental value of U in Eq. (27.61) fixes a relation between allowable values of β, λ: the available excitation energy defines the temperature and the chemical composition of hadronic fireballs. In Fig. 27.2a and b, these paths are shown for a choice of kinetic energies $E_{\text{k,lab}}/A$ in the μ, T plane and in the ν, T plane, respectively. In both cases, only the hadronic gas domain is shown. We wish to note several features of the curves shown in Fig. 27.2 that will be relevant in later considerations:

1. Beginning at the critical curve, the chemical potential first drops rapidly when T decreases and then rises slowly as T decreases further (Fig. 27.2a). This cor-

responds to a monotonically falling baryon density with decreasing temperature (Fig. 27.2b), but implies that, in the initial expansion phase of the fireball, the chemical composition changes more rapidly than the temperature.

2. The baryon density in Fig. 27.2b is of the order of 1–1.5 of normal nuclear density. This is a consequence of the choice of $\mathscr{B}^{1/4} = 145$ MeV. Were $\mathscr{B}$ three times as large, i.e., $\mathscr{B}^{1/4} = 190$ MeV, which is so far not excluded, then the baryon densities in this figure would triple to 3–5ν_0. Furthermore, we observe that, along the critical curve of the hadronic gas, the baryon density falls with rising temperature. This is easily understood as, at higher temperature, more volume is taken up by the numerous mesons.
3. Inspecting Fig. 27.2b, we see that, at given U, the temperatures at the critical curve and those at about $\nu_0/2$ differ little (10%) for low U, but more significantly for large U. Thus, highly excited fireballs cool down more before dissociation ('freeze out'). As particles are emitted all the time while the fireball cools down along the lines of Fig. 27.2, they carry kinetic energies related to various different temperatures. The inclusive single particle momentum distribution will yield only averages along these cooling lines.

Another remark which does not follow from the curves shown is:

4. Below about 1.8 GeV, an important portion of the total energy is in the collective (hydrodynamical) motion of hadronic matter, whence the cooling curves at constant excitation energy do not properly describe the evolution of the fireball.

Calculations of this kind can also be carried out for the quark plasma. They are, at present, uncertain due to the unknown values of α_s and $\mathscr{B}^{1/4}$. Fortunately, there is one particular property of the equation of state of the quark-gluon plasma that we can easily exploit. Combining Eq. (27.57) with Eq. (27.62), we obtain

$$P = \frac{1}{3}(U\nu - 4\mathscr{B}) \,. \tag{27.63}$$

Thus, for a given U (the available energy per baryon in a heavy ion collision), Eq. (27.63) describes the pressure–volume ($\sim 1/\nu$) relation. By choosing to measure P in units of $\mathscr{B}$ and ν in units of normal nuclear density $\nu_0 = 0.14/\mathrm{fm}^3$, we find

$$\frac{P}{\mathscr{B}} = \frac{4}{3}\left(\gamma\frac{U}{m_\mathrm{N}}\frac{\nu}{\nu_0} - 1\right) \,,$$

with

$$\gamma := \frac{m_\mathrm{N}\nu_0}{4\mathscr{B}} = 0.56 \,, \quad \mathscr{B}^{1/4} = 145\ \mathrm{MeV} \,, \quad \nu_0 = 0.14/\mathrm{fm}^3 \,. \tag{27.64}$$

Here, γ is the ratio of the energy density of normal nuclei ($\varepsilon_\mathrm{N} = m_\mathrm{N}\nu_0$) and of quark matter or of a quark bag ($\varepsilon_q = 4\mathscr{B}$). In Fig. 27.3a, this relation is shown for three projectile energies: $E_{\mathrm{k,lab}}/A = 1.80$ GeV, 3.965 GeV, and 5.914 GeV, corresponding to $U = 1.314$ GeV, 1.656 GeV, and 1.913 GeV, respectively. We observe that,

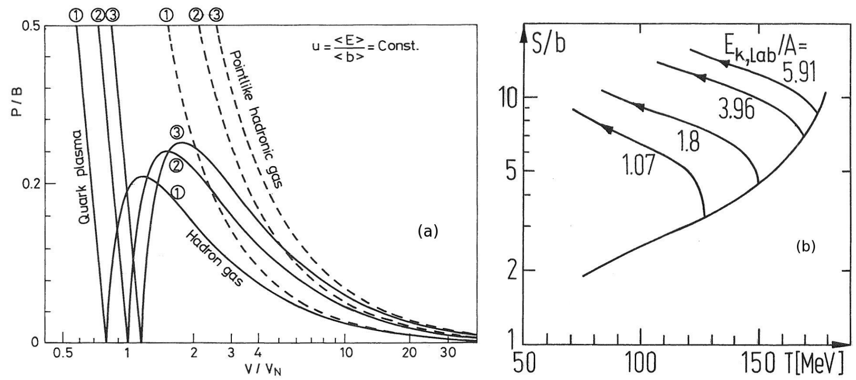


Fig. 27.3 (**a**) P,V diagram of 'cooling curves' belonging to different kinetic laboratory energies per nucleon: (1) 1.8 GeV, (2) 3.965 GeV, (3) 5.914 GeV. In the history of a collision, the system comes down the quark lines and jumps somewhere over to the hadron curves (Maxwell). *Broken lines* show the diverging pressure of pointlike bootstrap hadrons. (**b**) The total specific entropy per baryon in the hadronic gas phase. Same energies per nucleon as in (**a**) and a fourth value 1.07 GeV

even at the lowest energy shown, the quark pressure is zero near the baryon density corresponding to 1.3 normal nuclear density, given the current value of $\mathscr{B}$.

Before discussing this point further, we note that the hadronic gas branches of the curves in Fig. 27.3 show a quite similar behaviour to that shown at constant temperature in Fig. 27.1b. Remarkably enough, the two branches meet each other at $P = 0$, since both have the same energy density $\varepsilon = 4\mathscr{B}$ and therefore $V(P = 0) \sim 1/\nu = U/\varepsilon = U/4\mathscr{B}$. However, what we cannot see by inspecting Fig. 27.3 is that there will be a discontinuity in the variables μ and T at this point, except if parameters are chosen so that the critical curves of the two phases coincide. Indeed, near to $P = 0$, the results shown in Fig. 27.3a should be replaced by points obtained from the Maxwell construction. The pressure in a nuclear collision will never fall to zero. It will correspond to the momentary vapour pressure of the order of $0.2\mathscr{B}$ as the phase change occurs.

A further aspect of the equations of state for the hadronic gas is also illustrated in Fig. 27.3a. Had we ignored the finite size of hadrons (one of the van der Waals effects) in the hadron gas phase then, as shown by the dash-dotted lines, the phase change could never occur because the point particle pressure would diverge where the quark pressure vanishes. In our opinion, one cannot say it often enough: inclusion of the finite hadronic size and of the finite temperature when considering the phase transition to quark plasma lowers the relevant baryon density (from 8–$14\nu_0$ for cold point-nucleon matter) to 1–$5\nu_0$ (depending on the choice of $\mathscr{B}$) in 2–5 GeV/A nuclear collisions [21].

The physical picture underlying our discussion is an explosion of the fireball into vacuum with little energy being converted into collective motion, e.g., hydrodynamical flow, or being taken away by fast pre-hadronization particle emission. Thus the conserved internal excitation energy can only be shifted between thermal (kinetic) and chemical excitations of matter. 'Cooling' thus really means that, during the explosion, the thermal energy is mostly convered into chemical energy, e.g., *pions are produced*.

While it is at present hard to judge the precise amount of expected deviation from the cooling curves shown in Fig. 25.3, it is possible to show that they are entirely inconsistent with the notion of reversible adiabatic, i.e., entropy conserving, expansion. As the expansion proceeds along $U =$ const. lines, we can compute the entropy per participating baryon using Eqs. (27.38) and (27.39), and we find a significant growth of total entropy. As shown in Fig. 27.3b, the entropy rises initially in the dense phase of the matter by as much as 50–100% due to the pion production and resonance decay. Amusingly enough, as the newly produced entropy is carried mostly by pions, one will find that the entropy carried by protons remains constant. With this remarkable behaviour of the entropy, we are in a certain sense, victims of our elaborate theory. Had we used, e.g., an ideal gas of Fermi nucleons, then the expansion would seem to be entropy conserving, as pion production and other chemistry were forgotten. Our fireballs have no tendency to expand reversibly and adiabatically, as many reaction channels are open. A more complete discussion of the entropy puzzle can be found in [1].

Inspecting Fig. 27.2 again, it seems that a possible test of the equations of state for the hadronic gas consists in measuring the temperature in the hot fireball zone, and doing this as a function of the nuclear collision energy. The plausible assumption made is that the fireball follows the 'cooling' lines shown in Fig. 27.2 until final dissociation into hadrons. This presupposes that the surface emission of hadrons during the expansion of the fireball does not significantly alter the available energy per baryon. This is more likely true for sufficiently large fireballs. For small ones, pion emission by the surface may influence the energy balance. As the fireball expands, the temperature falls and the chemical composition changes. The hadronic clusters dissociate and more and more hadrons are to be found in the 'elementary' form of a nucleon or a pion. Their kinetic energies are reminiscent of the temperature found at each phase of the expansion.

To compute the experimentally observable final temperature [1, 13], we shall argue that a time average must be performed along the cooling curves. Not knowing the reaction mechanisms too well, we assume that the temperature decreases approximately linearly with the time in the significant expansion phase. We further have to allow that a fraction of particles emitted can be reabsorbed in the hadronic cluster. This is a geometric problem and, in a first approximation, the ratio of the available volume $\varDelta$ to the external volume V_{ex} is the probability that an emitted particle not be reabsorbed, i.e., that it can escape:

$$R_{\mathrm{esc}} = \frac{\varDelta}{V_{\mathrm{ex}}} = 1 - \frac{\varepsilon(\beta,\lambda)}{4\mathscr{B}} \, . \tag{27.65}$$

The relative emission rate is just the integrated momentum spectrum

$$R_{\mathrm{emis}} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathrm{e}^{-\sqrt{p^2+m^2}/T+\mu/T} = \frac{m^2 T}{2\pi^2} K_2(m/T) \mathrm{e}^{\mu/T} \, . \tag{27.66}$$

The chemical potential acts only for nucleons. In the case of pions, it has to be dropped from the above expression. For the mean temperature, we thus find

$$\langle T \rangle = \frac{\int_{\mathrm{c}} R_{\mathrm{esc}} R_{\mathrm{emis}} T \,\mathrm{d}T}{\int_{\mathrm{c}} R_{\mathrm{esc}} R_{\mathrm{emis}} \,\mathrm{d}T} \,, \tag{27.67}$$

where the subscript c on the integral indicates here a line integral along that particular cooling curve in Fig. 27.2 which belongs to the energy per baryon fixed by the experimentalist.

In practice, the temperature is most reliably measured through the measurement of mean transverse momenta of the particles. It may be more practical therefore to calculate the average transverse momentum of the emitted particles. In principle, to obtain this result we have to perform a similar averaging to the one above. For the average transverse momentum at given T, μ, we find [8]

$$\langle p_\perp(m,T,\mu)\rangle_p = \frac{\int p_\perp \mathrm{e}^{-\sqrt{p^2+m^2}-\mu)/T} \mathrm{d}^3 p}{\int \mathrm{e}^{-\sqrt{p^2+m^2}-\mu)/T} \mathrm{d}^3 p} = \frac{\sqrt{\pi m T/2}\, K_{5/2}(m/T)\mathrm{e}^{\mu/T}}{K_2(m/T)\mathrm{e}^{\mu/T}} \,. \tag{27.68}$$

The average over the cooling curve is then

$$\Big\langle \langle p_\perp(m,T,\mu)\rangle_p \Big\rangle_{\mathrm{c}} = \frac{\int_{\mathrm{c}} \frac{\Delta}{V_{\mathrm{ex}}} T^{3/2} \sqrt{\pi m/2}\, K_{5/2}(m/T)\mathrm{e}^{\mu/T}\, dT}{\int_{\mathrm{c}} \frac{\Delta}{V_{\mathrm{ex}}} T K_2(m/T)\mathrm{e}^{\mu/T}\, dT} \,. \tag{27.69}$$

We did verify numerically that the order of averages does not matter:

$$\Big\langle p_\perp(m, \langle T\rangle_{\mathrm{c}}, \mu)\Big\rangle_p \approx \Big\langle \langle p_\perp(m,T,\mu)\rangle_p \Big\rangle_{\mathrm{c}} \,, \tag{27.70}$$

which shows that the mean transverse momentum is also the simplest (and safest) method of determining the average temperature (indeed better than fitting ad hoc exponential type functions to $p_\perp$ distributions).

In the presented calculations, we chose the bag constant $\mathscr{B} = (145\ \mathrm{MeV})^4$, but we now believe that a larger $\mathscr{B}$ should be used. As a consequence of our choice and the measured pion temperature of $\langle T\rangle_\pi^{\mathrm{ex}} = 140$ MeV at highest ISR energies, we have to choose the constant H such that $T_0 = 190$ MeV [see Eq. (27.46)].

The average temperature, as a function of the range of integration over T, reaches different limiting values for different particles. The limiting value obtained thus is the observable 'average temperature' of the debris of the interaction, while the initial temperature T_{cr} at given $E_{\mathrm{k,lab}}$ (full line in Fig. 27.4) is difficult to observe. When integrating along the cooling line as in Eq. (27.67), we can easily, at each point, determine the average hadronic cluster mass. The integration for protons is interrupted (protons are 'frozen out') when the average cluster mass is about half the

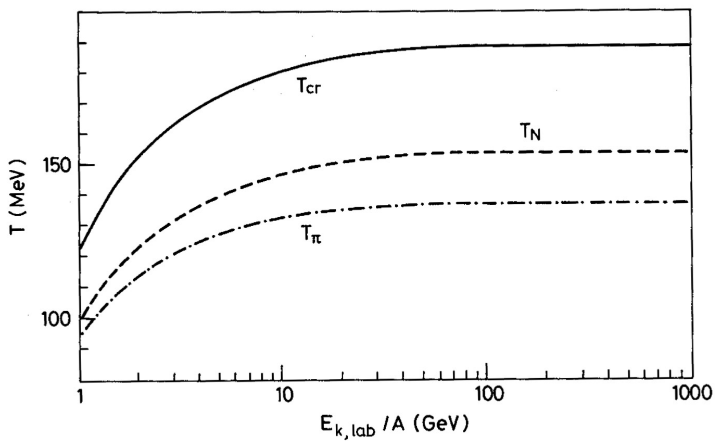


Fig. 27.4 Mean temperatures for nucleons and pions together with the critical temperature belonging to the point where the 'cooling curves' start off the critical curve (see Fig. 27.2a). The mean temperatures are obtained by integrating along the cooling curves. Note that T_N is always greater than T_π

nucleon isobar mass. We have also considered baryon density dependent freeze-out, but such a procedure depends strongly on the unreliable value of $\mathscr{B}$.

Our choice of the freeze-out condition was made in such a way that the nucleon temperature at $E_{k,lab}/A = 1.8$ GeV is about 120 MeV. The model dependence of our freeze-out introduces an uncertainty of several MeV in the average temperature. In Fig. 27.4, the pion and nucleon average temperatures are shown as a function of the heavy ion kinetic energy. Two effects contributed to the difference between the π and N temperatures:

1. The particular shape of the cooling curves (Fig. 27.2a). The chemical potential drops rapidly from the critical curve, thereby damping relative baryon emission at lower T. Pions, which do not feel the baryon chemical potential, continue being created also at lower temperatures.
2. The freeze-out of baryons occurs earlier than the freeze-out of pions.

A third effect has been so far omitted – the emission of pions from two-body decay of long-lived resonances [1] would lead to an effective temperature which is lower in nuclear collisions.

In Fig. 27.5, we show the dependence of the average transverse momenta of pions and nucleons on the kinetic energy of the heavy ion projectiles.

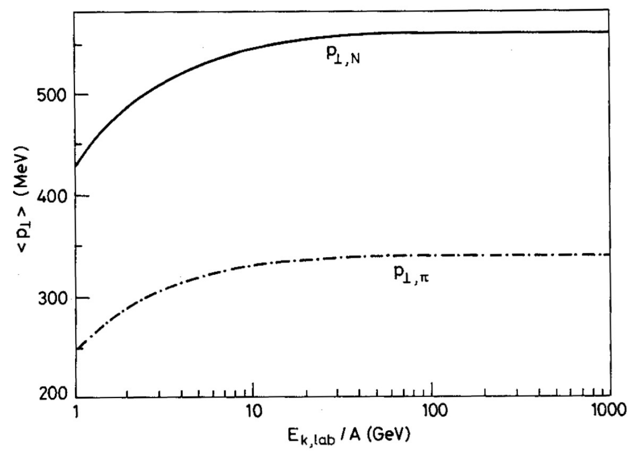


Fig. 27.5 Mean transverse momenta of nucleons and pions found by integrating along the 'cooling curves'.

27.6 Strangeness in Heavy Ion Collisions

From the averaging process described here, we have learned that the temperatures and transverse momenta of particles originating in the hot fireballs are more reminiscent of the entire history of the fireball expansion than of the initial hot compressed state, perhaps present in the form of quark matter. We may generalize this result and then claim that most properties of inclusive spectra are reminiscent of the equations of state of the hadronic gas phase and that the memory of the initial dense state is lost during the expansion of the fireballs as the hadronic gas rescatters many times while it evolves into the final kinetic and chemical equilibrium state.

In order to observe properties of quark-gluon plasma, we must design a thermometer, an isolated degree of freedom weakly coupled to the hadronic matter. Nature has, in principle (but not in practice) provided several such thermometers: leptons and heavy flavours of quarks. We would like to point here to a particular phenomenon perhaps quite uniquely characteristic of quark matter. First we note that, at a given temperature, the quark-gluon plasma will contain an equal number of strange (s) quarks and antistrange ($\bar{s}$) quarks, naturally assuming that the hadronic collision time is much too short to allow for light flavour weak interaction conversion to strangeness. Thus, assuming equilibrium in the quark plasma, we find the density of the strange quarks to be (two spins and three colors)

$$\frac{s}{V} = \frac{\bar{s}}{V} = 6\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathrm{e}^{-\sqrt{p^2+m_s^2}/T} = 3\frac{Tm_s^2}{\pi^2} K_2(m_s/T)\,, \tag{27.71}$$

neglecting for the time being the perturbative corrections and, of course, ignoring weak decays. As the mass m_s of the strange quarks in the perturbative vacuum is believed to be of the order of 280–300 MeV, the assumption of equilibrium for

$m_s/T \sim 2$ may indeed be correct. In Eq. (27.71), we were able to use the Boltzmann distribution again, as the density of strangeness is relatively low. Similarly, there is a certain light antiquark density ($\overline{q}$ stands for either $\overline{u}$ or $\overline{d}$):

$$\frac{\overline{q}}{V} = 6 \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathrm{e}^{-|p|/T - \mu_q/T} = \mathrm{e}^{-\mu_q/T} T^3 \frac{6}{\pi^2} \,, \tag{27.72}$$

where the quark chemical potential is $\mu_q = \mu/3$, as given by Eq. (27.49). This exponent suppresses the $\mathrm{q}\overline{q}$ pair production.

What we intend to show is that there are many more $\overline{s}$ quarks than antiquarks of each light flavour. Indeed,

$$\frac{\overline{s}}{\overline{q}} = \frac{1}{2} \left(\frac{m_s}{T} \right)^2 K_2 \left(\frac{m_s}{T} \right) \mathrm{e}^{\mu/3T} \,. \tag{27.73}$$

The function $x^2 K_2(x)$ is, for example, tabulated in [22]. For $x = m_s/T$ between 1.5 and 2, it varies between 1.3 and 1. Thus, we almost always have more $\overline{s}$ than $\overline{q}$ quarks and, in many cases of interest, $\overline{s}/\overline{q} \sim 5$. As $\mu \to 0$, there are about as many $\overline{\mathrm{u}}$ and $\overline{q}$ quarks as there are $\overline{s}$ quarks.

When the quark matter dissociates into hadrons, some of the numerous $\overline{s}$ may, instead of being bound in a $\mathrm{q}\overline{s}$ kaon, enter into a $\overline{q}\,\overline{q}\,\overline{s}$ antibaryon and, in particular[2], a $\overline{\Lambda}$ or $\overline{\Sigma}^0$. The probability for this process seems to be comparable to the similar one for the production of antinucleons by the antiquarks present in the plasma. What is particularly noteworthy about the $\overline{s}$-carrying antibaryons is that they can conventionally only be produced in direct pair production reactions. Up to about $E_{\mathrm{k,lab}}/A = 3.5$ GeV, this process is very strongly suppressed by energy–momentum conservation because, for free pp collisions, the threshold is at about 7 GeV. We would thus like to argue that a study of the $\overline{\Lambda}$ and $\overline{\Sigma}^0$ in nuclear collisions for $2 < E_{\mathrm{k,lab}}/A < 4$ GeV could shed light on the early stages of the nuclear collisions in which quark matter may be formed.

Let us mention here another effect of importance in this context: the production rate of a pair of particles with a conserved quantum number like strangeness will usually be suppressed by the Boltzmann factor $\mathrm{e}^{-2m/T}$, rather than a factor $\mathrm{e}^{-m/T}$ as is the case in thermomechanical equilibrium (see, for example, the addendum in [8]). As relativistic nuclear collisions are just on the borderline between those two limiting cases, it is important when considering the yield of strange particles to understand the transition between them. We will now show how one can describe these different cases in a unified statistical description [23].

As we have already implicitly discussed [see Eq. (27.13)], the logarithm of the grand partition function Z is a sum over all different particle configurations, e.g., expressed with the help of the mass spectrum. Hence, we can now concentrate in particular on that part of $\ln Z$ which is exclusively associated with the strangeness.

As the temperatures of interest to us and which allow appreciable strangeness production are at the same time high enough to prevent the strange particles from

[2] $\overline{\Sigma}^0$ decays into $\overline{\Lambda}$ by emitting a photon and is always counted as part of a $\overline{\Lambda}$ abundance.

being thermodynamically degenerate, we can restrict ourselves again to the discussion of Boltzmann statistics only.

The contribution to Z of a state with k strange particles is

$$Z_k = \frac{1}{k!}\left[\sum_s Z_I^s(T,V)\right]^k , \tag{27.74}$$

where the one-particle function Z_1 for a particle of mass m_s is given in Eq. (27.17). To include both particles *and* antiparticles as two thermodynamically independent phases in Eq. (27.74), the sum over s in Eq. (27.74) must include them both. As the quantum numbers of particles (p) and antiparticles (a) must always be present with *exactly* the same total number, not each term in Eq. (27.74) can contribute. Only when $n = k/2 =$ number of particles = number of antiparticles is exactly fulfilled do we have a physical state. Hence,

$$Z_{2n}^{\text{pair}} = \frac{1}{(2n)!}\binom{2n}{n}\left(\sum_{s_\mathrm{p}} Z_1^{s_\mathrm{p}}\right)^n\left(\sum_{s_\mathrm{a}} Z_1^{s_\mathrm{a}}\right)^n . \tag{27.75}$$

We now introduce the fugacity factor f^n to be able to count the number of strange pairs present. Allowing an arbitrary number of pairs to be produced, we obtain

$$Z_s(\beta,V;f) = \sum_{n=0}^{\infty}\frac{f^n}{n!n!}\left(\sum_{s_\mathrm{p}} Z_1^{s_\mathrm{p}}\right)^n\left(\sum_{s_\mathrm{a}} Z_1^{s_\mathrm{a}}\right)^n = I_0(\sqrt{4y}) , \tag{27.76}$$

where I_0 is the modified Bessel function and

$$y = f\left(\sum_{s_\mathrm{p}} Z_1^{s_\mathrm{p}}\right)\left(\sum_{s_\mathrm{a}} Z_1^{s_\mathrm{a}}\right) . \tag{27.77}$$

We have to maintain the difference between the particles (p) and antiparticles (a), as in nuclear collisions the symmetry is broken by the presence of baryons and there is an associated need for a baryon fugacity (chemical potential μ) that controls the baryon number. We obtain

$$Z_1^{\mathrm{p,a}} := \sum_{s_{\mathrm{p,a}}} Z_1^{s_{\mathrm{p,a}}} = \frac{VT^3}{2\pi^2}\left\{2W(x_\mathrm{K}) + 2\mathrm{e}^{\pm\mu/T}\left[W(x_\Lambda) + 3W(x_\Sigma)\right]\right\} , \tag{27.78}$$

for particles $(+\mu)$ and antiparticles $(-\mu)$, where $W(x) = x^2K_2(x)$, $x_i = m_i/T$, and all kaons and hyperons are counted. In the quark phase, we have

$$Z_{1,q}^{\mathrm{p,a}} = \frac{VT^3}{2\pi^2}\left[6\,\mathrm{e}^{\pm\mu/3T}W(x_s)\right] , \tag{27.79}$$

with $Tx_s = m_s \sim 280$ MeV. We note in passing that the baryon chemical potential cancels out in y of Eq. (27.77) when Eq. (27.79) is inserted in the quark phase [compare with Eq. (27.71)].

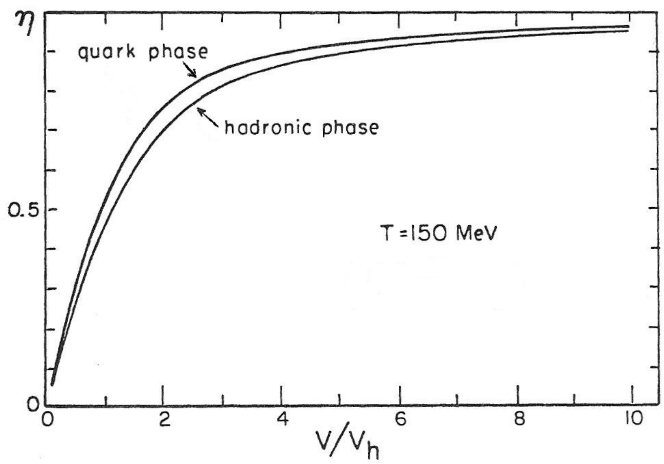


Fig. 27.6 The quenching factor for strangeness production as a function of the active volume V/V_{h}, where $V_{\mathrm{h}} = 4\pi/3\ \mathrm{fm}^3$, the hadron curve was obtained for baryochemical potential $\mu = 550\,\mathrm{MeV}$.

By differentiating $\ln Z_s$ of Eq. (27.76) with respect to f, we find the strangeness number present at given T and V :

$$\langle n\rangle_s = f\frac{\partial}{\partial f}\ln Z_s\Big|_{f=1} = \frac{I_1(\sqrt{4y})}{I_0(\sqrt{4y})}\sqrt{y}\,. \tag{27.80}$$

For large y, that is, at given T for large volume V, we find $\langle n\rangle_s = \sqrt{y} \sim \mathrm{e}^{-m/T}$, as expected. For small y, we find $\langle n\rangle_s = y \sim \mathrm{e}^{-2m/T}$. In Fig. 27.6, we show the dependence of the quenching factor $I_1/I_0 = \eta$ as a function of the volume V measured in units of $V_{\mathrm{h}} = 4\pi/3\ \mathrm{fm}^3$ for a typical set of parameters: $T = 150$, $\mu = 550$ MeV (hadronic gas phase).

The following observations follow from inspection of Fig. 27.6:

1. The strangeness yield is a qualitative measure of the hadronic volume in thermodynamic equilibrium.
2. Total strangeness yield is not an indicator of the phase transition to quark plasma, as the enhancement ($\sqrt{\eta_q/\eta} = 1.25$) in yield can be reinterpreted as being due to a change in hadronic volume.
3. We can expect that, in nuclear collisions, the active volume will be sufficiently large to allow the strangeness yield to correspond to that of 'infinite' volume for reactions triggered on 'central collisions'. Hence, e.g., Λ production rate will significantly exceed that found in pp collisions.

Our conclusions about the significance of $\overline{\Lambda}$ as an indicator of the phase transition to quark plasma remain valid as the production of $\overline{\Lambda}$ in the hadronic gas phase will only be possible in the very first stages of the nuclear collisions, if sufficient centre of mass energy is available.

27.7 Summary

Our aim has been to obtain a description of hadronic matter valid for high internal excitations. By postulating the kinetic and chemical equilibrium, we have been able to develop a thermodynamic description valid for high temperatures and different chemical compositions. In our work we have found two physically different domains: firstly, the hadronic gas phase, in which individual hadrons can exist as separate entities, but are sometimes combined into larger hadronic clusters, while in the second domain, individual hadrons dissolve into one large cluster consisting of hadronic constituents, viz., the quark-gluon plasma.

In order to obtain a theoretical description of both phases, we have used some 'common' knowledge and plausible interpretations of currently available experimental observations. In particular, in the case of hadronic gas, we have completely abandoned a more conventional Lagrangian approach in favour of a semi-phenomenological statistical bootstrap model of hadronic matter that incorporates those properties of hadronic interaction that are, in our opinion, most important in nuclear collisions.

In particular, the attractive interactions are included through the rich, exponentially growing hadronic mass spectrum $\tau(m^2, b)$, while the introduction of the finite volume of each hadron is responsible for an effective short-range repulsion. Aside from these manifestations of strong interactions, we only satisfy the usual conservation laws of energy, momentum, and baryon number. We neglect quantum statistics since quantitative study has revealed that this is allowed above $T \approx 50$ MeV. But we allow particle production, which introduces a quantum physical aspect into the otherwise 'classical' theory of Boltzmann particles.

Our approach leads us to the equations of state of hadronic matter which reflect what we have included in our considerations. It is the *quantitative* nature of our work that allows a detailed comparison with experiment. This work has just begun and it is too early to say if the features of strong interactions that we have chosen to include in our considerations are the most relevant ones. It is important to observe that the currently predicted pion and nucleon mean transverse momenta and temperatures show the required substantial rise (see Fig. 27.5) as required by the experimental results available at $E_{\mathrm{k,lab}}/A = 2$ GeV (BEVALAC, see [19]) and at 1000 GeV (ISR, see [18]). Further comparisons involving, in particular, particle multiplicities and strangeness production are under consideration.

We also mention the internal theoretical consistency of our two-fold approach. With the proper interpretation, the statistical bootstrap leads us, in a straightforward fashion, to the postulate of a phase transition to the quark-gluon plasma. This second phase is treated by a quite different method. In addition to the standard Lagrangian quantum field theory of weakly interacting particles at finite temperature and density, we also introduce the phenomenological vacuum pressure and energy density $\mathscr{B}$.

Perhaps the most interesting aspect of our work is the realization that the transition to quark matter will occur at much lower baryon density for highly excited hadronic matter than for matter in the ground state ($T = 0$). The precise baryon density of the phase transition depends somewhat on the bag constant, but we estimate it to be at about 2–4ν_0 at $T = 150$ MeV. The detailed study of the different aspects of this phase transition, as well as of possible characteristic signatures of quark matter, must still be carried out. We have given here only a very preliminary report on the status of our present understanding.

We believe that the occurrence of the quark plasma phase is observable and we have proposed therefore a measurement of the $\bar{\Lambda}/\overline{p}$ relative yield between 2 and 10 GeV/N kinetic energies. In the quark plasma phase, we expect a significant enhancement of $\bar{\Lambda}$ production which will most likely be visible in the $\bar{\Lambda}/\overline{p}$ relative rate.

Acknowledgements

Many fruitful discussions with the GSI/LBL Relativistic Heavy Ion group stimulated the ideas presented here. I would like to thank R. Bock and R. Stock for their hospitality at GSI during this workshop. As emphasized before, this work was performed in collaboration with R. Hagedorn.

References

1. R. Hagedorn, J. Rafelski: Manuscript in preparation for Physics Reports – This manuscript was never completed, the present volume comprises most if not all material that would have been included.; see also: preprints CERN-TH-2947 and CERN-TH-2969 "From hadron gas to quark matter I & II", in the *Proceedings of the International Symposium on Statistical Mechanics of Quarks and Hadrons*, Bielefeld, Germany, August 1980, ed. by H. Satz, (North Holland Publishing Company)
2. The many-body theory for QCD at finite temperatures has been discussed in:
   B.A. Freedman, L.D. McLerran: Phys. Rev. D **16**, 1169 (1977);
   P.D. Morley, M.B. Kislinger: Phys. Rep. **51**, 63 (1979);
   J.I. Kapusta: Nucl. Phys. B **148**, 461 (1979);
   E.V. Shuryak: Phys. Lett. B **81**, 65 (1979); ibid. Phys. Rep. **61**, 71 (1980);
   O.K. Kalashnikov, V.V. Klimov: Phys. Lett. B **88**, 328 (1979)
3. B. Touschek: Nuovo Cimento B **58**, 295 (1968)
4. W. Marciano, H. Pagels: Phys. Rep. **36**, 137 (1978)
5. K. Johnson: The MIT bag model. Acta Phys. Polon. B **6**, 865 (1975)
6. A. Chodos, R.L. Jaffe, K. Johnson, C.B. Thorn, V.F. Weisskopf: Phys. Rev. D **9**, 3471 (1974)
7. R. Hagedorn: Suppl. Nuovo Cimento **3**, 147 (1965)
8. R. Hagedorn: In *Lectures on the Thermodynamics of Strong Interactions*, CERN Yellow Report 71-12 (1971)
9. R. Hagedorn, I. Montvay, J. Rafelski: Lecture at Erice Workshop, *Hadronic Matter at Extreme Energy Density*, ed. by N. Cabibbo and L. Sertorio, Plenum Press (New York 1980), p. 49; and see Chapter 23 in this volume.

10. R. Hagedorn: How to Deal with Relativistic Heavy Ion Collisions, preceding Chapter 26 in this volume.
11. H. Grote, R. Hagedorn, J. Ranft: *Atlas of Particle Production Spectra*, CERN 1970
12. J. Rafelski, H.Th. Elze, R. Hagedorn: CERN preprint TH 2912 (August 1980) in *Proceedings of the Fifth European Symposium on Nucleon–Antinucleon Interactions*, Bressanone, Italy, June 1980, ed. by M. Cresti, CLEUP publishers, Padova, p. 357
13. R. Hagedorn, J. Rafelski: Phys. Lett. B **97**, 136 (1980)
14. F. Karsch, H. Satz: Phys. Lett. **21**, 1168 (1980). The 'solidification' of hadronic matter, a purely geometric effect, discussed in this reference should not be confused with the transition to the quark plasma phase discussed here
15. R. Hagedorn and J. Rafelski: Commun. Math. Phys. **83**, 563 (1982).
16. S. Frautschi: Phys. Rev. D **3**, 2821 (1971)
17. R. Hagedorn, I. Montvay: Nucl. Phys. B **59**, 45 (1973)
18. G. Giacomelli, M. Jacob: Phys. Rep. **55**, no. 1, 1 (1979)
19. A. Sandoval, R. Stock, H.E. Stelzer, R.E. Renfordt, J.W. Harris, J. Brannigan, J.B. Geaga, L.J. Rosenberg, L.S. Schroeder, K.L. Wolf:Phys. Rev. Lett. **45**, 874 (1980)
20. R. Hagedorn, J. Ranft: Suppl. Nuovo Cimento **6**, 169 (1968);
    W.D. Myers: Nucl. Phys. A **296**, 177 (1978);
    J. Gosset, J.L. Kapusta, G.D. Westfall: Phys. Rev. C **18**, 844 (1978);
    R. Malfliet: Phys. Rev. Lett. **44**, 864 (1980)
21. The possible formation of quark-gluon plasma in nuclear collisions was first discussed quantitatively by
    S.A. Chin: Phys. Lett. B **78**, 552 (1978);
    see also N. Cabibbo, G. Parisi: Phys. Lett. B **59**, 67 (1975)
22. M. Abramowitz, I.A. Stegun (Eds.): *Handbook of Mathematical Functions*, NBS (1964).
23. J. Rafelski, M. Danos: Phys. Lett. B **97**, 279 (1980)

Chapter 28
Hot Quark Plasma in ISR Nuclear Collisions – January 1981

Johann Rafelski

Abstract In 1980/81 the ISR community of Physicists at CERN was preparing for a heavy ion experimental program. My lecture was moved-up from a later *AA*-meeting after another speaker bowed-out from the α-meeting. Before describing my presentation, I provide a few circumstantial details of potential interest.

An Invitation to ISR-discussion meeting at CERN read:

Discussion Meeting
$\alpha\alpha$ and αp Interactions

ISR Amphitheatre
Thursday, 22 January 1981
14:00 hours

The purpose of this meeting is to review and discuss present information about $\alpha\alpha$ and αp interactions following the analysis of the data collected during the runs of July 1980. Whilst this meeting will focus on low $p_\perp$ physics another meeting, scheduled for 19 February, will discuss large $p_\perp$ results.

Introductory talks will be given by[1]:

D. Lloyd-Owen (R210) on elastic scattering
T.J.M. Symons (R418) on elastic scattering
S. Frankel (R807) on inelastic interactions
R. Szwed (R418) on inelastic interactions at low $p_\perp$
and
J. Rafelski (Frankfurt) who will review theoretical models[2]

Department of Physics, The University of Arizona, Tucson, AZ 85721, USA

[1] The numeral in parentheses indicates the ISR experiment reference.
[2] I was invited as replacement for L. Bertocchi (CTP Trieste).

This announcement is sent to contact persons only. Please post or circulate it. For questions or comments, please contact M. Albrow (5924) or M. Jacob (2414).

Each introductory talk is scheduled to last about 30 minutes with ample time for discussion. The meeting is expected to be over by 18:00 and will include a coffee break.

Shortly after my lecture, I found in my CERN mailbox a note from Maurice Jacob : *Thank you for your beautiful talk. I think the meeting was quite lively and it was good to give the field momentum.*

I do hope that you can leave me something for the proceedings. At least your μ/T *figure with an extensive caption and an explanation of the LBL/ISR behaviors is almost a must. Can you leave me at least that before you depart.*

I left a handwritten response before departing in early morning: *This is for the ISR meeting on 22 January, 1981; consult R. Hagedorn (2138) for unreadable words and insertion of formulas.* I never saw the ISR report, the following transcript is from my own correspondance records.

Write-up for the ISR-report:

Hot Quark Plasma in ISR Nuclear Collisions

As nucleons consist of three quarks trapped in their perturbative vacuum domain, there is a non-vanishing probability that in high energy heavy nuclear collisions sufficient temperatures and compressions will be reached to form a quark gluon plasma. The experiments currently in progress at LBL, Dubna and ISR may be capable of producing this new form of matter.

The thermodynamic properties of a hadronic fireball created in such collisions are best characterized by the following three parameters: Volume V, Temperature T and the baryon chemical potential μ that controls the baryon density in the fireball. In the Fig. 28.1 a summary of the current qualitative knowledge about hadronic matter is described. Further details can be found in Refs. [1, 2].

For relatively small temperatures, i.e. $50 < T < T_0$, hadronic matter will consist of individual hadrons, mesons for small μ and also nucleons brought into the reaction for $\mu \sim 500\,\mathrm{MeV}$. This part of the phase diagram is shown dashed in Fig. 28.1. For $\mu \to 1\,\mathrm{GeV}$ and $T \to 0$ we enter the dark-shaded domain of normal nuclear matter where effects other than those of interest here are relevant.

The phase transition from the hadronic gas to the quark-gluon plasma occurs when the number of hadrons at a given temperature and chemical potential is so large that their energy density corresponds to $4\mathscr{B}$, the value known from the quark bag models. $\mathscr{B}$ is the energy density of the perturbative vacuum as compared with the "true" vacuum state of QCD. At the same time $P_{\mathrm{vac}} = -\mathscr{B}$ is the pressure exercised by the true vacuum on the surface of the perturbative vacuum, balanced by the pressure of the quark-gluon plasma at the phase transition line where the total pressure of hadronic matter in comparison is small.

When the quark-gluon plasma is produced in nuclear collisions at some characteristic temperature T and chemical potential μ, it will expand against the vacuum pressure. The conservation laws of total energy and baryon number introduce two

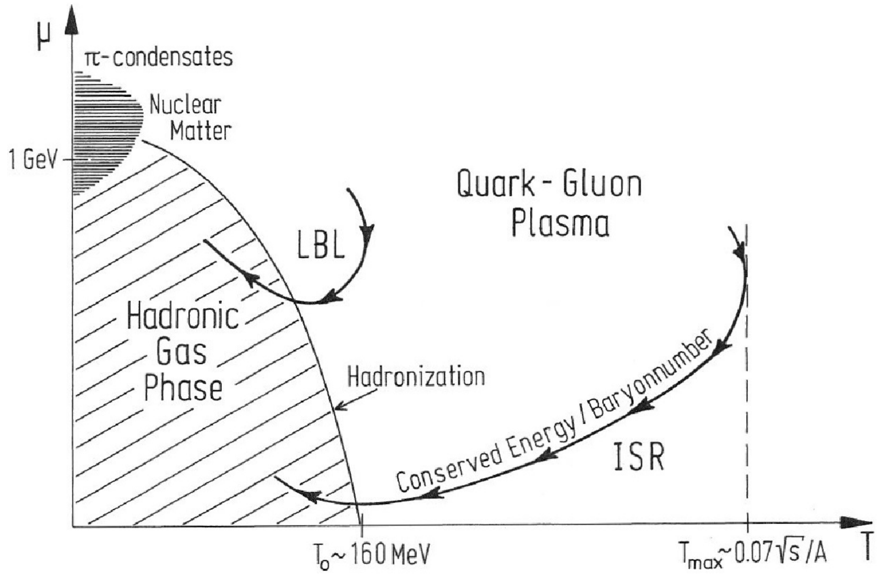


Fig. 28.1 See text; one non-explained item – a QGP fireball that equilibrates faster than it cools and expands at a prescribed energy and baryon content has $T_{\max}$ as shown on abscissa for $\alpha_s = 0.6$.

constraints between V, μ and T of the fireballs as a function of time. Assuming instantaneous thermal equilibrium, the fireballs can evolve only along the paths shown in the μ-T diagram. During this expansion, the entropy grows substantially. We note that in particular at ISR energies only the emission of particles from the fireballs that may lead to the high $p_\perp$ effects influence negligibly the energy and baryon number balance. The same is true for the energy of radial expansion mode.

The understanding of the quark-gluon plasma is not complete at present, but important qualitative insights can be gained by considering the effects of a Fermi-Bose gas with interaction of order α_s. Then at given collision energy at ISR, per nucleon, $\sqrt{s_{\rm NN}}/2 \sim 15\,\text{GeV}$ we find a relation

$$\sqrt{s_{\rm NN}} = 2\frac{(\pi T)^2}{\mu}\left[f(\alpha_s) \sim 1 + \frac{N_G}{N_q}\right], \tag{28.1}$$

which describes the initial quadratic rise of μ as function of T of the ISR path shown in Fig. 28.1.

As mentioned, the pressure is small and even vanishes at the phase boundary which leads to the relation

$$T_0 \simeq \mathscr{B}^{1/4}. \tag{28.2}$$

Consequently at ISR energies the chemical potential at the phase transition, where hadronization will occur, is

$$\mu_{\rm cr} = \frac{2\pi^2 \mathscr{B}^{1/2}}{\sqrt{s_{\rm NN}}} \sim 20\,\mathrm{MeV}. \tag{28.3}$$

In this number we recognize the main difference to the LBL Bevalac energies which lead to chemical potentials of the order and above 500 MeV at $T \sim (2/3)T_0$, see LBL path in [1].

When the hadronization occurs, the entropy of the fireball with $A = 4+4$

$$S = \ln Z + \frac{E - \mu A}{T} \tag{28.4}$$

can be well approximated for $\alpha\alpha$ ISR collisions as

$$\frac{S}{A} = \frac{\sqrt{s_{\rm NN}}}{2T_0} \sim 100 \tag{28.5}$$

given that $T \ln Z = PV \to 0$ and $\mu \ll \sqrt{s}$. This is an extremely high entropy per participating nucleon and it requires very high particle multiplicity by use of Boltzmann's relation $S \propto \ln W$. Hence we are led to the conclusion that the production of quark-gluon plasma at ISR must be characterized by very high multiplicities. The mean transverse momenta of the hadrons produced will show the known features of pp collisions as almost all particles are made in the final stages of the fireball explosion when the transition to the hadronic gas phase occurs.

I do not doubt that important signatures of quark-gluon plasma will be found, however we expect the relative particle yields and appearance of high $p_\perp$ particles to be more valuable indicators, rather than the inclusive particle spectra. I am not yet prepared to speculate further on possible characteristic features of the quark-gluon plasma formation in $\alpha\alpha$ collisions.

Finally let us stress the similarity of the physics at LBL-Bevalac and ISR, as shown in Fig. 28.1, despite different domains explored in the μ, T diagram and different type of experiments. It could be therfore desirable to have at ISR data with heavy nuclei (as compared with α's) at perhaps somewhat lower $\sqrt{s_{\rm NN}}$. This would close the gap between both available experiments, at the same time allowing for higher collectivity (higher number of nucleons A) and thus a much larger probability for production of the plasma.

I would like to thank R. Hagedorn for his interest, support and stimulating discussions.

References

1. R. Hagedorn and J. Rafelski: CERN preprints TH 2947, TH 2969 (1980) From hadron gas to quark matter I & II: in the *Proceedings of the International Symposium on Statistical Mechanics of Quarks and Hadrons*, Bielefeld, Germany, August 1980, ed. by H. Satz, (North Holland Publishing Company); see Extreme states of nuclear matter – 1980: Chapter 27 in this volume
2. E.V. Shuryak: QCD and the Theory of Superdense Matter, Physics Reports **61**, 71 (1980)

Chapter 29
Possible Experiments with Heavy Ions at the PS/SPS – CERN SPC 1982

Johann Rafelski, edit of the SPC Protocol

Abstract I present the heavy ion program development at CERN, reproducing much of the pivotal discussion at the 123th meeting of the CERN Scientific Policy Committee (SPC), Geneva – 21 and 22 June 1982, based on the Draft Minutes of the meeting (CERN/SPC/0490/Draft, 1982) and related clarifications as marked.

29.1 The Participants

The CERN Scientific Policy Committee meeting in June 1982 brought together a large invited group that included the international particle physics leadership.

Chairman: **Prof. V.L. Telegdi** Members:

Prof. I. Bergström	Prof. N. Cabibbo	Prof. P. Falk-Vairant
Prof. S.L. Glashow	Prof. E. Lohrmann	Prof. L.B. Okun
Prof. D.H. Perkins	Prof. A. Salam	Prof. G. Salvini
Dr. G.H. Stafford	Prof. W. Thirring	Prof. K. Tittel
Dr. R. Turlay		

Ex Officio Members:

Prof. G. Bellettini, Chairman – ISR Committee
Prof. P.G. Hansen, Chairman – PS/SC Committee
Prof. J. Lefrançois, Chairman – SPS Experiments Committee
Dr. J.H. Mulvey Invited in his capacity as Chairman of ECFA

Also present:

Prof. K.O. Nielsen – Chairman of the Finance Committee
Prof. J.C. Kluyver Prof. J. Lemonne Prof. P. Olesen Prof. A.C. Pappas

Former Members Invited:

Prof. E. Amaldi	Prof. M. Conversi	Prof. A.G. Ekspong
Prof. B. Hahn	Prof. W. Jentschke	Prof. A. Lehmann
Prof. L. Leprince-Ringuet	Prof. P.T. Matthews	Prof. W. Paul
Prof. F. Perrin	Prof. A. Rousset	Prof. S.A. Wouthuysen

Department of Physics, The University of Arizona, Tucson, AZ 85721, USA

CERN Officials: **Prof. H. Schopper** – CERN Director-General
Dr. G. Brianti – Technical Director
Dr. E. Gabathuler – Research Director
Prof. R. Klapisch – Research Director
Prof. E. Picasso – Director and LEP Project Leader
Invited: **Dr. M. Jacob** for the "Heavy Ion Collisions" item of the agenda

29.2 On Formation of QGP in Heavy Ion Collisions

Maurice Jacob begins his presentation at 11:20, 22 June, 1982.

"Heavy ion collisions offer the possibility to reach very high densities and very high temperatures over extended domains, many times larger than the size of a single hadron. The energy densities considered are of the order of 0.5 to 1.5 GeV/fm^3 and the relevant temperatures are in the 200 MeV range. The great interest of reaching such conditions originates from recent developments in Quantum Chromodynamics, QCD, which make it very plausible that, while color confinement should prevail under standard circumstances, deconfinement should occur at sufficiently high density and (or) sufficiently high temperature. Under such conditions a new phase of matter, a quark-gluon plasma, is likely to exist. This phase should be viewed as due to a coalescence, or perhaps a percolation, of hadrons into larger entities and not as an actual separation of free quarks! ...

"Over an extended volume where the required density or temperature conditions would prevail, one expects that the properties of the physical vacuum would be modified. While the normal vacuum excludes the gluon field, the color-equivalent of the dielectric constant being zero (or practically zero), one would get a new vacuum state where quarks and gluons could propagate while interacting perturbatively.

"The equivalent of the dielectric constant would now be unity. The required conditions may be reached at high enough densities, hadrons being squeezed into one another, or at high enough temperature, the calculation of the partition function no longer favoring confining configurations whereby a color flux tube of fixed cross section extends between two color sources. The temperature at which the phase transition is expected to occur depends on the density, or on the quark chemical potential. One may thus separate two phases, a hadron phase and a quark gluon plasma, on a density-temperature diagram....

"The presence of a phase transition could long be expected from phenomenological models with an exponentially increasing hadron spectrum. The limiting Hagedorn temperature, obtained as the specific heat of the hadron gas diverges, can be interpreted as a critical temperature beyond which the relevant description should be in terms of a quark-gluon plasma reaching eventually a Stephan-Boltzmann behavior. The actual presence of a phase transition finds however its strongest present support in lattice gauge calculations....

"Granting the fact that phase transition(s) exist(s), the next question is to assess whether or not the required conditions could be met in heavy ion collisions with center-of-mass energy in excess of 10 GeV/nucleon. At present there also appears to be a consensus that this is the case. . . .

"The expected mean energy density is of the order of 2 GeV/fm^3 for the (most favorable) case of head on U-U collisions and still of the order of 1.2 GeV/fm^3 for Fe-Fe collisions. This applies to the fragmentation region, considering the energy trapped in what remains of the projectile or target nucleus just after the collision. . . .

"Granting the fact that a thermalized quark gluon plasma is formed during the collision, it will very rapidly destroy itself through instabilities, expansion and cooling. One should then watch for specific signals which could be associated with its transient (but most interesting) presence. . . .

"Several signals have attracted particular attention.

1. One of them is provided by the prompt photon or lepton pairs radiated (a volume effect!) by the thermalized plasma, . . .
2. Another interesting signal may be provided by strange particles originating in relatively large number from the plasma, once it has reached chemical equilibrium.
3. There may also be more violent effects, with abnormal density fluctuations in the overall energy flow associated with secondaries.
4. Size and lifetime could be determined through pion/photon interferometry since each violent event with head on collision could produce pions in the thousands!"

29.3 Experimental Opportunities to Study QGP

At the recent Bielefeld workshop

". . . six working groups studied experimental questions from the point of view of physics goals and their technical realization.. . .

1. The group convened by S. Nagamiya and H. Specht studied measurement of inclusive particle distributions. It became clear that the desired measurements were single particle spectra, not necessarily truly inclusive, but with various triggers to select central collisions and those with large multiplicity or energy deposit in the target.

   From the physics side, it was established that a good way to investigate the effects in the quark-gluon plasma such as the suppression of u-quarks and the chemical equilibrium of s-quarks, should be to measure distributions of strange particles, mesons and especially strange and multiply strange baryons and antibaryons. . . . It was concluded that the presence of very high multiplicities does not present a major obstacle to these experiments.

   . . . It is certainly reasonable to expect to find many strange quarks (and some charmed quarks) lodged in the fragments of the projectile. The large Lorentz factor will then allow the use of beams similar to the existing hyperon beams to provide momentum analyzed, mass and charge identified hypernuclei. There should also be usable numbers of multiply strange hypernuclei. This is a radically new approach to the study of these particles and should give rise to a major step forward.

2. Experiments on correlations among a few particles were considered by the group convened by I. Otterlund and H. Boggild. The idea here was to handle the high multiplicities by using spectrometers with a solid angle just large enough to cover the angles between two particles in the range of interest, but small enough so that the number of particles to be measured is still close to that commonly encountered.. . .

   The topic of identical particle intensity interferometry by study of few particle correlations was given special attention. This technique has already proved its worth in nuclear collisions and is expected to be a major tool in high energy nucleus-nucleus interactions. It is used to measure the size and shape of the interaction volume,. . . .

   This apparatus also seems suited for studies of V0's. The group also designed a special spectrometer to study photon correlations. This group devoted a substantial effort to the study of various triggers to select central or peripheral events, including measurement of the forward particles and a "plastic ball" type of detector covering most of the solid angle of target fragmentation.
3. G. London and K. Nakai convened a group working on the production of leptons and photons. They established that several distinct kinematic regions seemed to be of interest. For intermediate and high mass muon pair production in the projectile fragment region, they showed that an experiment using the NA3 apparatus at the SPS with small modification could be very effective. . . .
4. A combined group convened by C. Fabjan, H. Gutbrod, A. Sandoval and A. Wagner studied colorimetric techniques and tracking devices in large solid angle detectors. The beauty of energy flow measurements with calorimeters is well recognized, but this group took the attitude that there would be powerful arguments for an apparatus which could make nearly complete measurements on an event by event basis and set out to investigate if it is technically feasible using methods presently available.. . .
5. Another working group convened by M. Faessler and S. Frankel studied the case of deuteron and alpha particle beams.. . .
6. A group convened by R. DeVries and H.G. Fischer worked on the subject of peripheral interactions. A major part of their time was devoted to the study of the experiments at Berkeley giving particles with very short interaction lengths . . . .

"The large variety of experiments devised by the working group indicates a need to run several experiments at one time. The intensity requirements of the experiments are such that this should be possible. . . .

"Concluding this rapid survey of the physics of heavy ion collisions, one may say that there is practically no doubt that a phase transition exists, even if the exact form which it takes is not yet precisely known. There is also practically no doubt that the energy density to be achieved in heavy ion collisions, with incident ion beams in the 200 GeV/nucleon energy range, should reach the critical value. . . ."

29.4 Discussion on Relativistic Heavy Ion Collisions

The chairman, **Prof. H. Schopper**, thanked Maurice Jacob for his presentation, and opened the discussion.

Replying to a question from **Prof. P.T. Matthews**, Maurice Jacob said that the fundamental purpose of heavy-ion collision experiments was to study matter at very high quark densities. It was thought that when such densities were created, a new phase of matter appeared which would signal its existence by an anomalous production of photons, lepton-pairs or strange particles. Heavy-ion collision experiments would therefore be designed to investigate this anomalous production. It was possible that even more peculiar effects could be associated with high quark densities, but he had concentrated on the conservative ones which one could expect to see from a blob of natter at a temperature of the order of 200 MeV. At this energy the blob would radiate photons and its gluons would transform favorably into pairs. Experiments would therefore be designed to observe and search for large fluctuations in specific parameters. It was expected that the production mechanism would show up clearly in heavy-ion collisions, whereas there was no evidence for, and little hope to reach, such energy densities over an extended domain in proton-nucleus collisions.

Prof. P.G. Hansen, PS/SC Committee Chair added that one of the essential aspects of any experiment would be to study the question at different energies to determine how much energy was required *for the formation of quark-gluon plasma, JR*. In such experiments there were three essential variables: the target mass, the projectile mass and the energy. Unfortunately, the projectile mass was not available at all energy scales, and therefore, for the time being, only relatively light projectiles at very high energies could be considered. This increased the importance of repeating the experiment at different energies to ascertain whether the signature variables showed any characteristic change which could indicate the existence of the phase transition. It was in this context that the discussion centered on the use of the PS as a step on the way to 200 GeV per nucleon.

Prof. D.H. Perkins observed that, as the atomic number of colliding ions increased, there must be a critical point where plasma effects became important, but it was difficult to see how this point could be determined owing to the large energy density fluctuations.

Maurice Jacob, replying, said that theoretical efforts were currently being concentrated on obtaining mean values of energy densities which could be expected in a collision of this kind. There were bound to be large fluctuations, and while there was some information about fluctuations in pp, p-nucleus and $\alpha\alpha$ collisions, as yet there was no information on how significant these fluctuations could be in the case of heavy-ion collisions. Information about such fluctuations was a very important reason for experimentation and indeed, density fluctuations towards large values were probably those needed for the phase transition to take place.

The chairman, **Prof. H. Schopper** pointed out that, with regard to the question of particle signature, it should first be established in what rare fraction of cases, using the standard theory, such phenomena would take place. It ought to be possible, for example, to predict the probability that 1000 pions would be produced at the reference energies without invoking such phenomena as the phase transition predicted by QCD.

Maurice Jacob replying, said that the standard theories would predict that the mean multiplicity would rise from between $A^{2/3}$ to $A^{4/3}$ according to the model. The observation of very large multiplicities, showing that a large amount of energy could be found in excitation energy, could be considered as a necessary condition for a phase transition.

Regarding fluctuations away from the mean, information was available in the case of proton/proton collisions where the fluctuations had been very well characterized in terms of the KNO distribution up to a certain value. Thereafter, practically nothing was known about collisions with extremely high multiplicities because they were so difficult to study. The results of the NA5 experiment had emphasized this point, showing, for example, that, when looking for large amounts of transverse energy, the production of a very large number of particles with medium $p_\perp$ might prove to be a more frequent phenomenon than the production of a few particles with large $p_\perp$ associated with jets.

Replying to questions from **Prof. D.H. Perkins** and the chairman, **Prof. H. Schopper**, Maurice Jacob said that collisions with a projectile with a large atomic number were required because the amount of deposited energy was proportional to the number of nucleons in the incident nucleus. Estimates suggested that, in the most optimistic case of head-on uranium/uranium collisions, energy densities of the order of 2 GeV/fm^3 would be obtained, whereas in the case of carbon/uranium collisions, this figure would fall to 1 GeV/fm^3.

Prof. G. Bellettini, ISR Committee Chairman, I observed that, not withstanding the obvious advantages of heavy-ion collisions, it would be interesting to ascertain experimentally whether anomalous phenomena could be observed with pp and/or $\alpha\alpha$ collisions.

Replying to a question from **Prof. E. Amaldi**, Maurice Jacob said that, with regard to the question of the time necessary for the plasma to achieve equilibrium, it was expected that there was a chance that some thermalization would take place at the level of the quarks and the gluons present in the plasma, many collisions having time to take place.

Replying to **Prof. N. Cabibbo**, Maurice Jacob said that the Helsinki group in particular had estimated lepton pair production in detail. In accordance with the standard thermodynamic formulae, the number of photons produced in the plasma depended upon the charged-particle density and the temperature. Since this would essentially be a volume effect, the larger the volume of the plasma the greater would be the increase in photon production with respect to pions. Consequently, the volume of the plasma was an important parameter to determine.

In reply to a question from **Prof. A.G. Ekspong**, about the anomalous effect termed the anomalon, Maurice Jacob said that its interpretation as the decay of a hyperfragment of a strange particle had now been rejected. Research at the Bevalac at Berkeley had revealed that, when observed close to production, some fragments seemed to have very large cross-sections for a given ionizing power, ... Purely experimental problems should, however, not be underestimated[1].

Replying to **Dr R. Turlay**, Maurice Jacob said that what was particularly new in this type of physics was the expected production of lepton pairs at large x. If a new state of matter existed, one could foresee that it would radiate lepton pairs and that their momentum would correspond essentially to the global motion of the blob of matter. Any experiment would therefore concentrate on looking for lepton pairs at large x with a thermal-type mass distribution as opposed to the $1/m^4$ distribution for the $d\sigma/dm^2$ distribution, associated with the Drell-Yan theory with a concentration at low x. One would expect to see a very sharp fall-off of the lepton-pair mass spectrum as compared to the Drell-Yan spectrum.

Replying to a question from **Prof. J. Lefrançois**, SPS Experiments Chairman, Maurice Jacob said that at 1 GeV/fm^3 the temperature of the plasma would be too low for significant production of charm and beauty particles.

In reply to a question from **Prof. N. Cabibbo**, Maurice Jacob said that the great merit of the QCD calculation using the lattice over the Hagedorn model was that it made direct exploration of the system possible over and beyond the phase transition, whereas the phenomenological model had been based on a separate study of the two phases. The two approaches were, however, complementary, in many respects. What the experimenters wished to do with heavy-ion collision experiments was to ascertain whether matter existed in a different form beyond the hadron gas.

The chairman, **Prof. H. Schopper**, in conclusion, said it was clear that any discussion of heavy-ion collision experiments raised as many questions as it attempted to resolve. However, before very long the Scientific Policy Committee would have to address itself to the question of heavy-ion collision experiments in a more formal way. When the proceedings of the Bielefeld Workshop had been published, the Committee would be in a position to brief itself more thoroughly in order to formulate an appropriate recommendation.

The Committee took note of the report, and of the further explanations provided by Maurice Jacob.

The chairman, **Prof. H. Schopper**, said that before concluding the proceedings he wished to ask any of the former members of the Committee whether they had any comments or statements of a general nature to make.

Prof. L. Leprince-Ringuet said that, although no longer directly associated with the affairs of CERN, he nevertheless continued to follow its development with great interest. In this respect, he particularly appreciated the opportunity afforded him by

[1] An analysis offered by I. Otterlund a year later (lecture at the *Sixth High Energy Heavy Ion Study*, Berkeley, 28 June – 1 July 1983) has shown that the bias of human eye-based-analysis was the source of the shortened reaction path observed; see also S. B. Beri *et al.* [Banaras-Chandigarh-Jaipur-Jammu-Lund Collaboration], "A Search for Anomalous Fragments in 1.8A Gev ^{40}Ar Reactions in Nuclear Emulsions," Phys. Rev. Lett. **54** (1985) 771. JR.

this meeting of the Scientific Policy Committee to become acquainted with the latest developments in particle physics research and to hear about the progress achieved in specific projects.

In general terms, however, he was increasingly bewildered by the size and complexity of CERN's activities and of individual experiments, which could involve hundreds of physicists and whose leaders were thus no longer experimentalists in the true sense of the word but administrators. He was concerned that this preoccupation with size and a concomitantly high degree of organization could have the effect of reducing flexibility and the ability of scientists to maintain an open-minded approach to the problems with which they were concerned.

Increasingly, it seemed, experimentalists were informed in advance of the phenomena they would encounter. This elevation of the theorist to pre-eminence could have the detrimental effect of reducing the receptiveness of the experimentalist to the unexpected, weighed down as he was by the sheer volume of data to be analyzed. It should never be forgotten that most of the major discoveries made in the field of particle physics during the century had been unforeseen.

Prof. E. Amaldi said that while he did not share Leprince-Ringuet's concern that the size of experiments must necessarily limit their success, for he was certain that new discoveries would emerge before long, he doubted whether a member of a modern collaboration of, say, 250 physicists could derive as much pleasure and satisfaction from an experiment as had physicists of his own generation.

On behalf of the Committee, the chairman, **Prof. H. Schopper**, expressed thanks to all former members of the Scientific Policy Committee for their contributions during the meeting, and to the three members now leaving the Committee - Prof. G. Salvini, Dr. G.H. Stafford and Prof. W. Thirring – for their work.

The meeting ended at 13.15 – after 1h 55 min mostly if not exclusively devoted to the discussion of the future heavy ion program at CERN.

Chapter 30
What Happened to 'Strangeness in Quark-Gluon Plasma – 1982'

Johann Rafelski

Abstract Due to mishaps, the following manuscript Chapter 31 did not appear in the proceedings of the QM2 meeting in Bielefeld, 10-14 May, 1982. It seems appropriate to show in this volume how things worked and what obstacles needed to be overcome in a new field of research that was recently born and rapidly advancing. The science policy decisions to be taken meant that 1-2 weeks were important, while a document could take as long as 4 weeks to travel between Geneva and Frankfurt.

The lectures given at the QM2 meeting play an important role in the report made by Maurice Jacob on 22 June 1982 to the CERN Scientific Policy Committee, see Chapter 29. Looking at the actual CERN SPC meeting protocol prepared the end of June 1982, one sees that Maurice's own QM2 contribution was not available but in first typed draft, with hand-drawn figures.

The story of my displaced manuscript: I left it behind in the care of a trusted student in Frankfurt, and I believed that my instructions were that upon final corrections in one-two days my contribution would be sent straight to Maurice Jacob at CERN. However, Mr. Günther Staadt felt uneasy submitting what I had not seen and on Wednesday, 14 July, 1982, translated from German, he wrote:

Enclosed you will find, as desired, a copy of your typed and corrected work, "Strangeness in Quark-Gluon-Plasma." Everything here is in order. As soon as the fifth chapter of your book is finished, I will send you a copy. I wish you a continued pleasant stay in America.

I was in Seattle. I reviewed the manuscript and after one small white-out change I sent it off the same day to CERN, with the note to Maurice Jacob:

Dear Maurice, I hope that you can enclose this late manuscript, "Strangeness in Quark-Gluon Plasma" into the proceedings of the Bielefeld meeting, Sincerely yours, Johann, PS Till 13 August in Seattle, from Sept 4th at Frankfurt

Department of Physics, The University of Arizona, Tucson, AZ 85718, USA

After I was already back a letter from Maurice Jacob dated and stamped at CERN on 12 August, 1982 arrived on 8 September, taking four weeks from Geneva to reach my mail bin in Frankfurt:

Dear Johann, Thank you very much for your letter and your manuscript. However even though your letter is dated "July" I received it only today (12 August, 1982) when everything is already in Singapore.

Since the Bertocci's session report was in an unsettled stage until the last minute I am forwarding your paper to Helmut who had straightened out the matter after discussing things with Bertocci. While there may still be time to include your contribution I am very much afraid it could already be very – too – late.

I remember calling Bielefeld that day and talking to a secretary. She had the contents of the book on her desk and informed me that my paper was not in proceedings. This conference paper was half review of published work and in part new and original with many ideas. This is a typical conference, and less typical manuscript to be submitted to a refereed journal. So what was I to do? I gave a copy of the manuscript to my friend John W. Clark who happened to visit just at the time of the call to Bielefeld.On 4 October, 1982 he wrote back:

Dear Jan, I turned (2 copies of) your paper "Strangeness in Quark-Gluon Plasma" over to Manuel de Llano, who will present it to the editor of KINAM. However, a cover letter stating explicitly that the paper is submitted for publication in KINAM will be needed. Please address the letter to . . .

After more lost mail (!) KINAM wrote on 2 March, 1983 by surface mail:

I am glad to inform you that your paper . . . has been recommended for publication after revision. Please find enclosed the referee's comments.

The request of the referee was justified: I had to either make this a review, or a research paper. I hand-wrote a response on 4 May, 1983:

Regret that in view of the enclosed review I am unable to satisfy the demands of the referee and withdraw the paper.

I presented the story as an anecdote when traveling in South Africa, and found sympathetic ears. The submission letter sent on 11 May, 1983 from Cape Town to Professor Chris Engelbrecht, the editor of the South African Journal of Physics reads:

The enclosed manuscript 'Strangeness in Quark-Gluon Plasma' was prepared some months ago with the intention that it should appear in proceedings of the Bielefeld Workshop, as mentioned on the first page. Unfortunately, it had arrived too late to be included in these proceedings. Aside from reviewing work published in Refs. 4, 11 (enclosed for your information) it also contains quantitative discussion of strangeness as a signal for plasma formation (Section 4) not available elsewhere. I would be delighted if you decide to publish this manuscript in your journal.

The reception date by publishers was 16 May, 1983. Note how well, in comparison, the mail in South Africa worked.

Chapter 31
Strangeness in Quark–Gluon Plasma – 1982

Johann Rafelski

Abstract It is argued that observation of the strange-particle abundance may lead to identification of the quark-gluon plasma and measurement of some of its properties. Approach to chemical equilibrium and competitive processes in the hadronic gas phase are discussed.

31.1 Overview

I would like to argue in this paper that the nature of the properties of quark-gluon plasma can be studied by observing the abundance of strange particles created in nuclear collisions [1]. Unlike hadron–hadron collisions, we anticipate that in an important fraction of nucleus–nucleus collisions, each participating quark will scatter many times before joining in an asymptotic hadronic state. The associated simplification of the physics involved arises because the well-established methods of statistical physics can be used in such a case in order to connect the microscopic world with effects and properties visible to the experimentalist's eyes. Only the presumption of an approximate thermochemical equilibrium, to be studied below in more detail, frees us from the dependence on details of quark wave functions.

As a consequence of the statistical equilibrium the available energy is equipartitioned among accessible degrees of freedom and, among other $s\bar{s}$ pairs. This means

Preprint UFTP-82-86 Sep 1982; prepared for proceedings of QM2: Bielefeld 10-14 May, 1982, missed due to mail mishaps the deadlines, printed later in South African Journal of Physics Volume 6 No 2 (1983) pp. 37-43. This abbreviated version omits material seen in other chapters.

Department of Physics, The University of Arizona, Tucson, AZ 85721, USA

that there exists a domain in space in which, in a proper Lorentz frame, the energy of the longitudinal motion has been largely transformed to transverse degrees of freedom. The basic question concerns the internal structure of this hadronic fireball: instead of consisting of individual hadrons, it may be formed by quarks and gluons. In this new physical phase, these colour-charged particles are deconfined and can move freely over the volume of the fireball. It appears that the phase transition from the hadronic gas phase to the quark-gluon plasma is mainly controlled by the energy density of the fireball. Several estimates [2] lead to 0.6–1 GeV/fm^3 for the critical energy density, to be compared with around 0.25 GeV/fm^3 inside individual hadrons. Many theoretical questions about strong interactions will be settled once the parameters and nature of the phase transition have been determined.

Further development of this new field of research depends on the ability to observe plasma creation and its detailed physical properties. It is quite difficult to insert a thermometer and to measure baryon density at $T = 150$ MeV. We must either use only electromagnetically interacting particles [3] (photons, lepton pairs) in order to get them out of the plasma, or study the heavy quark flavour abundance, in particular strangeness, generated in the collision [1]. To obtain a better impression of what is meant, imagine that strange quarks are very abundant in the plasma (and indeed they are!). Then, for example, since the (sss)-state is bound and stable in the hot perturbative QCD vacuum, it would be the most abundant baryon to emerge from the plasma. I doubt that such an omegazation of nuclear matter could leave any doubts about the occurrence of the phase transition. But even the enhancement of the more accessible abundance of $\overline{\Lambda}$ may already be sufficient for our purposes.

I will now explain in more detail why the strange-particle abundance is so useful [1] for observing properties of the quark-gluon plasma. First we note that, at a given temperature, the quark-gluon plasma will contain an equal number of strange (s) and antistrange ($\overline{s}$) quarks, naturally assuming that the hadronic collision time is much too short to allow for light-flavour weak-interaction conversion to strangeness. Thus, assuming equilibrium in the quark plasma (see Sect. 31.2), we find the density of the strange quarks to be (two spins and three colours)

$$\frac{s}{v} = \frac{\overline{s}}{v} = 6\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\exp(\sqrt{p^2+m_s^2}/T)+1} \approx 3\frac{T m_s^2}{\pi^2} K_2(m_s/T) \,, \tag{31.1}$$

neglecting for the time being the perturbative corrections. The mass m_s of the strange quarks in the perturbative vacuum is believed to be of the order of 180–300 MeV[1]. Since the phase space density of strangeness is not too high, the Boltzmann limit is used in Eq. (31.1). Similarly, there is a certain light antiquark density ($\overline{q}$ stands for either $\overline{u}$ or $\overline{d}$):

$$\frac{\overline{q}}{v} = 6\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\exp(|p|/T+\mu_q/T)+1} \approx \mathrm{e}^{-\mu_q/T} T^3 \frac{6}{\pi^2} \,, \tag{31.2}$$

[1] The 2014 reference value is $m_s(\mu = 2\,\mathrm{GeV}) = 95 \pm 5\,\mathrm{MeV}$.

where the quark chemical potential is $\mu_q = \mu_B/3$ and μ_B is the baryon chemical potential, we drop below the subscript 'B'. This exponent suppresses the $q\bar{q}$ pair production. It reflects the chemical equilibrium between q and $\bar{q}$ and the presence of a light quark density associated with the net baryon number.

Alternative, but physically equivalent, ways to understand these factors are the following two statements:

- $\bar{q}$ is Fermi-blocked, since in its production the partner q-quark has to go on top of a Fermi sphere ($T \to 0$ limit).
- $\bar{q}$ quarks are easily destroyed by the abundant q quarks in the plasma.

What we now intend to show is that there are often more $\bar{s}$ quarks than antiquarks of each light flavour. Indeed,

$$\frac{\bar{s}}{\bar{q}} = \frac{1}{2}\left(\frac{m_s}{T}\right)^2 K_2\left(\frac{m_s}{T}\right) \mathrm{e}^{\mu/3T} . \tag{31.3}$$

This ratio is shown in Fig. 31.1. Thus, we almost always have more $\bar{s}$ than $\bar{q}$ quarks and, in many cases of interest, $\bar{s}/\bar{q} \approx 5$. As $\mu \to 0$, there are about twice as many $\bar{u}$ or $\bar{d}$ quarks as there are $\bar{s}$ quarks at $T \gtrsim m_s$.

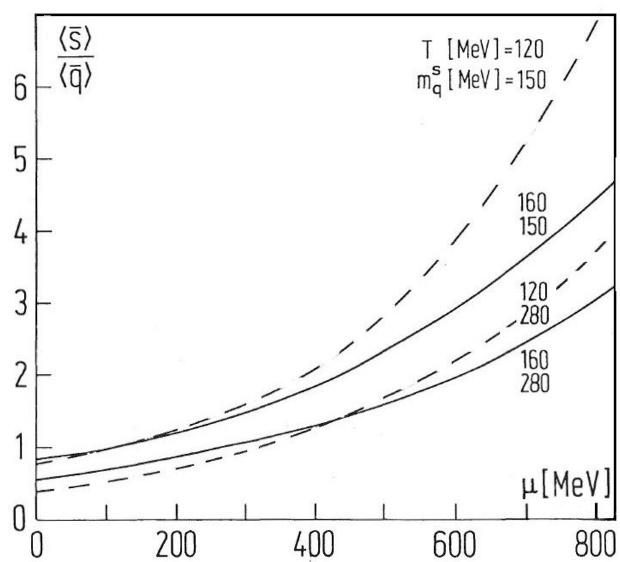


Fig. 31.1 Relative abundance of antistrange quarks $\bar{s}$ to light antiquark $\bar{q}$ as a function of μ for $T = 160\,$MeV (solid lines) and $T = 120$ (dashed lines), and strange quark mass $m_s = 150$, and 280 MeV, respectively.

When the quark matter dissociates into hadrons, some of the numerous $\bar{s}$ quarks may, instead of being bound in a $q\bar{s}$ kaon, enter into a $(\bar{q}\bar{q}\bar{s})$ or $(\bar{q}\bar{s}\bar{s})$ antibaryon and, in particular, a $\overline{\Lambda}$, $\overline{\Sigma}$, or $\overline{\Xi}$. The probability for this process seems to be comparable to the similar one for the production of $\overline{\Lambda}$, $\overline{\Sigma}$, $\overline{\Xi}$, or $\overline{\Omega}$ by the quarks present in the plasma. What is particularly noteworthy about the $\bar{s}$-carrying antibaryons is that they can conventionally only be produced in direct pair production reactions. Up to high energies, this process is suppressed by energy-momentum conservation and phase space considerations. This leads me to argue that a study of the $\overline{\Lambda}$, $\overline{\Sigma}$, $\overline{\Xi}$, and $\overline{\Omega}$ in high energy nuclear collisions could shed light on the early stages of the nuclear collisions in which quark matter may be formed.

As is apparent from the above remark, the crucial aspects of the proposal to use strangeness as a tag of quark-gluon plasma involve:

- assumption of thermal *and* chemical equilibrium (see next section),
- comparison between results anticipated in both hadronic phases at given T and μ, the chemical potential to be determined by other considerations (see Sect. 31.3).

The theoretical techniques required for the description of the two quite different hadronic phases, the hadronic gas and the quark-gluon plasma, must allow for the formation of numerous hadronic resonances, which then dissolve at sufficiently high partial density in the state consisting of its constituents. At this point we must appreciate the importance of, and help provided by finite temperature. To obtain high particle density we may, instead of compressing the matter (which as it turns out is quite difficult), heat it up; many pions are generated easily, leading to the occurrence of the transition at moderate (even vanishing) baryon density [1].

31.2 Strangeness Production in the Quark–Gluon Plasma

In this section, we investigate the abundance of strangeness as a function of the lifetime and excitation of the plasma state [4]. This investigation was motivated by the observation that light quarks could not by themselves lead to chemical equilibrium of strange quarks [5]. After identifying the strangeness-producing mechanisms, we compute the relevant rates as a function of the energy density ('temperature') of the plasma state and compare them with those for light u and d quarks.

In lowest order in perturbative QCD, $s\bar{s}$ quark pairs can be created in collisions of two gluons (Fig. 31.2a,b,c) and by annihilation of light quark-antiquark pairs (Fig. 31.2d). The averaged total cross-sections for these processes were calculated by B. Combridge [6]. For fixed invariant mass-squared $s = (k_1 + k_2)^2$, where k_i are the four-momenta of the incoming particles, below $w(s) = (1 - 4M^2/s)^{1/2}$,

$$\overline{\sigma}_{gg\to s\bar{s}} = \frac{2\pi\alpha_s^2}{3s}\left[\left(1+\frac{4M^2}{s}+\frac{M^4}{s^2}\right)\tanh^{-1} w(s) - \left(\frac{7}{8}+\frac{31}{8}\frac{M^2}{s}\right)w(s)\right] \quad (31.4a)$$

$$\overline{\sigma}_{q\bar{q}\to s\bar{s}} = \frac{8\pi\alpha_s^2}{27s}\left(1+\frac{2M^2}{s}\right)w(s)\ , \quad (31.4b)$$

For the mass of the strange quark, we explore the following cases:

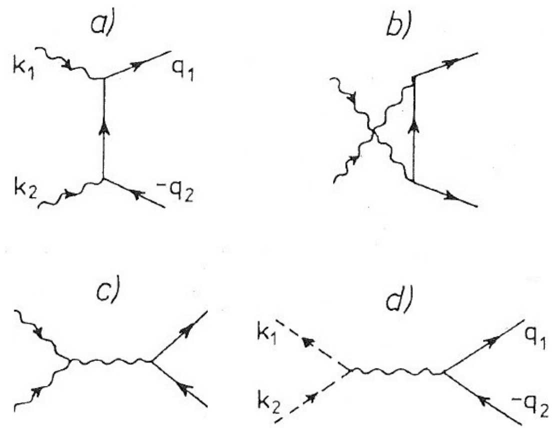


Fig. 31.2 Lowest order QCD diagrams for $s\bar{s}$ production: **a,b,c**) $gg \to s\bar{s}$, and **d**) $q\bar{q} \to s\bar{s}$.

- the value fitted within the MIT bag model: $M = 280$ MeV, and
- the value found in the study of quark currents: $M = 150$ MeV.

When discussing light quark production, we use $M = 15$ MeV. The effective QCD coupling constant $\alpha_s = g^2/4\pi$ is an average over space- and timelike domains of momentum transfers in the reactions shown in Fig. 31.2. We use (a) $\alpha_s = 2.2$, the value consistent with $M = 280$ MeV in the MIT bag model, and (b) the value $\alpha_s = 0.6$, expected at the involved momentum transfer, togather with $M = 150$ MeV.

Given the averaged cross-sections, it is easy to calculate the rate of events per unit time, summed over all final and initial states[2]:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{1}{2}\int \mathrm{d}^3x \int \frac{\mathrm{d}^3k_1}{(2\pi)^3|k_1|}\sum_i \rho_i(k_1,x)\int \frac{\mathrm{d}^3k_2}{(2\pi)^3|k_2|}\sum_i \rho_i(k_2,x) I k_1^{\mu} k_{2\mu}\overline{\sigma}(s). \tag{31.5}$$

The sum over initial states involves the discrete quantum numbers i (colour, spin, etc.) over which Eqs. (31.4a) and (31.4b) are averaged. The factor $k_1 \cdot k_2/|k_1||k_2|$ is the relative velocity for massless particles. We introduced a dummy integration $I \equiv \int_{4M^2}^{\infty} \mathrm{d}s\, \delta\big(s-(k_1+k_2)^2\big) = 1$ in order to facilitate the calculations. We now replace the phase space densities $\rho_I(k,x)$ by momentum distributions $f_g(k)$, $f_q(k)$, and $f_{\bar{q}}(k)$ of gluons, quarks, and antiquarks that can still have a parametric x dependence, i.e., through a space dependence of temperature $T = T(x)$. The (invariant) rate per unit time and volume for the elementary processes shown in Fig. 31.2 is then

[2] An additional factor 1/2 in the gluon production term is included in this printing: the wave function of two identical particles comprises the normalization factor $1/\sqrt{2}$ which when squared leads to $1/2$ in the thermal rate.

$$A = \frac{\mathrm{d}N}{\mathrm{d}t\mathrm{d}^3x} = \frac{1}{2}\int_{4M^2}^{\infty} s\mathrm{d}s\,\delta\big(s-(k_1+k_2)^2\big)\frac{\mathrm{d}^3k_1}{(2\pi)^3|k_1|}\frac{\mathrm{d}^3k_2}{(2\pi)^3|k_2|}$$
$$\Big[\frac{(2\times 8)^2}{2}f_g(k_1)f_g(k_2)\overline{\sigma}_{gg\to s\bar{s}} + 2(2\times 3)^2 f_q(k_1)f_{\bar{q}}(k_2)\overline{\sigma}_{q\bar{q}\to s\bar{s}}\Big]\,, \tag{31.6}$$

where the numerical factors count the spin, colour, and isospin degrees of freedom.

We furthermore assume that in the rest frame of the plasma, the distribution functions f depend only on the absolute value $|k| = k_0 \equiv k$ of the momentum. We then evaluate angular integrals in Eq. (31.6):

$$A = \frac{4}{\pi^4}\int_{4M^2}^{\infty} s\mathrm{d}s\,\overline{\sigma}_{gg\to s\bar{s}}\left[\int_0^{\infty}\mathrm{d}k_1\int_0^{\infty}\mathrm{d}k_2\Theta(4k_1k_2-s)f_g(k_1)f_g(k_2)\right]$$
$$+\frac{9}{4\pi^4}\int_{4M^2}^{\infty} s\mathrm{d}s\,\overline{\sigma}_{q\bar{q}\to s\bar{s}}\left[\int_0^{\infty}\mathrm{d}k_1\int_0^{\infty}\mathrm{d}k_2\Theta(4k_1k_2-s)f_q(k_1)f_{\bar{q}}(k_2)\right] \tag{31.7}$$

where the step function Θ requires that $k_1k_2 \geq s/4 \geq M^2$. We now turn to the discussion of the momentum distribution and related questions. We note that the anticipated lifetime of the plasma created in nuclear collisions is of the order 6 fm/$c = 2\times 10^{-23}$ s. After this time, the high internal excitation will most likely have dissipated to below the energy density required for the global restoration of the perturbative QCD vacuum state [7, 8]. The transition between the hadronic and the quark-gluon phase is expected at an energy density of approximately 1 GeV/fm^3. Under these conditions, it is possible to estimate that each perturbative quantum (light quark, gluon) in the plasma state will rescatter several times during the lifetime of the plasma. Hence the momentum distribution functions $f(p)$ can be approximated by the statistical Bose (Fermi) distribution functions:

$$f_g(p) \approx \left(\mathrm{e}^{\beta\cdot p}-1\right)^{-1} \qquad \text{(gluons)}\,, \tag{31.8}$$

$$f_{q/\bar{q}}(p) \approx \left[\mathrm{e}^{\beta\cdot p}\lambda^{(\pm)}+1\right]^{-1} \qquad \text{(quarks/antiquarks)}\,, \tag{31.9}$$

where $\beta\cdot p = \beta_0 E_p - \boldsymbol{\beta}\cdot\mathbf{p}$, $E_p \to |\mathbf{p}|$ for massless particles, $(\beta\cdot\beta)^{-1/2} = T$ is the temperature-like parameter, and $\lambda^{(\pm)}$ is the baryon number (antibaryon number) fugacity. In the rest frame of the plasma, $\beta\cdot p = E_p/T \to |\mathbf{p}|/T$. The distributions [Eqs. (31.8) and (31.9)] can only be taken seriously for $|\mathbf{p}|$ not much larger than T; to populate the high-energy tail of the distributions, too many collisions are required, for which there may not be enough time during the lifetime of the plasma. While in each individual nuclear collision, the momentum distribution may vary, the ensemble of many collisions may lead to better statistical distributions.

Finally, let us discuss the values of the fugacities $\lambda^{(\pm)}$ in Eq. (31.9). As quarks are brought into the reaction by the colliding nuclei, baryon number conservation makes it possible to relate the baryon density ν to the fugacities by integrating Eq. (31.9) over all momenta:

$$\nu(T,\lambda^+,\lambda^-) = \frac{1}{3}\times 12\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left[\left(\mathrm{e}^{|\mathbf{p}|/T}\lambda^+ + 1\right)^{-1} - \left(\mathrm{e}^{|\mathbf{p}|/T}\lambda^- + 1\right)^{-1}\right]. \quad (31.10)$$

The factor 1/3 takes into account the fractional baryon number of quarks. As we will show, the $gg \to q\bar{q}$ reaction time is much shorter than that for $q\bar{q} \to s\bar{s}$ production, since the light quark masses are only of the order of ≈ 15 MeV. Consequently, we may assume chemical equilibrium between q and $\bar{q}$ ($\mu = 3\mu_q$):

$$\lambda^+ = \frac{1}{\lambda^-} = \mathrm{e}^{-\mu_q/T}\,, \quad (31.11a)$$

$$\nu(T,\mu_q) = \frac{2}{3\pi^2}\left[\mu_q^3 + \mu_q(\pi T)^2\right]. \quad (31.11b)$$

As long as gluons dominate the $s\bar{s}$ formation in plasma state, conditions at the phase transition, such as abundance of q and $\bar{q}$, will not matter for the $s\bar{s}$ abundances at times comparable to the lifetime of the plasma. Hence, for the purpose of this study, we will use the value $\mu_q = 300$ MeV in order to estimate the quark densities at given temperature. We can now return to the evaluation of the rate integrals in Eq. (31.7).

In the glue part of the rate A, Eq. (31.7), the k_1, k_2 integral can be carried out exactly by expanding the Bose function Eq. (31.8) in a power series in $\exp(-k/T)$:

$$A_g = \frac{4}{\pi^4}T\int_{4M^2}^{\infty}\mathrm{d}s\, s^{3/2}\overline{\sigma}_{gg\to s\bar{s}}(s)\sum_{n,n'=1}(nn')^{-1/2}K_1\left(\frac{(nn's)^{1/2}}{T}\right). \quad (31.12)$$

In the quark contribution, an expansion of the Fermi function is not possible and the integrals must be evaluated numerically. It is found that the gluon contribution of Eq. (31.12) dominates the rate A. For $T/M \gtrsim 1$, we find

$$A \approx A_g = \frac{7}{6\pi^2}\alpha_s^2 M T^3 \mathrm{e}^{-2M/T}\left(1 + \frac{51}{14}\frac{T}{M} + \cdots\right). \quad (31.13)$$

Examples for A at $\alpha_s = 0.6$ and $M = 150$ MeV is shown in Fig. 31.3a. We note that in general the invariant production rate rises rapidly with T.

The abundance of $s\bar{s}$ pairs driven by A cannot grow forever. At some point the $s\bar{s}$ annihilation reaction will deplete the strange quark population. The $s\bar{s}$ pair annihilations may not only proceed via the two-gluon channel, but instead through γg final states [9]. The noteworthy feature of such a reaction is the production of relatively high energy γ's at the fixed energy of about 700–900 MeV ($T = 160$ MeV). These γ's will leave the plasma without further interactions. To some degree, this process is stimulated by coherent glue emission.

In any case, the $s\bar{s}$ annihilation loss term is proportional to the square of the density n_s of strange and antistrange quarks. With $n_s(\infty)$ being the saturation density at large times, the following differential equation determines n_s as a function of time [10]:

$$\frac{\mathrm{d}n_s}{\mathrm{d}t} \approx A\left\{1 - \left[\frac{n_s(t)}{n_g(\infty)}\right]^2\right\}. \quad (31.14)$$

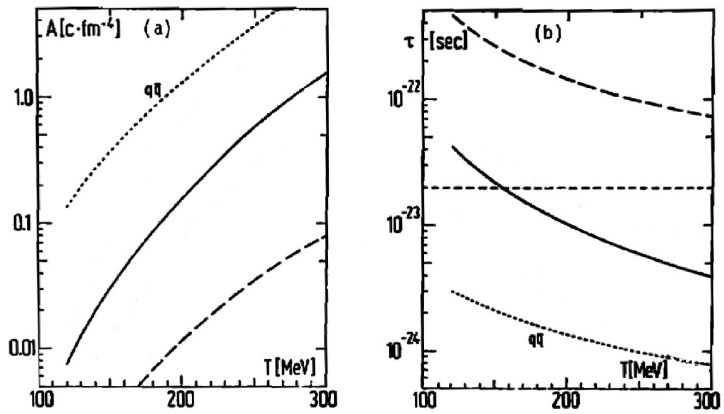


Fig. 31.3 (**a**) Rates A and (**b**) relaxation time constants τ, as a function of temperature T for $\alpha_s = 0.6$ and $M = 150$ MeV. *Full lines* total process: $gg \to s\bar{s} + q\bar{q} \to s\bar{s}$. *Dashed lines*: $q\bar{q} \to s\bar{s}$. *Dotted lines*: $gg \to q\bar{q}$ where $M = 15$ MeV. *Note that on left rates A for glue based processes are shown too large by factor two.*

The solution for initial value $n_s(t=0) \approx 0$ is,

$$n_s(t) = n_s(\infty) \tanh \frac{t}{2\tau} \;\to\; n_s(\infty)\left(1 - 2e^{-t/\tau}\right) , \tag{31.15}$$

where[3] $\tau = n_s(\infty)/2A$. $n_s(t)$ is a monotonically inreasing with temperature, saturating function, with asymptotic $t \to \infty$ behavior seen in Fig. 31.4b, controlled by the characteristic time constant τ. In a thermally equilibrated plasma, the asymptotic strangeness density $n_s(\infty)$ is that of a relativistic Fermi gas ($\lambda = 1$):

$$n_s(\infty) = \frac{2 \times 3}{2\pi^2} T M^2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} K_2(nM/T) , \tag{31.16}$$

provided the volume V is large.

For τ we find that the relaxation time[4]

$$\tau \approx \tau_g = \frac{9}{7}\left(\frac{\pi}{2}\right)^{1/2} \alpha_s^{-2} M^{1/2} T^{-3/2} \mathrm{e}^{M/T} \left(1 + \frac{99}{56}\frac{T}{M} + \cdots\right)^{-1} \tag{31.17}$$

is falling rapidly with increasing temperature as shown in Fig. 31.3b. While our results for strangeness production by light quarks agree in order of magnitude with those of Biró and Zimányi [5] (considering their choice of parameters), it is here obvious that gluonic strangeness production, which was not discussed by these authors, is the dominant process.

[3] The factor 2 in definition of τ had been inadvertently omitted.

[4] The following result has been always correct, however it combines two compensating, omitted factors 2 as noted above.

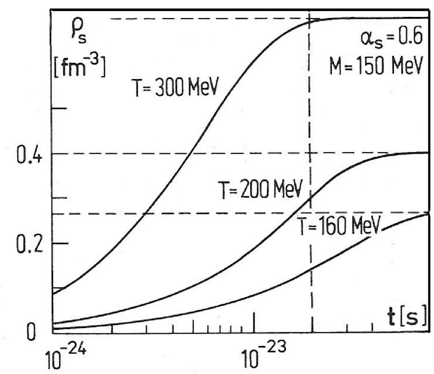


Fig. 31.4 Time-evolution of the strange quark density in the plasma for temperatures $T = 300, 200, 160$ MeV (top to bottom) for $m_s = 150$ MeV, $\alpha_s = 0.6$.

If we compare the time constant τ with the estimated lifetime of the plasma state, we find that the strangeness abundance will be chemically saturated for temperatures of 160 MeV and above, i.e., for an energy density above 1 GeV/fm^3. We note that τ is quite sensitive to the choice of the strange quark mass parameter and the coupling constant α_s and both must, however, be chosen consistently.

Also included in Figs. 31.3a and b are our results for gluon conversion into light quark–antiquark pairs. The shortness of τ for this process indicates that gluons and light quarks reach chemical equilibrium during the beginning stage of the plasma state, even if the quark/antiquark (i.e., baryon/meson) ratio was quite different in the prior hadronic compression phase.

The evolution of the density of strange quarks in Eq. (31.15) in the plasma state is shown in Fig. 31.4 for temperatures $T = 300, 200, 160$ MeV. The saturation of the abundance is clearly visible for $T \geq 160$ MeV. To obtain the measurable abundance of strange quarks, the corresponding values reached after the typical lifetime of the plasma state, 2×10^{-23} s, can be read off in Fig. 31.4 as a function of temperature. The strangeness abundance shows a pronounced threshold behaviour at $T \approx 120$–160 MeV.

I thus conclude that strangeness abundance saturates in sufficiently excited quark-gluon plasma ($T > 160$ MeV, $\varepsilon > 1$ GeV/fm^3).

31.3 Equilibrium Chemistry of Strange Particles in Hot Nuclear Matter

In order to establish the relevance of the strangeness signal for diagnosis of a possible formation of quark-gluon plasma, we must establish relevant particle rates originating from highly excited matter but consisting of individual hadrons – the hadronic gas phase, see Ref. [11] and Section 32.2. The main hypothesis which makes it possible to simplfiy the situation is to postulate the resonance dominance of hadron–hadron interactions [12] – in this case the hadronic gas phase is practically a superposition of an infinity of different hadronic gases and all information about the interaction is hidden in the mass spectrum $\tau(m^2, b)$ which describes the number of hadrons of baryon number b in a mass interval dm^2 [13]. When considering strangeness-carrying particles, all we need to consider is the baryon chemical potential established predominantly by the non-strange particles[5].

Here, we turn our interest directly to the rarest of all singly strange particles, and show in Fig. 31.5 the ratio $\langle n_{\overline{\Lambda}}\rangle / \langle n_{\Lambda}\rangle$. We notice an expected suppression of $\overline{\Lambda}$ due to the baryon chemical potential as well as strangeness chemistry. This ratio exhibits both a strong temperature dependence and a strong μ dependence. The remarkably small abundance of $\overline{\Lambda}$, e.g., $10^{-4}\Lambda$, under conditions likely to be reached in an experiment [$T \approx$ 120–180 MeV, $\mu \approx (4\text{–}6)T$] is characteristic of the nuclear nature of the hot hadronic matter phase under consideration here. Our estimates for quark-gluon plasma based on flavour content are two to three orders of magnitude higher. We note that $\mathrm{K}^+/\mathrm{K}^-$ abundance is a sensitive measure of the baryochemical potential, see Fig. 32.3.

In summary to this section, the relative abundance of strangeness-carrying antibaryons is greatly suppressed in the hadronic gas phase. Hence enhancements observed in nuclear collisions may be a useful indications of the reactions leading to the formation of the quark-gluon plasma. The study of multistrange hadrons is in progress.

31.4 Discussion

Only some selected aspects of the strangeness production in hot hadronic matter have been studied in detail. The results are quite encouraging and suggest interesting future perspectives. It was shown in Section 31.2 that strangeness abundance reaches chemical equilibrium in the plasma. The subsequent depletion of the strangeness during the plasma disintegration as well as its preferred hadronization channels have not yet been studied in detail. However, only if the plasma disintegration is an extremely slow process, lasting on the order of 10^{-22} s, can we anticipate a significant feedback on the high s abundance created at the maximum temperature reached in the collision. As shown in Fig. 31.3, the invariant rates drop quite rapidly with decreasing temperature, leading to a rapid increase in the equilibrium time constant τ; hence the strangeness abundance decouples from the equilibrium and remains a witness of the hot collision period.

While we cannot yet discuss in detail the abundance of multistrange antihadrons, which are influenced also by the possible ss, $\bar{s}\bar{s}$, sss, $\bar{s}\bar{s}\bar{s}$, and $s\bar{s}$ bound states in the plasma, it is apparent from the calculations performed in Section 31.3 that measurement of the production cross-section of the antistrange baryons could already

[5] To minimize duplication within this book we refer for the technical developments and the measurement of baryochemical potential μ to Section 32.2.

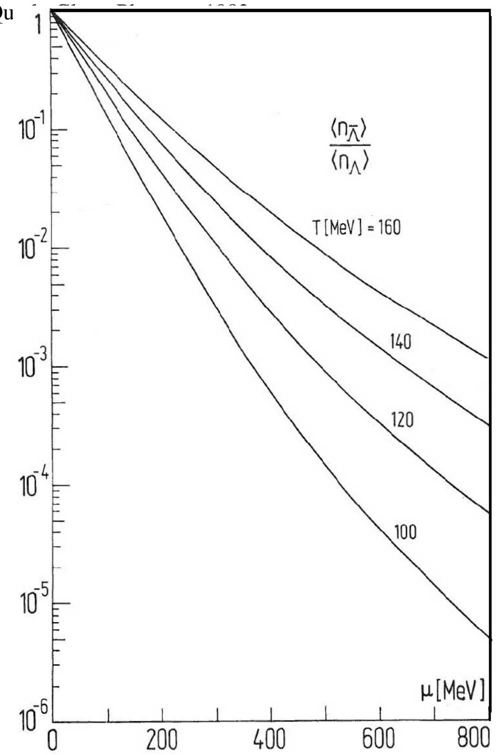


Fig. 31.5 The hadron gas ratio $\langle n_{\overline{\Lambda}} \rangle / \langle n_{\Lambda} \rangle$ as a function of μ for several temperatures $T = 100, 120, 140, 160\,\mathrm{MeV}$.

be quite helpful in the observation of the phase transition. The high suppression of these degrees of freedom in the hadronic gas phase for obvious physical reasons is not maintained in the plasma phase, where $\bar{s}$ abundance is larger than $\bar{u}$, $\bar{d}$ abundance, as shown in Section 31.1. Measurement of the relative $\mathrm{K}^+/\mathrm{K}^-$ yield, while indicative for the value of the chemical potential, see Section 32.2, may carry less specific information about the plasma.

The K/π ratio may indeed also contain relevant information. However, it will be much more difficult to decipher the message. The π abundance will originate from diverse sources needed to be understood for that purpose.

It is more appropriate to concentrate attention on those reaction channels which will be particularly strongly populated when the quark plasma dissociates into hadrons. Here, in particular, it appears that otherwise quite rare multistrange hadrons will be enhanced, on the one hand by the relatively high phase space density of strangeness in the plasma, on the other hand by the attractive ss QCD interaction in the $\overline{3}_{\mathrm{c}}$ and $\bar{s}s$ in 1_{c} channels. Hence we should search for the rise of the abundance of particles like Ξ, $\overline{\Xi}$, Ω, $\overline{\Omega}$, and ϕ, and perhaps in highly strange pieces of baryonic matter, rather than in the K channels. It seems that such experiments would uniquely determine the existence of the phase transition to the quark-gluon plasma.

It is important to appreciate that the experiments discussed above would certainly be complementary to the measurement with the help of electromagnetically interacting probes, e.g., dileptons or direct photons. Strangeness-based measurements have the advantage that they are based on the observation of a strongly interacting particle ($s, \bar{s}$ quark) originating from the hot plasma phase; these are much more abundant than the electromagnetic particles.

Acknowledgements

While much theoretical work remains for the future, I would like to thank all who have, through collaboration or stimulative discussions, helped in the development of this research field: M. Danos, W. Greiner, R. Hagedorn, P. Koch, B. Müller, H. Rafelski, and G. Staadt. This work was in part supported by the Deutsche Forschungsgemeinschaft.

References

1. J. Rafelski, R. Hagedorn: In *Thermodynamics of Quarks and Hadrons*, ed. by H. Satz, North Holland, Amsterdam (1981); CERN preprint TH 2969 (1980);
   H. Rafelski, J. Rafelski: unpublished;
   J. Rafelski: Exteme states of nuclear matter – 1980, Chapter 27 in this volume.
2. See, e.g., E.V. Shuryak: Phys. Rep. **61**, 71 (1980)
3. K. Kajantie, H.I. Mietinnen: Z. Phys. C **9**, 341 (1981);
   G. Domokos, J.I. Goldman: Phys. Rev. D **23**, 203 (1981);
   K. Kajantie, H.I. Mietinnen: Z. Phys. C **14**, 357 (1982)
4. J. Rafelski, B. Müller: Phys. Rev. Lett. **48**, 1066 (1982)
5. T. Biró, J. Zimányi: Phys. Lett. B **113**, 6 (1982)
6. B.L. Combridge: Nucl. Phys. B **151**, 429 (1979)
7. See also R. Anishetty, P. Koehler, L. McLerran: Phys. Rev. D **22**, 2793 (1980)
8. M. Danos, J. Rafelski: Phys. Rev. D **27**, 671 (1983)
9. G. Staadt, J. Rafelski, W. Greiner: in preparation [later published: Phys.Rev. D **33** 66 (1986)]
10. R. Hagedorn: How to Deal with Relativistic Heavy Ion Collisions – 1980, Chapter 26 in this volume
11. P. Koch, J. Rafelski, W. Greiner: Phys. Lett. B **123**, 151 (1983)
12. See, for example, R. Hagedorn: Nuovo Cimento Suppl. 3, 147 (1964); Nuovo Cimento **6**, 311 (1968); and R. Hagedorn: [10], p. 236
13. R. Hagedorn, I. Montvay, J. Rafelski: Hadronic matter at extreme energy density , lecture at Erice Workship, ed. by N. Cabibbo, Plenum Press, New York (1980), p. 49;
    R. Hagedorn, J. Rafelski: Phys. Lett. B **97**, 136 (1980) and:
    J. Rafelski and R. Hagedorn: Thermodynamic of Hot Nuclear Matter in the Statistical Bootstrap Model – 1978, Chapter 23 in this volume

Chapter 32
Strangeness and Phase Changes in Hot Hadronic Matter – 1983

Johann Rafelski

Abstract Two phases of hot hadronic matter are described with emphasis put on their distinction. Here the role of strange particles as a characteristic observable of the quark-gluon plasma phase is particularly explored.

32.1 Phase Transition or Perhaps Transformation: Hadronic Gas and the Quark-Gluon Plasma

I explore here consequences of the hypothesis that the energy available in the collision of two relativistic heavy nuclei, at least in part of the system, is equally divided among the accessible degrees of freedom. This means that there exists a domain in space in which, in a suitable Lorentz frame, the energy of the longitudinal motion has been largely transformed to transverse degrees of freedom. The physical variables characterizing such a 'fireball' are energy density, baryon number density, and total volume. The *basic* question concerns the *internal structure* of the fireball. It can consist either of individual hadrons, or instead, of quarks and gluons in a new physical phase, the plasma, in which they are deconfined and can move freely over the volume of the fireball. It appears that the phase transition from the hadronic gas phase to the quark-gluon plasma is controlled mainly by the energy density of the fireball. Several estimates [1] lead to 0.6–1 GeV/fm^3 for the critical energy density, to be compared with 0.16 GeV/fm^3 in nuclear matter.

We first recall that the unhandy extensive variables, viz., energy, baryon number, etc., are replaced by intensive quantities. To wit, the temperature T is a measure of energy per degree of freedom; the baryon chemical potential μ controls the mean baryon density. The statistical quantities such as entropy (= measure of the number of available states), pressure, heat capacity, etc., will also be functions of T

Invited lectures at the *Sixth High Energy Heavy Ion Study*, Berkeley, 28 June – 1 July 1983. In LBL-16281; UC-34C; CONF-830675 pp. 489-510; also preprint CERN-TH-3685 available at https://cds.cern.ch/record/147343/files/198311019.pdf. This abbreviated version omits material seen in other chapters.

Department of Physics, The University of Arizona, Tucson, AZ 85721, USA

and μ, and will have to be determined. The theoretical techniques required for the description of the two quite different phases, viz., the hadronic gas and the quark-gluon plasma, must allow for the formulation of numerous hadronic resonances on the one side [2], which then at sufficiently high energy density dissolve into the state consisting of their constituents. At this point, we must appreciate the importance and help by a finite, i.e., nonzero temperature in reaching the transition to the quark-gluon plasma: to obtain a high particle density, instead of only compressing the matter (which as it turns out is quite difficult), we also heat it up; many pions are generated in a collision, allowing the transition to occur at moderate, even vanishing baryon density [3].

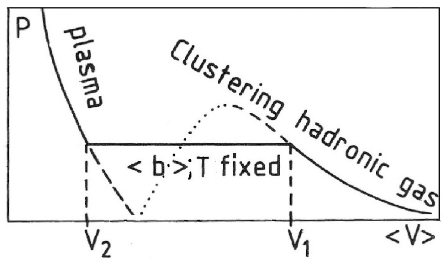


Fig. 32.1 p, V diagram for the gas–plasma first order transition, with the *dotted curve* indicating a model-dependent, unstable domain between overheated and undercooled phases.

Consider, as an illustration of what is happening, the p, V diagram shown in Fig. 32.1. Here we distinguish three domains. The hadronic gas region is approximately a Boltzmann gas where the pressure rises with reduction of the volume. When the internal excitation rises, the individual hadrons begin to cluster. This reduces the increase in the Boltzmann pressure, since a smaller number of particles exercises a smaller pressure. In a complete description of the different phases, we have to allow for a coexistence of hadrons with the plasma state in the sense that the internal degrees of freedom of each cluster, i.e., quarks and gluons, contribute to the total pressure even before the dissolution of individual hadrons. This does indeed become necessary when the clustering overtakes the compressive effects and the hadronic gas pressure falls to zero as V reaches the proper volume of hadronic matter. At this point the pressure rises again very quickly, since in the absence of individual hadrons, we now compress only the hadronic constituents. By performing the Maxwell construction between volumes V_1 and V_2, we can in part account for the complex process of hadronic compressibility alluded to above.

As this discussion shows, and detailed investigations confirm [4], we cannot escape the conjecture of a first order phase transition in our approach. This conjecture of [1, item (g)] has been criticized, and only more recent lattice gauge theory calculations have led to the widespread acceptance of this phenomenon, provided that an internal SU(3) (colour) symmetry is used – SU(2) internal symmetry leads to a second order phase transition [1, item (i)]. It is difficult to assess how such hypothetical changes in actual internal particle symmetry would influence phenomenological

Table 32.1 Phase transition of hot hadronic matter in theoretical physics

Object	$\longrightarrow$	Observational hypothesis	$\longrightarrow$	Theoretical consequence
Nature	$\longrightarrow$	Internal SU(3) symmetry	$\longrightarrow$	First order phase transition (on a lattice)
Nature	$\longrightarrow$	Bootstrap $\widehat{=}$ resonance dominance of hadronic interactions	$\longrightarrow$	First order phase transition in a phenomenological bootstrap approach
?	$\longrightarrow$	Internal SU(2) symmetry	$\longrightarrow$	Second order phase transition (on a lattice)

descriptions based on an observed picture of nature. For example, it is difficult to argue that, were the colour symmetry SU(2) and not SU(3), we would still observe the resonance dominance of hadronic spectra and could therefore use the bootstrap model. *All* present understanding of phases of hadronic matter is based on approximate models, which requires that Table 32.1 be read from left to right.

I believe that the description of hadrons in terms of bound quark states on the one hand, and the statistical bootstrap for hadrons on the other hand, have many common properties and are quite complementary. Both the statistical bootstrap and the bag model of quarks are based on quite equivalent phenomenological observations. While it would be most interesting to derive the phenomenological models quantitatively from the accepted fundamental basis – the Lagrangian quantum field theory of a non-Abelian SU(3) 'glue' gauge field coupled to coloured quarks – we will have to content ourselves in this report with a qualitative understanding only. Already this will allow us to study the properties of hadronic matter in both aggregate states: the hadronic gas and the state in which individual hadrons have dissolved into the plasma consisting of quarks and of the gauge field quanta, the gluons.

It is interesting to follow the path taken by an isolated quark-gluon plasma fireball in the μ, T plane, or equivalently in the ν, T plane. Several cases are depicted in Fig. 32.2. In the Big Bang expansion, the cooling shown by the dashed line occurs in a Universe in which most of the energy is in the radiation. Hence, the baryon density ν is quite small. In normal stellar collapse leading to cold neutron stars, we follow the dash-dotted line parallel to the ν axis. The compression is accompanied by little heating.

In contrast, in nuclear collisions, almost the entire ν, T plane can be explored by varying the parameters of the colliding nuclei. We show an example by the full line, and we show only the path corresponding to the cooling of the plasma, i.e., the part of the time evolution after the termination of the nuclear collision, assuming a plasma formation. The figure reflects the circumstance that, in the beginning of the cooling phase, i.e., for 1–1.5×10^{-23} s, the cooling happens almost exclusively by the mechanism of pion radiation [5]. In typical circumstances, about half of the available energy has been radiated away before the expansion, which brings the surface temperature close to the temperature of the transition to the hadronic phase. Hence a possible, perhaps even likely, scenario is that in which the freezing

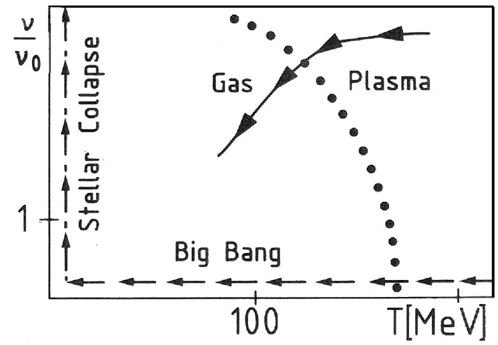


Fig. 32.2 Paths taken in the ν, T plane by different physical events.

out and the expansion happen simultaneously. These highly speculative remarks are obviously made in the absence of experimental guidance. A careful study of the hadronization process most certainly remains to be performed.

In closing this section, let me emphasize that the question whether the transition hadronic gas $\longleftrightarrow$ quark-gluon plasma is a phase transition (i.e., discontinuous) or continuous phase transformation will probably only be answered in actual experimental work; as all theoretical approaches suffer from approximations unknown in their effect. For example, in lattice gauge computer calculations, we establish the properties of the *lattice* and not those of the continuous space in which we live.

The remainder of this report is therefore devoted to the study of strange particles in different nuclear phases and their relevance to the observation of the quark-gluon plasma.

32.2 Strange Particles in Hot Nuclear Gas

My intention in this section is to establish quantitatively the different channels in which the strangeness, however created in nuclear collisions, will be found. In our following analysis (see Ref. [7]) a tacit assumption is made that the hadronic gas phase is practically a superposition of an infinity of different hadronic gases, and all information about the interaction is hidden in the mass spectrum $\tau(m^2, b)$ which describes the number of hadrons of baryon number b in a mass interval $\mathrm{d}m^2$ and volume $V \sim m$. When considering strangeness-carrying particles, all we then need to include is the influence of the non-strange hadrons on the *baryon chemical potential* established by the non-strange particles.

The total partition function is approximately multiplicative in these degrees of freedom:

$$\ln Z = \ln Z^{\text{non-strange}} + \ln Z^{\text{strange}} \,. \tag{32.1}$$

For our purposes, i.e., in order to determine the particle abundances, it is sufficient to list the strange particles separately, and we find

$$\ln Z^{\text{strange}}(T,V,\lambda_s,\lambda_q) = C\Big\{2W(x_{\mathrm{K}})(\lambda_s\lambda_q^{-1}+\lambda_s^{-1}\lambda_q) + 2\big[W(x_\Lambda)+3W(x_\Sigma)\big](\lambda_s\lambda_q^2+\lambda_s^{-1}\lambda_q^{-2})\Big\}\,, \tag{32.2}$$

where

$$W(x_i) = \left(\frac{m_i}{T}\right)^2 K_2\left(\frac{m_i}{T}\right)\,. \tag{32.3}$$

We have $C = VT^3/2\pi^2$ for a fully equilibrated state. However, strangeness-creating ($x \to s+\bar{s}$) processes in hot hadronic gas may be too slow (see below) and the total abundance of strange particles may fall short of this value of C expected in *absolute strangeness chemical equilibrium*. On the other hand, strangeness exchange cross-sections are very large (e.g., the $\mathrm{K}^- p$ cross-section is ~ 100 mb in the momentum range of interest), and therefore any momentarily available strangeness will always be distributed among all particles in Eq. (32.2) according to the values of the fugacities $\lambda_q = \lambda_{\mathrm{B}}^{1/3}$ and λ_s. Hence we can speak of a *relative strangeness chemical equilibrium*. Henceforth we omit the subscript ‘B’ when refering to barychemical properties and symbols.

We neglected to write down quantum statistics corrections as well as the multistrange particles Ξ and Ω^-, as our considerations remain valid in this simple approximation [6]. Interactions are effectively included through explicit reference to the baryon number content of the strange particles, as just discussed. Non-strange hadrons influence the strange faction by establishing the value of λ_q at the given temperature and baryon density.

The fugacities λ_s and λ_q as introduced here control the strangeness and the baryon number, respectively. While λ_s counts the strange quark content, the up and down quark content is counted by $\lambda_q = \lambda^{1/3}$.

Using the partition function Eq. (32.2), we calculate for given μ, T, and V the mean strangeness by evaluating

$$\langle n_s - n_{\bar{s}}\rangle = \lambda_s \frac{\partial}{\partial\lambda_s}\ln Z^{\text{strange}}(T,V,\lambda_s,\lambda_q)\,, \tag{32.4}$$

which is the difference between strange and antistrange components. This expression must be equal to zero due to the fact that the strangeness is a conserved quantum number with respect to strong interactions. From this condition, we get

$$\lambda_s = \lambda_q \left|\frac{W(x_{\mathrm{K}})+\lambda^{-1}\big[W(x_\Lambda)+3W(x_\Sigma)\big]}{W(x_{\mathrm{K}})+\lambda\big[W(x_\Lambda)+3W(x_\Sigma)\big]}\right|^{1/2} \equiv \lambda_q F\,, \tag{32.5}$$

a result contrary to intuition: $\lambda_s \neq 1$ for a gas with total $\langle s\rangle = 0$. We notice a strong dependence of F on the baryon number. For large μ, the term with λ^{-1} will tend to zero and the term with λ will dominate the expression for λ_s and F. As a consequence, the particles with fugacity λ_s and strangeness $S=-1$ (note that by convention strange quarks s carry $S=-1$, while strange antiquarks $\bar{s}$ carry $S=1$) are suppressed by a factor F which is always smaller than unity. Conversely, the production of particles which carry the strangeness $S=+1$ will be favoured by F^{-1}. This is a consequence of the presence of nuclear matter: for $\mu=0$, we find $F=1$.

In nuclear collisions, the mutual chemical equilibrium, that is, a proper distribution of strangeness among the strange hadrons, will most likely be achieved. By studying the relative yields, we can exploit this fact and eliminate the absolute normalization C [see Eq. (32.2)] from our considerations. We recall that the value of C is uncertain for several reasons:

i V is unknown.
ii C is strongly (t,r)-dependent, through the spacetime dependence of T.
iii Most importantly, the value $C=VT^3/2\pi^2$ assumes absolute chemical equilibrium, which is not achieved owing to the shortness of the collision.

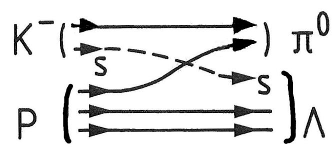
Indeed, we have [see Eq. (32.31) for in plasma strangeness formation and further details and solutions, also see Section 31.2]

$$\frac{\mathrm{d}C}{\mathrm{d}t}=A_{\mathrm{H}}\left[1-\frac{C(t)^2}{C(\infty)^2}\right], \tag{32.6}$$

and the time constant $\tau_{\mathrm{H}}=C(\infty)/2A_{\mathrm{H}}$ for strangeness production in nuclear matter can be estimated to be 10^{-21} s [8]. Thus C does not reach $C(\infty)$ in plasmaless nuclear collisions. If the plasma state is formed, then the relevant $C>C(\infty)$ (since strangeness yielod in plasma is above strangeness yield in hadron gas, see Chapter 31 and below).

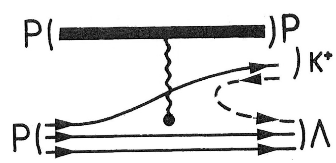
Now, why should we expect *relative* strangeness equilibrium to be reached faster than *absolute* strangeness equilibrium [7]? Consider the strangeness *exchange* interaction

$\mathrm{K^-p} \longrightarrow \Lambda\pi^0$ (32.7)



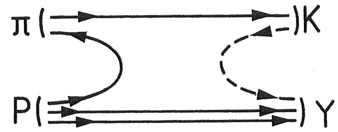
which has a cross-section of about 10 mb at low energies, while the $s\bar{s}$ 'strangeness creating' associate production

$\mathrm{pp} \longrightarrow \mathrm{p\Lambda K^+}$ (32.8)



has a cross-section of less than 0.06 mb, i.e., 150 times smaller. Since the latter reaction is somewhat disfavoured by phase space, consider further the reaction

$$\pi p \longrightarrow YK \qquad (32.9)$$



where Y is any hyperon (strange baryon). This has a cross-section of less than 1 mb, still 10 times weaker than *one* of the *s*-exchange channels in Eq. (32.7). Consequently, I expect the relative strangeness equilibration time to be about ten times shorter than the absolute strangeness equilibration time, namely 10^{-23} s, in hadronic matter of about twice nuclear density.

We now compute the relative strangeness abundances expected from nuclear collisions. Using Eq. (32.5), we find from Eq. (32.2) the grand canonical partition sum for zero average strangeness:

$$\begin{aligned}\ln Z_0^{\text{strange}} = C\Big[&2W(x_{\mathrm{K}})\big(F\lambda_{\mathrm{K}}+F^{-1}\lambda_{\overline{\mathrm{K}}}\big)+2W(x_{\Lambda})\big(F\lambda\lambda_{\Lambda}+F^{-1}\lambda^{-1}\lambda_{\overline{\Lambda}}\big)\\ &+6W(x_{\Sigma})\big(F\lambda\lambda_{\Sigma}+F^{-1}\lambda^{-1}\lambda_{\overline{\Sigma}}\big)\Big]\,, \end{aligned} \qquad (32.10)$$

where, in order to distinguish different hadrons, dummy fugacities λ_i, $i = \mathrm{K}, \overline{\mathrm{K}}, \Lambda, \overline{\Lambda}, \Sigma, \overline{\Sigma}$ have been written. The strange particle multiplicities then follow from

$$\langle n_i\rangle = \lambda_i \frac{\partial}{\partial\lambda_i}\ln Z_0^{\text{strange}}\Big|_{\lambda_i=1}\,. \qquad (32.11)$$

Explicitly, we find (notice that the power of F follows the *s*-quark content):

$$\langle n_{\mathrm{K}^\pm}\rangle = CF^{\mp}W(x_{\mathrm{K}})\,, \qquad (32.12)$$

$$\langle n_{\Lambda/\Sigma^0}\rangle = CF^{+1}W(x_{\Lambda/\Sigma^0})\mathrm{e}^{+\mu/T}\,, \qquad (32.13)$$

$$\langle n_{\overline{\Lambda}/\overline{\Sigma}^0}\rangle = CF^{-1}W(x_{\overline{\Lambda}/\overline{\Sigma}^0})\mathrm{e}^{-\mu/T}\,. \qquad (32.14)$$

In Eq. (32.14) we have indicated that the multiplicity of antihyperons can only be built up if antibaryons are present according to their (small) phase space. This still seems an unlikely proposition, and the statistical approach may be viewed as providing an upper limit on their multiplicity.

From the above equations, we can derive several very instructive conclusions. In Fig. 32.3 we show the ratio $\langle n_{\mathrm{K}^+}\rangle/\langle n_{\mathrm{K}^-}\rangle = F^{-2}$ as a function of the baryon chemical potential μ for several temperatures that can be expected and which are seen experimentally. We see that this particular ratio is a good measure of the baryon chemical potential in the hadronic gas phase, provided that the temperatures are approximately known. The mechanism for this process is as follows: the strangeness exchange reaction of Eq. (32.7) tilts to the left (K^-) or to the right (abundance $F \sim \mathrm{K}^+$), depending on the value of the baryon chemical potential.

In the Fig. 32.4 the long dashed line shows the upper limit for the abundance of $\overline{\Lambda}$ as measured in terms of Λ abundances. Clearly visible is the substantial rel-

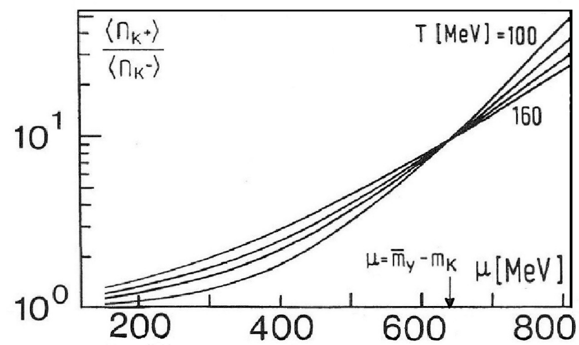


Fig. 32.3 The ratio $\langle n_{K^+}\rangle / \langle n_{K^-}\rangle \equiv F^{-2}$ as a function of the baryon chemical potential μ, for $T = 100, (20), 160$ MeV. The lines cross where $\mu = \overline{m}_Y - m_K$; $\overline{m}_Y$ is the mean hyperon mass.

ative suppression of $\overline{\Lambda}$, in part caused by the baryon chemical potential factor of Eq. (32.14), but also by the strangeness chemistry (factor F^2), as in K^+K^- above. Indeed, the actual relative number of $\overline{\Lambda}$ will be even smaller, since Λ are in relative chemical equilibrium and $\overline{\Lambda}$ in hadron gas are not: the reaction $K^+\overline{p} \rightarrow \overline{\Lambda}\pi^0$, analogue to Eq. (32.7), will be suppressed by *low* $\overline{p}$ abundance. Also indicated in Fig. 32.4 by shading is a rough estimate for the $\overline{\Lambda}$ production in the plasma phase, which suggests that anomalous $\overline{\Lambda}$ abundance may be an interesting feature of highly energetic nuclear collisions [9], for further discussion see Section 32.5 below.

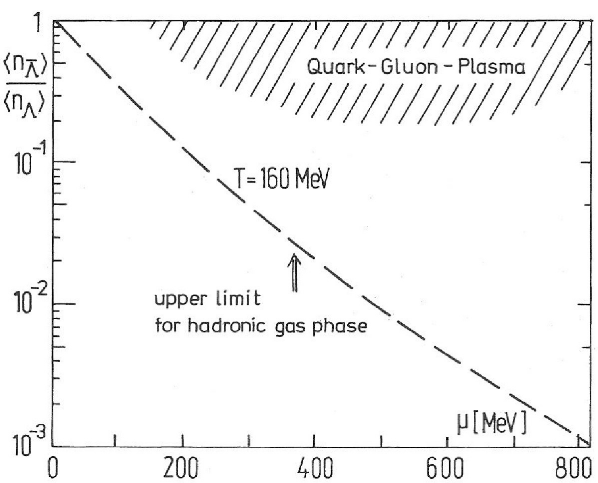


Fig. 32.4 Relative abundance of $\overline{\Lambda}/\Lambda$. The actual yield from the hadronic gas limit may still be 10–100 times smaller than the statistical value shown.

32.3 Quark-Gluon Plasma

From the study of hadronic spectra, as well as from hadron–hadron and hadron–lepton interactions, there has emerged convincing evidence for the description of hadronic structure in terms of quarks [10]. For many purposes it is entirely satisfactory to consider baryons as bound states of three fractionally charged particles, while mesons are quark–antiquark bound states. The Lagrangian of quarks and gluons is very similar to that of electrons and photons, except for the required summations over flavour and colour:

$$L = \overline{\psi}\big[F \cdot (p - gA) - m\big]\,\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\ . \tag{32.15}$$

The flavour-dependent masses m of the quarks are small. For u, d flavours, one estimates $m_{\mathrm{u,d}} \sim$ 5–20 MeV. The strange quark mass is usually chosen at about 150 MeV [11]. The essential new feature of QCD, not easily visible in Eq. (32.15), is the non-linearity of the field strength F in terms of the potentials A. This leads to an attractive glue–glue interaction in select channels *and*, as is believed, requires an improved (non-perturbative) vacuum state in which this interaction is partially diagonalized, providing for a possible perturbative approach.

The energy density of the perturbative vacuum state, defined with respect to the true vacuum state, is by definition a positive quantity, denoted by $\mathscr{B}$. This notion has been introduced originally in the MIT bag model [12], logically, e.g., from a fit to the hadronic spectrum [12], which gives

$$\mathscr{B} = \big[(140\text{–}210)\ \mathrm{MeV}\big]^4 = (50\text{–}250)\ \mathrm{MeV/fm}^3\ . \tag{32.16}$$

The central assumption of the quark bag approach is that, inside a hadron where quarks are found, the true vacuum structure is displaced or destroyed. One can turn this point around: quarks can only propagate in domains of space in which the true vacuum is absent. This statement is a reformulation of the quark confinement problem. Now the remaining difficult problem is to show the incompatibility of quarks with the true vacuum structure. Examples of such behaviour in ordinary physics are easily found; e.g., a light wave is reflected from a mirror surface, magnetic field lines are expelled from superconductors, etc. In this picture of hadronic structure and quark confinement, all *colourless* assemblies of quarks, antiquarks, and gluons can form stationary states, called a quark bag. In particular, all higher combinations of the three-quark baryons (qqq) and quark–antiquark mesons ($q\bar{q}$) form a permitted state.

As the u and d quarks are almost massless inside a bag, they can be produced in pairs, and at moderate internal excitations, i.e., temperatures, many q$\bar{q}$ pairs will be present. Similarly, $s\bar{s}$ pairs will also be produced. We will return to this point at length below. Furthermore, real gluons can be excited and will be included here in our considerations.

Thus, what we are considering here is a *large* quark bag with substantial, equilibrated internal excitation, in which the interactions can be handled (hopefully) perturbatively. In the large volume limit, which as can be shown is valid for baryon number $b \gtrsim 10$, we simply have for the light quarks the partition function of a Fermi gas which, for practically massless u and d quarks can be given analytically (see ref. [1, item (b)] and [13]), even including the effects of interactions through first order in $\alpha_s = g^2/4\pi$:

$$\ln Z_q(\beta,\mu) = \frac{gV}{6\pi^2}\beta^{-3}\left\{\left(1-\frac{2\alpha_s}{\pi}\right)\left[\frac{1}{4}(\mu\beta)^4+\frac{\pi^2}{2}(\mu\beta)^2\right]+\left(1-\frac{50}{21}\frac{\alpha_s}{\pi}\right)\frac{7\pi^4}{60}\right\} . \tag{32.17}$$

Similarly, the glue is a Bose gas:

$$\ln Z_g(\beta,\lambda) = V\frac{8\pi^2}{45}\beta^{-3}\left(1-\frac{15}{4}\frac{\alpha_s}{\pi}\right) , \tag{32.18}$$

while the term associated with the difference to the true vacuum, the bag term, is

$$\ln Z_{\text{bag}} = -\mathscr{B}V\beta . \tag{32.19}$$

It leads to the required positive energy density $\mathscr{B}$ within the volume occupied by the coloured quarks and gluons and to a negative pressure on the surface of this region. At this stage, this term is entirely phenomenological, as discussed above. The equations of state for the quark-gluon plasma are easily obtained by differentiating

$$\ln Z = \ln Z_q + \ln Z_g + \ln Z_{\text{vac}} , \tag{32.20}$$

with respect to β, μ, and V.

An assembly of quarks in a bag will assume a geometric shape and size such as to make the total energy $E(V,b,S)$ as small as possible at fixed given baryon number and fixed total entropy S. Instead of just considering one bag we may, in order to be able to use the methods of statistical physics, use the microcanonical ensemble. We find from the first law of thermodynamics, viz.,

$$\mathrm{d}E = -P\mathrm{d}V + T\mathrm{d}S + \mu\mathrm{d}b , \tag{32.21}$$

that

$$P = -\frac{\partial E(V,b,S)}{\partial V} . \tag{32.22}$$

We observe that the stable configuration of a single bag, viz., $\partial E/\partial V = 0$, corresponds to the configuration with vanishing pressure P in the microcanonical ensemble. Rather than work in the microcanonical ensemble with fixed b and S, we exploit the advantages of the grand canonical ensemble and consider P as a function of μ and T :

$$P = -\frac{\partial}{\partial V}\left[T\ln Z(\mu,T,V)\right] , \tag{32.23}$$

with the result

$$P = \frac{1}{3}(\varepsilon - 4\mathscr{B}) , \tag{32.24}$$

where ε is the energy density:

$$\varepsilon = \frac{6}{\pi^2}\left\{\left(1-\frac{2\alpha_s}{\pi}\right)\left[\frac{1}{4}\left(\frac{\mu}{3}\right)^4+\frac{1}{2}\left(\frac{\mu}{3}\right)^2(\pi T)^2\right]+\left(1-\frac{50}{21}\frac{\alpha_s}{\pi}\right)\frac{7}{60}(\pi T)^4\right\} + \frac{8}{15\pi^2}(\pi T)^4\left(1-\frac{15}{4}\frac{\alpha_s}{\pi}\right)+B\,. \tag{32.25}$$

In Eq. (32.24), we have used the relativistic relation between the quark and gluon energy density and pressure:

$$P_q = \frac{1}{3}\varepsilon_q\,,\qquad P_g = \frac{1}{3}\varepsilon_g\,. \tag{32.26}$$

From Eq. (32.24), it follows that, when the pressure vanishes in a static configuration, the energy density is $4\mathscr{B}$, independently of the values of μ and T which fix the line $P = 0$. We note that, in both quarks and gluons, the interaction conspires to *reduce* the effective available number of degrees of freedom. At $\alpha_s = 0$, $\mu = 0$, we find the handy relation

$$\varepsilon_q + \varepsilon_g = \left(\frac{T}{160\,\mathrm{MeV}}\right)^4\left[\frac{\mathrm{GeV}}{\mathrm{fm}^3}\right]\,. \tag{32.27}$$

It is important to appreciate how much entropy must be created to reach the plasma state. From Eq. (32.20), we find for the entropy density $\mathscr{S}$ and the baryon density ν:

$$\mathscr{S} = \frac{2}{\pi}\left(1-\frac{2\alpha_s}{\pi}\right)\left(\frac{\mu}{3}\right)^2\pi T+\frac{14}{15\pi}\left(1-\frac{50}{21}\frac{\alpha_s}{\pi}\right)(\pi T)^3+\frac{32}{45\pi}\left(1-\frac{15}{4}\frac{\alpha_s}{\pi}\right)(\pi T)^3\,, \tag{32.28}$$

$$\nu = \frac{2}{3\pi^2}\left\{\left(1-\frac{2\alpha_s}{\pi}\right)\left[\left(\frac{\mu}{3}\right)^3+\frac{\mu}{3}(\pi T)^2\right]\right\}\,, \tag{32.29}$$

which leads for $\mu/3 = \mu_q < \pi T$ to the following expressions for the entropy per baryon [including the gluonic entropy second T^3 term in Eq. (32.28)]:

$$\frac{\mathscr{S}}{\nu} \approx \frac{37}{15}\pi^2\frac{T}{\mu_q} \quad \stackrel{T\sim\mu_q}{\longleftrightarrow} \quad 25\,! \tag{32.30}$$

As this simple estimate shows, plasma events are extremely entropy-rich, i.e., they contain very high particle multiplicity. In order to estimate the particle multiplicity, one may simply divide the total entropy created in the collision by the entropy per particle for massless black body radiation, which is $S/n = 4$. This suggests that, at $T \sim \mu_q$, there are roughly six pions per baryon.

32.4 Strange Quarks in Plasma

In lowest order in perturbative QCD, $s\bar{s}$ quark pairs can be created by gluon fusion processes, Fig. 31.2a,b,c; and by annihilation of light quark-antiquark pairs, see Fig. 31.2d. The averaged total cross-sections for these processes were calculated by Brian Combridge [14]. *Note that in this book the thermal in quark-gluon plasma strangness production rates were evaluated in Section 31.2 closely following the original presentation in Ref. [15] and thus we skip much of the original presentation.*

The loss term of the strangeness population is proportional to the square of the density n_s of strange and antistrange quarks. $s\bar{s}$ pair annihilations proceeds via the two-gluon channel, quark-antiquark channel, and occasionally through γG final states [16]. With $n_s(\infty)$ being the saturation density at large times, the following differential equation determines n_s as a function of time [15]:

$$\frac{\mathrm{d}n_s}{\mathrm{d}t} \approx A\left\{1-\left[\frac{n_s(t)}{n_s(\infty)}\right]^2\right\} . \tag{32.31}$$

Thus we find

$$n_s(t) = n_s(\infty)\frac{\tanh(t/2\tau)+\frac{n_s(0)}{n_s(\infty)}}{1+\frac{n_s(0)}{n_s(\infty)}\tanh(t/2\tau)} , \qquad \tau = \frac{n_s(\infty)}{2A} . \tag{32.32}$$

For $\alpha_s \sim 0.6$ and $M \sim T$, we find, see Section 31.2 that $\tau \sim 4\times 10^{-23}$ s. τ falls off rapidly with increasing temperature. Figure 32.5 shows the approach of $n_s(t)$, normalized with baryon density, to the fully saturated phase space as a function of time. We note the high abundance of strangeness relative to baryon number seen in Fig. 32.5 – here, baryon number was computed assuming $T \sim \mu_q = \mu/3$ [see Eq. (32.29)]. These two facts, namely:

1. high relative strangeness abundance in plasma,
2. practical saturation of available phase space,

have led me to suggest the observation of strangeness as a possible signal of quark-gluon plasma [9].

There are two elements in point (1) above: firstly, strangeness in the quark-gluon phase is practically as abundant as the anti-light quarks $\bar{\mathrm{u}} = \bar{\mathrm{d}} = \bar{q}$, since both phase spaces have similar suppression factors, see Section 31.1. Note that the chemical potential of quarks suppresses the $\bar{q}$ density. This phenomenon reflects on the chemical equilibrium between $q\bar{q}$ and the presence of a light quark density associated with the net baryon number. Secondly, strangeness in the plasma phase is more abundant than in the hadronic gas phase (even if the latter phase space is saturated) when compared at the *same* temperature and baryon chemical potential in the phase transition region. The rationale for the comparison at fixed thermodynamic variables, rather than at fixed values of microcanonical variables such as energy density and baryon density, is outlined in the next section. I record here only that the abundance

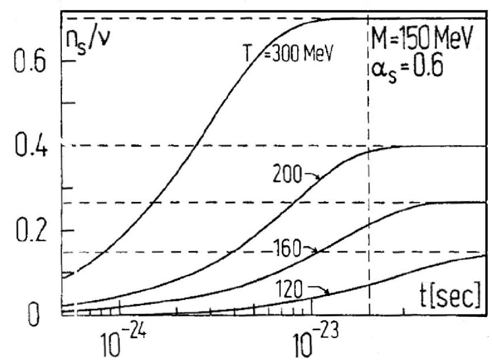


Fig. 32.5 Time evolution of the strange quark to baryon number abundance in the plasma for various temperatures $T \sim \mu_q = \mu/3$. $M = 150$ MeV, $\alpha_s = 0.6$.

of strangeness in the plasma is well above that in the hadronic gas phase space (by factors 1–6, see Fig. 31.1) and the two become equal only when the baryon chemical potential μ is so large that abundant production of hyperons becomes possible. This requires a hadronic phase at an energy density of 5–10 GeV/fm^3.

32.5 How to Discover the Quark–Gluon Plasma

Here only the role of the strange particles in the anticipated discovery will be discussed. My intention is to show that, under different possible transition scenarios, characteristic anomalous strange particle patterns emerge. Examples presented are intended to provide some guidance to future experiments and are not presented here in order to imply any particular preference for a reaction channel. I begin with a discussion of the observable quantities.

The temperature and chemical potential associated with the hot and dense phase of nuclear collision can be connected with the observed particle spectra, and, as discussed here, particle abundances. The last grand canonical variable – the volume – can be estimated from particle interferences. Thus, it is possible to use these measured variables, even if their precise values are dependent on a particular interpretational model, to uncover possible rapid changes in a particular observable. In other words, instead of considering a particular particle multiplicity as a function of the collision energy $\sqrt{s}$, I would consider it as a function of, e.g., mean transverse momentum $\langle p_\perp \rangle$, which is a continuous function of the temperature (which is in turn continuous across any phase transition boundary).

To avoid possible misunderstanding of what I want to say, here I consider the (difficult) observation of the width of the K$^+$ two-particle correlation function in momentum space as a function of the average K$^+$ transverse momentum obtained at given $\sqrt{s}$. Most of K$^+$ would originate from the plasma region, which, when it is created, is relatively small, leading to a comparatively large width. (Here I have assumed a first order phase transition with substantial increase in volume as matter changes from plasma to gas.) If, however, the plasma state were not formed, K^+ originating from the entire hot hadronic gas domain would contribute a relatively large volume which would be seen; thus the width of the two-particle correlation function would be small. Thus, a first order phase transition implies a jump in the K^+ correlation width as a function of increasing $\langle p_\perp \rangle_{K^+}$, as determined in the same experiment, varying $\sqrt{s}$.

From this example emerges the general strategy of my approach: search for possible discontinuities in observables derived from discontinuous quantities (such as volume, particle abundances, etc.) as a function of quantities measured experimentally and related to thermodynamic variables always continuous at the phase transition: temperature, chemical potentials, and pressure. This strategy, of course, can only be followed if, as stated in the first sentence of this report, approximate local thermodynamic equilibrium is also established.

Strangeness seems to be particularly useful for plasma diagnosis, because its characteristic time for chemical equilibration is of the same order of magnitude as the expected lifetime of the plasma: $\tau \sim 1\text{–}3 \times 10^{-23}$ s. This means that we are dominantly creating strangeness in the zone where the plasma reaches its hottest stage – freezing over the abundance somewhat as the plasma cools down. However, the essential effect is that the strangeness abundance in the plasma is greater, by a factor of about 30, than that expected in the hadronic gas phase at the same values of μ, T. Before carrying this further, let us note that, in order for strangeness to disappear partially during the phase transition, we must have a *slow* evolution, with time constants of $\sim 10^{-22}$ s. But even so, we would end up with strangeness-saturated phase space in the hadronic gas phase, i.e., roughly ten times more strangeness than otherwise expected. For similar reasons, i.e., in view of the rather long strangeness production time constants in the hadronic gas phase, strangeness abundance survives practically unscathed in this final part of the hadronization as well. Facit:

> If a phase transition to the plasma state has occurred, then on return to the hadron phase, there will be most likely significantly more strange particles around than there would be (at this T and μ) if the hadron gas phase had never been left.

In my opinion, the simplest observable proportional to the strange particle multiplicity is the rate of V-events from the decay of strange baryons (e.g., Λ) and mesons (e.g., K_s) into two charged particles. Observations of this rate require a visual detector, e.g., a streamer chamber. To estimate the multiplicity of V-events, I reduce the total strangeness created in the collision by a factor 1/3 to select only neutral hadrons and another factor 1/2 for charged decay channels. We thus have

$$\langle n_V \rangle \approx \frac{1}{6} \frac{\langle s \rangle + \langle \bar{s} \rangle}{\langle b \rangle} \langle b \rangle \sim \frac{\langle b \rangle}{15} \,, \tag{32.33}$$

where I have taken $\langle s \rangle / \langle b \rangle \sim 0.2$ (see Fig. 32.5). Thus for events with a large baryon number participation, we can expect to have several V's per collision, which is 100–

1000 times above current observation for Ar-KCl collision at 1.8 GeV/Nuc kinetic energy [17].

Due to the high $\bar{s}$ abundance, we may further expect an enrichment of strange antibaryon abundances [9]. I would like to emphasize here $\bar{s}\bar{s}\bar{q}$ states (anticascades) created by the accidental coagulation of two $\bar{s}$ quarks helped by a gluon $\to \bar{q}$ reaction. Ultimately, the $\bar{s}\bar{s}\bar{q}$ states become $\bar{s}\bar{q}\bar{q}$, either through an $\bar{s}$ exchange reaction in the gas phase or via a weak interaction much, much later. However, half of the $\bar{s}\bar{q}\bar{q}$ states are then visible as $\overline{\Lambda}$ decays in a visual detector. This anomaly in the apparent $\overline{\Lambda}$ abundance is further enhanced by relating it to the decreased abundance of antiprotons, as described above.

Unexpected behaviour of the plasma–gas phase transition can greatly influence the channels in which strangeness is found. For example, in an extremely particle-dense plasma, the produced $s\bar{s}$ pairs may stay near to each other – if a transition occurs without any dilution of the density, then I would expect a large abundance of $\phi(1020)$ $s\bar{s}$ mesons, easily detected through their partial decay mode (1/4%) to a $\mu^+\mu^-$ pair.

Contrary behaviour will be recorded if the plasma is cool at the phase transition, and the transition proceeds slowly – major coagulation of strange quarks can then be expected with the formation of sss and $\bar{s}\bar{s}\bar{s}$ baryons and in general $(s)^{3n}$ clusters. Carrying this even further, supercooled plasma may become 'strange' nuclear (quark) matter [18]. Again, visual detectors will be extremely successful here, showing substantial decay cascades of the same heavy fragment.

In closing this discussion, I would like to give warning about the pions. From the equations of state of the plasma, we have deduced in Sect. 32.3 a very high specific entropy per baryon. This entropy can only increase in the phase transition and it leads to very high pion multiplicity in nuclear collisions, probably created through pion radiation from the plasma [5] and sequential decays. Hence by relating anything to the pion multiplicity, e.g., considering K/π ratios, we dilute the signal from the plasma. Furthermore, pions are not at all characteristic for the plasma; they are simply indicating high entropy created in the collision. However, we note that the K/π ratio *can* show substantial deviations from values known in *pp* collisions – but the interpretations of this phenomenon will be difficult.

It is important to appreciate that the experiments discussed above would certainly be quite complementary to the measurements utilizing electromagnetically interacting probes, e.g., dileptons, direct photons. Strangeness-based measurements have the advantage that they have much higher counting rates than those recording electromagnetic particles.

Acknowledgements

I would like to thank R. Hagedorn, B. Müller, and P. Koch for fruitful and stimulating discussions, and R. Hagedorn for a thorough criticism of this manuscript. Thie work was in part supported by Deutsche Dorschungsgemeinschaft.

References

1. An incomplete list of quark-gluon plasma papers includes:
(a) B.A. Freedman, L.C. McLerran: Phys. Rev. D **16**, 1169 (1977);
(b) S.A. Chin: Phys. Lett. B **78**, 552 (1978);
(c) P.D. Morley, M.B. Kislinger: Phys. Rep. **51**, 63 (1979);
(d) J.I. Kapusta: Nucl. Phys. B **148**, 461 (1978);
(e) O.K. Kalashnikov, V.V. Kilmov: Phys. Lett. B **88**, 328 (1979);
(f) E.V. Shuryak: Phys. Lett. B **81**, 65 (1979); also Phys. Rep. **61**, 71 (1980);
(g) J. Rafelski, R. Hagedorn: From hadron gas to quark matter II. In *Thermodynamics of Quarks and Hadrons*, ed. by H. Satz, North Holland, Amsterdam (1981);
(h) J. Rafelski, M. Danos: *Perspectives in high energy nuclear collisions,* NBS-IR-83-2725; US Department of Commerce, National Technical Information Service (NTIS) Accession Number PB83-223982 (1983)
(i) H. Satz: Phys. Rep. **88**, 321 (1982)
2. These ideas originate in Hagedorn's statistical bootstrap theory:
(a) R. Hagedorn: Suppl. Nuovo Cimento **3**, 147 (1964); Nuovo Cimento **6**, 311 (1968);
(b) R. Hagedorn: How to Deal with Relativistic Heavy Ion Collisions – 1980, Chapter 26 in this volume
3. R. Hagedorn, J. Rafelski: Phys. Lett. B **97**, 136 (1980); see also [1, item (g)]
4. (a) R. Hagedorn: Z. Phys. C **17**, 265 (1983)see also Chapter 24 in this volume;
(b) R. Hagedorn, I. Montvay, J. Rafelski: *Hadronic Matter at Extreme Energy Density*, ed. by N. Cabibbo, Plenum Press, NY (1980); and J. Rafelski and R. Hagedorn: Thermodynamics of hot nuclear matter in the statistical bootstrap model – 1978, Chapter 23 in this volume
5. J. Rafelski, M. Danos: Pion radiation by hot quark-gluon plasma, CERN preprint TH 3607 (1983); M. Danos, J. Rafelski: Phys. Rev. D **27**, 671 (1983)
6. P. Koch: Diploma thesis, Universität Frankfurt (1983)
7. P. Koch, J. Rafelski, W. Greiner: Phys. Lett. B **123**, 151 (1983)
8. A.Z. Mekjian: Nucl. Phys. B **384**, 492 (1982)
9. J. Rafelski: Extreme states of nuclear matter – 1980, Chaper 27 in this volume
10. See, e.g., S. Gasiorowicz, J.L. Rosner: Hadron spectra and quarks, Am. J. Phys. **49**, 954 (1981)
11. (a) P. Langacker, H. Pagels: Phys. Rev. D **19**, 2070 (1979) and references therein;
(b) S. Narison, N. Paver, E. de Rafael, D. Treleani: Nucl. Phys. B **212**, 365 (1983)
12. (a) A. Chodos, R.L. Jaffe, K. Johnson, C.B. Thorn, V.F. Weisskopf: Phys. Rev. D **9**, 3471 (1974);
(b) K. Johnson: Acta Phys. Polon. B **6**, 865 (1975);
(c) T. De Grand, R.L. Jaffe, K. Johnson, J. Kiskis: Phys. Rev. D **12**, 2060 (1975)
13. H.Th. Elze, W. Greiner, J. Rafelski: J. Phys. G **6**, L149 (1980); J. Rafelski, H.Th. Elze, R. Hagedorn: Hot hadronic and quark matter in $\overline{\mathrm{p}}$ annihilation on nuclei, CERN preprint TH 2912 (1980), in *Proceedings of Fifth European Symposium on Nucleon–Antinucleon Interactions*, Bressanone, 1980, CLEUP, Padua (1980)
14. B.L. Combridge: Nucl. Phys. B **151**, 429 (1979)
15. J. Rafelski, B. Müller: Phys. Rev. Lett. **48**, 1066 (1982)
16. G. Staadt: Diploma thesis, Universität Frankfurt (1983), [later published in: Phys.Rev. D **33** 66 (1986)]
17. J.W. Harris, A. Sandoval, R. Stock, et al.: Phys. Rev. Lett. **47**, 229 (1981)
18. S.A. Chin, A.K. Kerman: Phys. Rev. Lett. **43**, 1292 (1979)

Chapter 33
Melting Hadrons, Boiling Quarks

Johann Rafelski

Abstract The events presented in this book happened more than three decades ago. At that time we did not know how long it would take for the experimental program to come to be, and to make the discovery happen. Looking back, and looking at the present I can say that a vast majority of physicists studying relativistic heavy ion collisions agree today that the new quark-gluon plasma phase has been discovered and the discovery of more than a decade ago has been confirmed by the more recent results obtained at LHC. Given this circumstance, as a final word, I answer a few pertinent questions which I have heard often as related directly to the contents of this book – there are many other questions each answer generates.

33.1 The Concepts: Hadron Side

What is Hagedorn temperature?

Hagedorn temperature $T_{\rm H} \simeq 1.8 \times 10^{12}$ K is the maximum temperature at which matter can exist in the usual form. At $T > T_{\rm H}$ all individual material particles dissolve into the quark-gluon plasma. This transformation can occur at a lower temperature in the presence of dense nuclear matter. At densities an order of magnitude greater than the nuclear density this transformation probably can occur near to, or even at, zero temperature.

The value of $T_{\rm H}$ is measured by the way of the exponential growth of the hadron mass spectrum,

$$\rho(m) \propto m^{-a} \exp(m/T_{\rm H}). \tag{33.1}$$

$T_{\rm H}$ is thus uniquely defined independent of the question, if the conversion of matter into quark-gluon plasma is a sharp boundary, or a continuous transformation. The index 'a' of the pre-exponential factor determines the nature of the transformation, see Table 23.1.

Department of Physics, The University of Arizona, Tucson, AZ 85721, USA

$T_{\rm H}$ is not a maximum temperature in the Universe. A further heating of the quark-gluon plasma 'liquid' can and will continue. We understand today $T_{\rm H}$ as the boiling point of a hot gas made of hadrons, i.e. Hadron Gas (HG), dissolving into the quark-gluon plasma (QGP), a liquid phase made of Debye screened color-ionic quarks and gluons.

What is the Statistical Bootstrap Model?

SBM is based on the hypothesis that the exponential in particle mass growth of the density of hadron states generates a state of matter in which practically every strongly interacting particle produced is distinguishable – one way to think about this situation is to omit in the statistical evaluation the Boltzmann pre-factor $1/n!$. The SBM relies on the model hypothesis of which the most prominent is, see Eq. (20.4)

$$\frac{\log\rho(m)}{\log\sigma(m)} \underset{m\to\infty}{\longrightarrow} 1\,, \tag{33.2}$$

where $\sigma(m)$ is the density of states of the system, from which the shape of the exponential mass spectrum $\rho(m)$, Eq. (33.1) emerges. It is important to note the relation to Eq. (33.1) which thus characterizes $\sigma(m)$, and keep in mind that Hagedorn temperature and SBM are two separate ideas.

The pre-exponential power index a in Eq. (33.1) is dependent on additional technical details, see Hagedorn's discussion in Chapter 25, below Eq. (25.16) on page 286. By 1972 in a Lorentz-covariant SBM a value $a=3$ emerges, replacing the value $a=2.5$ that Hagedorn considered in 1965, see Chapter 20. The compressibility of the finite size hadron fireballs embedded in dense matter plays an important role producing other values of a discussed in Chapter 21: For incompressible hadrons of finite size one finds $a=7/2$, while allowing compressibility leads to $a>7/2$. How the value of a controls the singular behavior near phase boundary is shown in table 23.1 on page 253.

SBM can evolve with our understanding of the strongly interacting matter and provide a deeper understanding of the results of lattice-QCD: for example introducing strange quark related scale into characterization of the hadron volume, or making baryons more compressible as compared to mesons in consideration of the interaction scale of QCD. In this way one can embrace in detail the current emerging lattice-QCD paradigm predicting a critical point at finite baryon density and a phase transition for higher baryon density.

What is hadron resonance gas?

While SBM produces the shape of the mass spectrum $\rho(m)$, this is a description that includes averaging of the hadron spectrum features. This can be avoided: given the availability of computers of ever greater power, it is opportune to employ an experimentally known spectral composition including all observed hadrons as explicit partial fractions. This is the Hadron (equivalently, Hagedorn) Resonance Gas (HRG), represented by a discrete sum, see Section 7.4.

The emphasis here is on ‘resonances’, reminding us that all hadrons, stable and unstable, must be included. Hagedorn went to great length to justify how the inclusion of unstable hadrons, i.e. resonances, accounts for the dominant part of the interaction between all hadronic particles. His theoretical insight can be tested today by comparing HRG results with lattice-QCD. One finds good agreement, see Chapters 7 and 21: within 10% precision we have ab-initio confirmation that Hagedorn developed a properly working model of strongly interacting particles for $T < T_{\rm H}$. I believe, based on my own tedious study of the experimental particle yields and fireball properties within the SHM (see next), that the experimentally available discrete hadron mass spectrum is sufficient to achieve accurate description of physical phenomena for $T < 145\,\mathrm{MeV}$ at a precision level that exceeds the numerical precision of lattice-QCD results.

Still, there is something that can be done better: not all ‘high’ mass hadron resonances are known, with the current experimental limit implying that ‘high’ means about twice the proton mass. The physical relevance of such experimentally undiscovered Hagedorn states depends on the temperature of the system. Thus for higher values of T in the direct vicinity of $T_{\rm H}$, such additional heavy resonances could play a significant role in the comparison of lattice results with the HRG model.

What is the statistical hadronization model?

The statistical hadronization model (SHM) was invented to characterize, using Fermi-Hagedorn statistical particle evaporation methods, how a blob of primordial matter falls apart into individual hadrons. The SHM is in essence a complete and careful implementation of the Fermi-Hagedorn picture of particle production using the observed discrete hadron mass spectrum.

The SHM analysis relies on the hypothesis that a hot fireball will ‘hadronize’, populating all available phase space cell proportional to their respective size. This is the Fermi hypothesis which is now implemented using the semi-grand-canonical Hagedorn method. In the present day implementation all known exact (baryon number for example) and approximate (entropy) conservation laws can be respected.

This analysis of particle production allows the inference of both the statistical canonical parameters as well as the extensive and intensive microcanonical physical properties of the fireball source. Importantly, among the observables we note the entropy content and strangeness content of the emerging multiplicity of hadronic particles. These properties originate at a far earlier fireball evolution stage compared to the hadronization process itself. Therefore performing a SHM analysis of all hadrons produced we obtain a deeper look into the history of the expanding QGP fireball.

33.2 The Concepts: Quark Side

Why are quarks confined?

Quark confinement can be seen as an expression of the incompatibility of quark and gluon color-electrical fields with the vacuum structure. This feature was inherent in the work on quark confinement by Ken Wilson [1]. A clear statement of how this mechanism works, with a description of confinement of color charge, is seen for the first time in the September 28, 1979 lecture by T.D. Lee [2]. Quark confinement within a bound state with other quarks is explained as result of a transport property of the vacuum state surrounding us, and is not a direct consequence of the nature of an inter-quark force.

This understanding of confinement is convenient for the understanding of the quark-gluon plasma as a domain in space in which this vacuum structure is dissolved, and chromo-electric field lines can exist.

With their color field lines expelled from the vacuum, quarks can only exist in colorless cluster states: mesons $q\bar{q}$ and baryons qqq (and antibaryons $\overline{qqq}$). These are bubbles with the electric field lines contained in small space domain.

To make the mechanism of confinement in lattice-QCD visible we can pose the question; what is the interaction energy between a heavy pair of a quark and an antiquark? Such a particle pair interacts in terms of color-Coulomb force. Such a force can be for $T > T_{\mathrm{H}}$ similar to the normal electric-Coulomb $1/r$ force when the pair is in a global colorless state. For $T < T_{\mathrm{H}}$ the color field lines are, however, confined. When we place heavy quarks relatively far apart, the field lines are according to above squeezed into a cigar-like shape where the field occupied volume grows linearly with long axis of the cigar. The expected heavy quark potential will therefore have more linear than Coulomb $1/r$ character. Potential shape can be studied as a function of quark separation and of temperature, demonstrating how the potential properties change when deconfinement sets in [3, 4].

What is the quark bag model?

A popular model implementation of quark-confinement is the so-called quark-bag model where by imposing boundary conditions we find quark wave functions in a localized bound state. This model works akin to the localization of quantum states in an infinite square-well potential. Since there have been quite a few variants of the quark-bag model I present in qualitative terms the main common ideas. A key new ingredient is that the domain occupied by quarks and their chromo-electrical fields has a higher energy density called bag constant $\mathscr{B}$: the deconfined state is the state of higher energy compared to the conventional confining vacuum state. Variants of such a model including a contributing "surface energy" are not viable phenomenologically.

The physical volume size V_h of a deconfined domain containing quarks forming a hadron 'h' arises from the balance of the vacuum energy $V_\mathrm{h}\mathscr{B}$ with the quark energy $\propto n/V_\mathrm{h}^{1/3}$ inside the bag. n is the number of valance quarks and antiquarks. Optimization of the total energy reveals an optimum size for each hadron $V_\mathrm{h} \propto \mathscr{B}^{-3/4}$. The larger is $\mathscr{B}$, the smaller and more compressed are hadron volume bubbles. In such a simplified model with just one scale parameter $\mathscr{B}$, the mass of each hadron can be written as being proportional to the particle volume: $M_\mathrm{h} = 4V_\mathrm{h}\mathscr{B}$. Knowledge of the hadronic size of the proton (a ball of radius 1fm) allows an estimate of $\mathscr{B}$.

The growth in energy of the quark bound state with the volume occupied by the field means that as a 'kicked' quark attempts an exit, as described in above discussion on confinement, pulling its field lines in a cigar-shaped geometry. As result there is a linearly rising attachment energy as function of the length of the cigar-shaped field lines. Ultimately, one can expect that the field line connection snaps, producing a quark-antiquark pair. Instead of a free 'kicked' quark, a colorless meson escapes from the colorless bound state that remains colorless. The field lines connecting the quark to its source, along with the modification of the vacuum that arise, are called a 'QCD string'.

This explanation of quark confinement as a confinement of the color-electrical field lines takes us to the question: how can there be a vacuum structure that expels color-electric field lines? Can we invoke as a justification the present day results of lattice-QCD computations? If you attempt a search on-line you will be mostly disappointed. This is so because lattice-QCD produces values of static observables, and not interpretation of confinement in terms of moving quarks and dynamics of the color-electric field lines.

What does quark-gluon plasma mean precisely?

Quark-Gluon Plasma (QGP) in the contemporary use of the language is a nearly free gas of quarks and gluons at thermal (kinetic) and close to chemical (abundance) equilibrium. Even today not everybody likes this 'QGP' name, as an example see Chapter 9. Léon Van Hove wrote a report in which he refers in title to "QGP, also called Quark Matter".

Let us look within lattice-QCD at strongly interacting matter in the domain of temperature which is large compared to Hagedorn temperature, yet not beyond the range of experiments that can be conducted today, $T \simeq 4 \times T_\mathrm{H}$. We look at results of references seen in Chapters 7 and 21, such as the behavior of the pressure which follows the Stefan-Boltzmann law,

$$P_\mathrm{QCD} = ST^4, \qquad S = g_\mathrm{QCD}\frac{\pi^2}{90} \tag{33.3}$$

S is the QCD Stefan-Boltzmann constant, and g_QCD describes all effectively massless 'radiation' particles which can be excited at temperature T. g_QCD includes

$2_s \times 8_c = 16$ gluons and $2_s \times 2_p \times 3_c \times 3_f \times 7/8|_{\rm F} = 31.5$ u,d,s-quarks, where indices stand for: s=spin (=2), c=color (=3, or =8), p-particle and antiparticle (=2), f-flavor (=3), and F-Fermi as compared to Bose particle reference in the Stefan-Boltzmann constant S.

Based on perturbative thermal, QCD properties, we expect and find a 10–20% reduction in $g_{\rm QCD} \simeq 40$ instead of 47.5 due to effects of interaction in $\mathscr{O}(\alpha_s/\pi)$. α_s is the QCD energy scale dependent coupling constant, which in the domain of T we consider is about $\alpha_s \simeq 0.5$.

Given that lattice-QCD is a correct description of strongly interacting particles, we can conclude that the state of strongly interacting matter at $T \simeq 4T_{\rm H}$ is composed of the expected number of nearly free quarks and gluons, and the count of these particles emerges exactly as expected in results of lattice-QCD. We can say that in this numerical work, QGP emerges to be *the phase of strongly interacting matter which manifests its physical properties in terms of nearly free dynamics of practically massless gluons and quarks.* The 'practically massless' is inserted also for gluons as we must remember that in dense matter all color charged particles including gluons acquire an effective in medium mass.

As temperature decreases towards and below $T_{\rm H}$, the color charge of quarks and gluons literally freezes, and for $T < 0.8T_{\rm H}$ the properties of strongly interacting matter are now fully characterized by a HRG, see Chapters 7 and 21. As these results of lattice-QCD demonstrate, the modern meaning of "quark-gluon plasma" is a phase of matter comprising color charged particles (gluons and quarks) that can move nearly freely so as to create ambient pressure close to the Stefan-Boltzmann limit. The properties of QGP that we check for are thus:

1. kinetic equilibrium – that is a meaningful definition of temperature;
2. dominance by effectively massless particles assuring that $P \propto T^4$;
3. both quarks and gluons must be present in conditions near chemical (yield) equilibrium with their color charge 'open' so that the count of their number produces the correctly modified Stefan-Boltzmann constant of QCD.

How do we connect this simple result to experiment? The path to measuring P in plasma and for that matter the local energy density ε goes via the dynamics of the expansion of the QGP phase. It is important to note that the smallness of the QCD interaction effects that one sees in the behavior of $P(T)$ indicates that the color-ionic charges are screened; the viscosity entering flow models should be, in relative terms, small. Thus we expect that a QGP blob formed at a high value of $T \gg T_{\rm H}$ will expand in a way similar to a gas of non-interacting quarks and gluons, but with a reduced by interaction value of $g_{\rm QCD}$.

How did the name "QGP" come into use?

Quite often in physics names attached to important insights appear late and even sometimes attribute the discovery to the wrong person. The situation is similar with the naming of hot interacting quark-gluon matter as QGP: we call QGP today what appeared in many early articles under a different name 'quark matter', while yesterday QGP used to denote something else, a Feynman parton gas.

In my memory, the use of "QGP" to describe the strongly interacting quark-gluon interacting thermal equilibrium matter was adopted following the title of a paper by Kalashnikov and Klimov [5] of July 1979. However, let me stress that the work by Kalashnikov-Klimov [5] did *not* invent QGP, neither in the content, and the name already existed

- We see the key results of Kalashnikov-Klimov in a year earlier, July 1978, work of S.A. Chin [6] presented under the name "Hot Quark Matter" *and* including hot gluons and their interaction with quarks and with themselves, which is the important pivotal element missing in many other papers.
- Kalashnikov-Klimov may have borrowed the term from another work, of March 1978, by E. Shuryak [7]. Shuryak at that time also used 'QGP' in his title addressing *pp* collisions as a source of photons, dileptons and charmonium. With time one notices Shuryak's *pp* work cited in the modern *AA* QGP meaning context. This was also done in some of our citations both by Hagedorn and myself.

Why is quark-gluon plasma of interest?

Several fundamental questions come together in the study of the deconfined phase of matter, QGP:

- All agree that QGP was the Big-Bang stuff that filled the Universe before matter formed. The experimental exploration of the QGP properties solidifies our models of the Big Bang when the Universe was younger than 20 microseconds. We learn about the material content of the Universe and what happened near the end of the quark Universe era.
- In relativistic heavy ion collisions the kinetic energy of ions feeds the quark population in the QGP phase. These quarks later turn into material particles. This means that we study experimentally the mechanisms that lead to the conversion of energy into matter. In an as yet unknown way, this could lead to a better understanding of the stability of matter and conversion of matter into energy.
- While as noted above, the mass of quarks is believed to originate in the Higgs field, the mass of nucleons, a 'bag' of three confined quarks is about 40 times larger than the sum of constituent quark masses. Nucleons dominate the mass of matter by a factor 2000. For this reason, the origin of the mass of matter is recognized to be caused by the confinement of quarks, compressed to a relatively small, hadron volume - this confinement mass effect dominates the Higgs effect by a large factor [8, 9]. Therefore, the vacuum structure which causes confine-

ment of color is responsible for the inertia of matter. We can hope to learn how to use this deep insight in the future.

- In the standard model there are three families of particles which duplicate in essence all their properties, except for their mass-generating interaction with the Higgs field. They are thus distinguished only by three different sets of elementary particle masses. At present we do not have a good explanation why this is so.

  There have been few experiments possible to study this situation since in experiments involving elementary particle collisions, we deal with a few if not only one pair of newly created 2nd, or 3rd family at a time. A new situation arises in the QGP formed in relativistic heavy ion collisions. QGP includes a large number of particles from the second family: the strange quarks, and in fact also, the yet heavier charmed quarks; and from third family at the LHC also bottom quarks.

  The new ability to study a large number of these 2nd and 3rd generation particles present together in a different vacuum structure of QGP could help answer the riddle about the meaning and origin of the three particle families. "Could" means that a proposal has not emerged on how to approach this fundamental question.

33.3 Quark-Gluon Plasma and Relativistic Heavy Ion Collisions

How did RHI collisions and QGP come together?

In October 1980, I answered this question as follows, see Chapter 27, Ref. [21]: "The possible formation of quark-gluon plasma in nuclear collisions was first discussed *quantitatively* by S.A. Chin: Phys. Lett. B **78**, 552 (1978); see also N. Cabibbo, G. Parisi: Phys. Lett. B **59**, 67 (1975)."

I now have second thoughts about this answer. The work by Cabibbo and Parisi, though pointing to the need to develop SBM to include melting of hadrons, does not mention or allude to nuclear collisions directly or indirectly. And, the paper by Chin, of July 1978, in its Reference [7] grants the origin of the idea to Chapline and Kerman [10], manuscript of March 1978.

The contents of the never-published Chapline-Kerman's manuscript is qualitative. The authors did not pursue further development of their idea; I note that a year later Chin and Kerman presented the idea of strangelets [11], cold drops of quark matter containing a large strangeness content. This proposal will anchor the resources of the BNL-AGS program for many years. A few years later Chapline considers the possibility of 'warm' high baryon density quark matter being produced in RHI collisions, which the experiments did not confirm.

Here some partial regrets: an ‘idea’ paper equivalent to Ref. [10] introducing bootstrap of hot hadron matter could have been written by Hagedorn and myself in October 1977. But, as already discussed in Chapter 1, Hagedorn would never write such a paper without working out a good model. After 10 months of further effort we wrote a 99 page long paper [12], as well as a shorter version, presented in Chapter 23.

How and what happens when QGP is created in the laboratory?

The reaction path into QGP in some early work has been a line placed in the temperature-baryon density plane, such as the one shown in Fig. 32.2 on page 394, with an arrow pointing from a hot thermal hadron phase into the QGP domain. For RHI collisions capable of forming QGP, such a picture can only apply if a mechanism of entropy production exists at hadron collision level that creates the thermally equilibrated hadron phase. While new particles are formed, this state dissolves into QGP.

This process requires conversion of directed motion energy into locally equilibrated matter. Moreover, the system proceeds via a non-equilibrium stage where neither the particle abundance nor their spectra are close to conditions that are associated with the phase diagram properties. Thus a locally equilibrated matter emerges first amongst quark and gluon degrees of freedom. Presentation in the phase diagram of a RHI collision entrance path into QGP domain is thus not appropriate.

In fact, for very high RHIC and LHC energies all scattering processes occur at quark-gluon (parton) level. Thus there is no connection whatsoever with models of hadron-hadron scattering that sometimes decorate in an explanatory way the *AA* collision process. At much lower energies, near to the presumed threshold of QGP formation, the reaction path at least in part involves hadron based processes described within kinetic non-equilibrium approaches. The question one may wonder about in this case is how Hagedorn could interpret hadron production, introducing his limiting temperature.

“Why the Hadronic Gas Description of Hadronic Reactions Works” is the title of a work suggesting an explanation long ago [13]: it is the formation of nearly equilibrated QGP that is, partonic gas, and the evaporation of hadrons from QGP fireball that produces the near equilibrium hadron particle abundances observed. I believe this is practically the case for all strong interaction reaction processes, including *pp* and *pA* (proton-Nucleus) scattering, aside of *AA* nucleus-nucleus (heavy ion) collisions, all of these differ only in the degree of equilibration that is achieved.

The present consensus about the formation of an equilibrium state characterized by a high entropy density contents in relativistic heavy ion collision at RHIC and LHC is that it is much more likely to be produced in the context of parton collisions. Among the first to address a parton based entropy production quantitatively within a kinetic collision model was Klaus Geiger [14, 15], see Fig. 14.2 on page 112. Klaus built computer cascade models at parton level, and studied thermalization as a collision based process which opens a Pandora box of questions in regard to decoherence of investigated processes. Thus more than 20 years later a search and exploration of fast entropy generating mechanism properly described within QCD continues, see for example Ref. [16].

When and where was QGP discovered?

Both CERN and BNL have held press conferences describing their experimental work. In Fig. 33.1 a screenshot shows how CERN advertised its position in February 2000 to a wider public. The document for scientists agreed to by those representing the seven CERN experiments read: "A common assessment of the collected data leads us to conclude that we now have compelling evidence that a new state of matter has indeed been created, ... The new state of matter found in heavy ion collisions at the SPS features many of the characteristics of the theoretically predicted quark-gluon plasma."

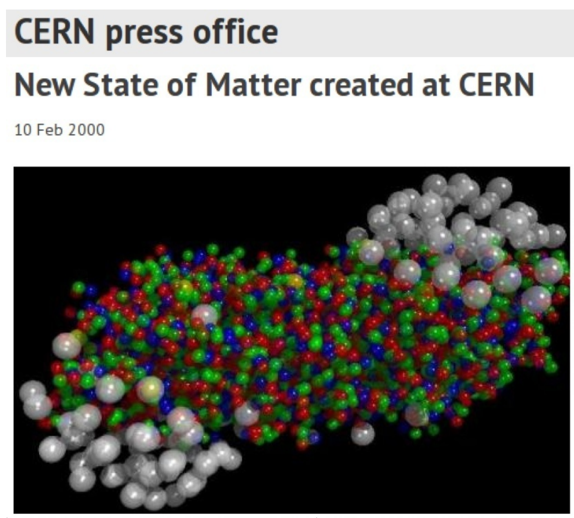


Fig. 33.1 The press release text: "At a special seminar on 10 February 2000, spokespersons from the experiments on CERN's Heavy Ion programme presented compelling evidence for the existence of a new state of matter in which quarks, instead of being bound up into more complex particles such as protons and neutrons, are liberated to roam freely."

At the April 2005 meeting of the American Physical Society, held in Tampa, Florida a press conference took place on Monday, April 18, 9:00 local time. The public announcement read: At RHIC ". . . two beams of gold atoms are smashed together, the goal being to recreate the conditions thought to have prevailed in the universe only a few microseconds after the Big Bang, so that novel forms of nuclear matter can be studied. At this press conference, RHIC scientists will sum up all they have learned from several years of observing the worlds most energetic collisions of atomic nuclei. The four experimental groups operating at RHIC will present a consolidated, surprising, exciting new interpretation of their data." The participants at the conference obtained "Hunting for Quark-Gluon Plasma" report, of which the cover in Fig. 33.2 shows the four BNL experiments, which reported on the QGP physical properties that have been obtained at BNL.

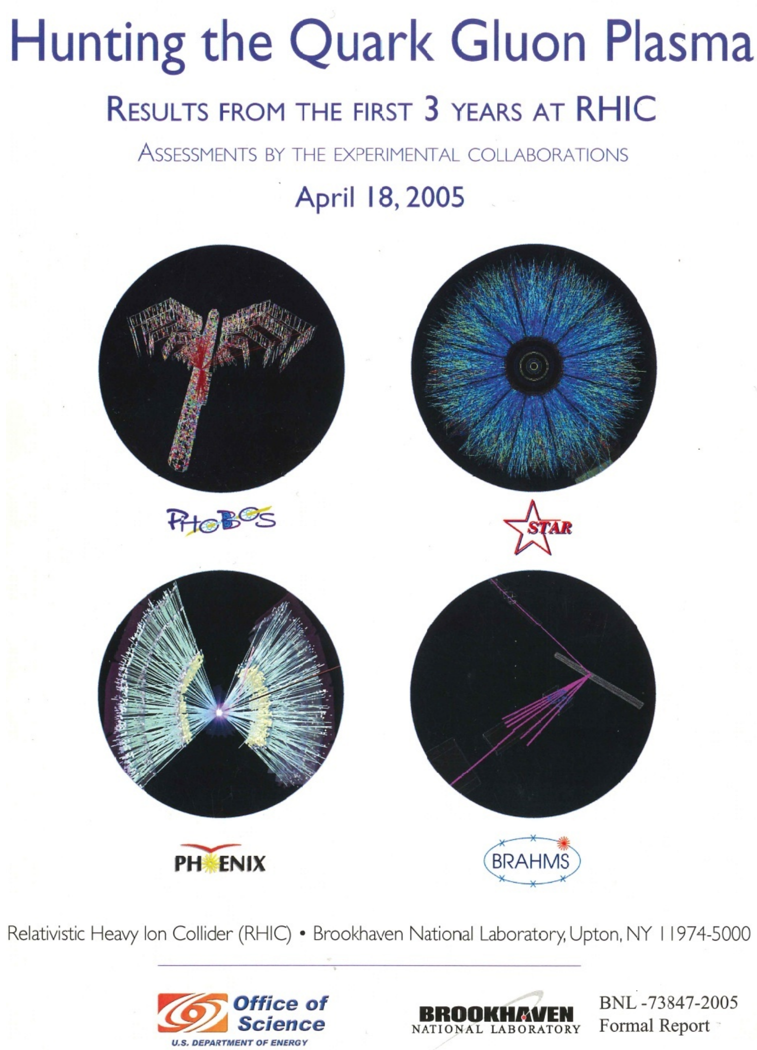


Fig. 33.2 The cover of the BNL-73847-2005 Formal Report prepared by the Brookhaven National Laboratory, on occasion of the RHIC experimental program press conference April 2005. The cover identified the four RHIC experiments.

33.4 Hadrons and Quark-Gluon Plasma

What controls kinetic energy conversion into material particles?

Particles emerging in hadronization of QGP carry entropy. In the temporal sequence of events, entropy contents must increase. Conversely, the final yield of particles produced is thus dependent on how much entropy will be created when heavy ions collide. Most of the entropy production is, when considered in quantitative fashion, related to the process of color deconfinement and thermalization of quarks and gluons in QGP.

Entropy is produced in the processes that occur when partons collide forming dense matter. These mechanisms continue at first when the system expands. The massless light quark and gluon abundances all grow, substantially. Thus at least in the beginning the dense matter fireball explodes in a non-adiabatic fashion, forming additional entropy in the process of the creation of new particle populations, such as strange quark pairs.

Once local thermal and chemical equilibrium is achieved, the explosive flow of QGP, the micro-bang, should be largely adiabatic, not much different from the picture that emerges in the study of the Big-Bang QGP dynamics in the early Universe. The main difference between the big- and micro-bangs is that in the laboratory experiments the time frame in which dense matter exists is so short that the electromagnetic and weakly interacting particles remain far from equilibrium.

In this adiabatic expansion involving dilution of particle density and adiabatic cooling of temperature thermal energy is transferred into the energy of the kinetic flow of matter. This, as well as the potential radiation effects reduce the temperature of QCD matter fireball, ultimately leading to the freezing of quarks and gluons back into hadrons. This latter process is described in the Statistical Hadronization Model described earlier.

All of the entropy produced in this time line of QGP formation and hadronization turns at the end into a hadronic matter-antimatter, meson, particle gas, just as was the case in the the evolution of the early Universe. A remarkable outcome of the QGP formation is that by way of the formation of a large entropy content when breaking color bonds and deconfining quarks and gluons we convert the kinetic energy of the colliding nuclei into abundantly produced entropy that needs to emerge at the end in the form of material particles.

What is special about heavy quarks?

Strangeness

In order to produce the large abundance of strange quark pairs that can be present in QGP, the initial collisions of partons do not suffice. One can see this by considering

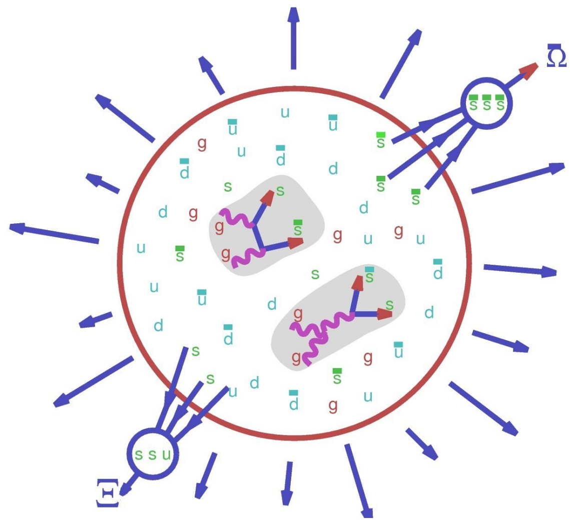


Fig. 33.3 Multistrange (anti)baryons as signature of QGP, see text for further discussion.

the strangeness yield as a function of reaction energy and size of QGP formed: the relative population of strangeness grows as the collision volume increases and/or the energy increases. Strange quark pairs: s and antiquarks $\bar{s}$, are for most part produced after QGP formation, in processes called gluon fusion [17] illustrated in the center of Fig. 33.3, see Section 31.2. Processes based on light quark collisions contribute fewer $s\bar{s}$-pairs by nearly a factor 10. Thus, the abundance of strangeness is considered a signature of the formation of a thermal gluon medium.

The fireball of QCD matter, driven by its internal pressure, changes rapidly and with this the rates of production and reannihilation of massive quarks change. In the early stage it is likely that a sequence of chemical equilibration processes is present, with gluons being first to equilibrate in their number and momentum distribution, and then gluon based processes driving the equilibration of quarks, first light to later, heavy.

Once produced, strangeness evolves with light $(u,\bar{u},d,\bar{d})$ quarks and gluons g until the time of hadronization, when the remaining particles seed the formation of hadrons observed in the experiment. In QGP, s and $\bar{s}$ can move freely and their large abundance leads to unexpectedly large yields of particles with a large s and $\bar{s}$ content, as is illustrated exterior of the QGP domain in Fig. 33.3.

Strange antibaryons

In regard to strange antibaryon signature: in the 1982 discussion of the possible and forthcoming CERN SPS experiments I said [18], see Section 31.4:

"... we should search for the rise of the abundance of particles like Ξ, $\overline{\Xi}$, Ω, $\overline{\Omega}$, and ϕ, ... such experiments would uniquely determine the existence of the phase transition to the quark-gluon plasma.... Strangeness-based measurements have the advantage that they are based on the observation of a strongly interacting particle ($s, \bar{s}$ quark) originating from the hot plasma phase; these are much more abundant than the electromagnetic particles (dileptons or direct photons)."

Léon Van Hove, the former DG (1976-1980), characterized the strange antibaryon situation *after* the 1982 presentation as follows [19]:

In the "Signals for Plasma" section: ... implying (production of) an abnormally large antihyperon to antinucleon ratio when plasma hadronizes. The qualitative nature of this prediction is attractive, all the more so that no similar effect is expected in the absence of plasma formation.

These remarks became the intellectual cornerstone of the experimental strangeness program carried out at the CERN SPS in the last decade of the XX century.

Production and annihilation of flavor

The initial on-impact production of charm c, $\bar{c}$ and yet heavier bottom quark pairs $b, \bar{b}$ increases with AA collision energy. From some collision energy on, dependent on the heavy quark mass, the initial production yields thus corresponds to an abundance which exceeds the chemical equilibrium yield of heavy quark pairs at the later hadronization condition of the QGP fireball. This happens because the ratio of heavy quark mass to the hadronization temperature enters in the exponential, and the mass of charmed (and bottom) quarks is much larger compared to the final temperature at which QGP fireball breaks up into hadrons. I expect that there is enough charm produced at LHC to allow that during evolution of the QGP fireball charm yield undergoes the thermal pair annihilation processes decreasing in yield down towards the chemical equilibrium abundance.

After noting this anomalous charm behavior, one wonders if to a smaller measure a similar above chemical abundance yield can occur also for strangeness: if the QGP formed in the AA high energy collision is very hot, thermally produced strangeness reaches its highest abundance at a transient high temperature condition in the QGP fireball. Later, as QGP expands and cools, strangeness pair yield is above chemical equilibrium, just as charm is. The difference between strangeness and charm is that for strangeness both production and later annihilation is by thermal in-plasma reaction processes. Once above chemical equilibrium, strangeness in QGP is decreasing towards chemical equilibrium. Depending on dynamical evolution details, strangeness can hadronize from a state above QGP chemical equilibrium.

The final state abundance of all heavy quark flavors: strangeness, charm and yet heavier bottom quark pairs $b, \bar{b}$ is thus a part of the ongoing investigation of the time evolution of a QGP fireball.

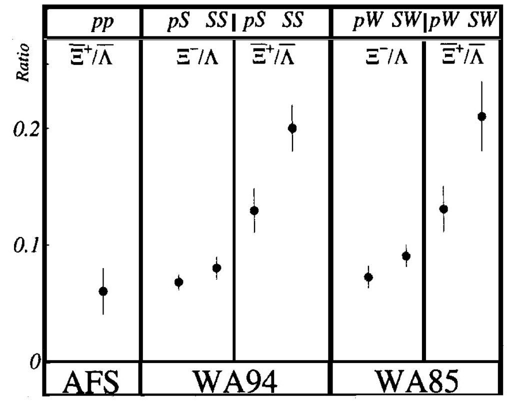


Fig. 33.4 Results obtained at the CERN-SPS Ω'-spectrometer for Ξ/Λ-ratio in fixed target S-S and S-Pb at 200 A GeV/c; results from compilation in Ref. [20].

Was the predicted strange (anti)baryon enhancement found?

SPS results

Given the quark combination reactions shown in Fig. 33.3 that create multistrange baryons and antibaryons, these particles are naturally a sensitive probe of the hadronization strangeness density. Experiments explored the production of multi strange nucleons – 'Cascades' $\Xi^-(ssd)$ and 'Omegas' $\Omega^-(sss)$ and, importantly, their antiparticles, the multi-strange anti-nucleons $\overline{\Xi}, \overline{\Omega}$. The study of single strange mesons – kaons $K^+(u\bar{s})$, $K^-(\bar{u}s)$, and single strange nucleons – the Lambda particles $\Lambda^0(sud)$ set a comparison base-line.

The studies in AA collisions at the CERN-SPS Ω'-spectrometer, see Section 15.3 thus measured the production of higher strangeness content baryons and antibaryons, as compared to lower strangeness content particles, Ξ/Λ and $\bar{\Xi}/\bar{\Lambda}$. These early SPS experiments clearly confirmed the QGP prediction in a systematic fashion, as we see in the compilation of the pertinent results in Fig. 33.4, see Ref. [20].

In these experiments WA85 and WA94 the sulfur ions S with 200 A GeV hit stationary laboratory targets, S, W (tungsten), respectively, with reference date from pp (AFS-ISR) and p on S shown for comparison. The Ξ/Λ and $\bar{\Xi}/\bar{\Lambda}$ ratio enhancement rises with the size of the reaction volume measured in terms of target A, and is larger for antimatter as compared to matter particles. This agrees very well with qualitative model predictions [18], Chapter 31 and their quantitative model consideration [21], much of this work was carried out with Berndt Müller, Fig. 33.5.

When a thermal QGP fireball domain is not formed, the production of such complex multistrange (anti)baryons is less probable for two reasons:

1. When new particles are produced in color string breaking process, strangeness is known to be produced less often by a factor 3 compared to lighter quarks.
2. The generation of multistrange content requires multiple such suppressed steps.

Fig. 33.5 Berndt Müller (left) with Johann Rafelski work on hadronization of QGP in 1984/85, the Physics Reports article [21]. *Image credit: Johann Rafelski and University of Cape Town.*

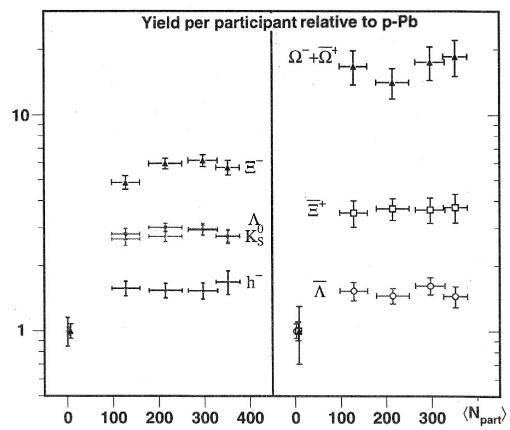


Fig. 33.6 Results obtained at the CERN-SPS Ω'-spectrometer for multi-strangeness enhancement at mid-rapidity $|y_{\mathrm{CM}}| < 0.5$ in fixed target Pb-Pb collisions at $158\,A\,\mathrm{GeV}/c$ at CERN SPS as a function of the mean number of participants $\langle N_{\mathrm{part}} \rangle$, from Ref. [22].

Thus the conclusion is that with increasing strangeness content the production by string processes of strange hadrons is progressively more suppressed.

Conversely, comparing pp, pA to AA absolute yield results, the enhanced production of higher strangeness content baryons and antibaryons in AA collisions increases with the particle strangeness content. To make this comparison fairly, one normalizes the yields to be per unit of hadronization volume measured in terms of the number of collision participating nucleons. The number of 'participants' $\langle N_{\mathrm{part}} \rangle$ is obtained from geometric models of reaction based on energy and particle flows.

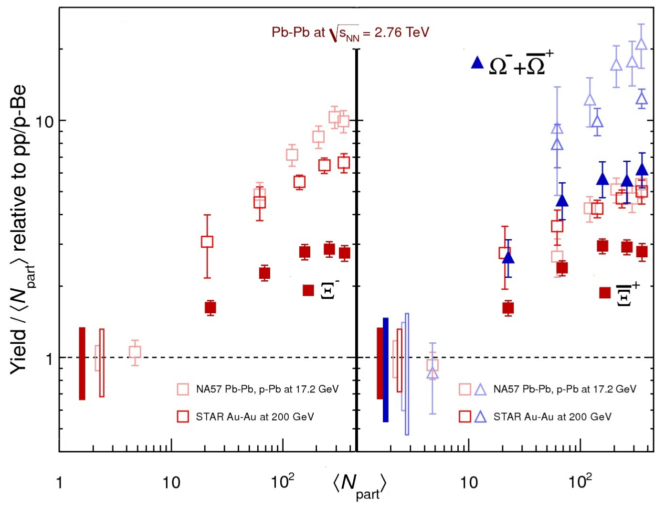


Fig. 33.7 Enhancements of Ξ^-, $\overline{\Xi}^+$, $\Omega^- + \overline{\Omega}^+$ in the rapidity range $|y_{CM}| < 0.5$ as a function of the mean number of participants $\langle N_{part} \rangle$: LHC-ALICE: full symbols; RHIC-STAR and SPS-NA57: open symbols. The LHC reference data use interpolated in energy *pp* reference values. Results at the dashed line (at unity) indicate statistical and systematic uncertainties on the *pp* or *p*Be (at SPS) reference. Error bars on the data points represent the corresponding uncertainties for all the heavy-ion measurements. Results presented and compiled in Ref. [23].

The results obtained for the top SPS energy Pb (lead) beam of 156 *A* GeV are shown in Fig. 33.6. On the right are considered particles made only of quarks and antiquarks that are created in the collision. On the left some of the particle valence quarks can be from matter brought into the reaction volume. The number of participants is large, greater than 100, a point to remember. The particles made entirely from newly created quarks are up to 20 times more abundant. This enhancement falls with decreasing strangeness content and increasing contents of valence quarks which are brought into collision. The results at yield ratio '1' provide the error measure for the *pA* reference measurement.

All these results are in excellent agreement with the deconfined QGP fireball as the source of strange baryons and antibaryons. These results provided key evidence for the formation of a new state of matter at the CERN-SPS energies, which CERN announced in a press release in February 2000. Much has been learned about the QGP fireball properties from ongoing analysis of these and other related hadron production results [24–28].

RHIC and LHC confirmation

Some of the above presented discoveries are now nearly 20 years old. They have been confirmed by further results obtained at SPS, at RHIC, and at the LHC. The present day experimental summary is shown in the figure Fig. 33.7. We see results obtained by the collaborations:

SPS	NA57 for collision energy $\sqrt{s_{\rm NN}} = 17.2\,\text{GeV}$ (lighter open symbols);
RHIC	STAR for collision energy $\sqrt{s_{\rm NN}} = 200\,\text{GeV}$ (darker open symbols);
LHC	Alice for collision energy $\sqrt{s_{\rm NN}} = 2760\,\text{GeV}$ (filled symbols).

These results span a range of collision energies that differ by factor 160.

Comparing results of Fig. 33.7 with those seen in Fig. 33.6 we note that $\langle N_{\rm part}\rangle$ is now on a logarithmic scale: the results of Fig. 33.6 which show that the enhancement is volume independent are in Fig. 33.7 compressed to a relatively small domain on the right in both panels. The new SPS results seen in Fig. 33.7 are in agreement with the earlier SPS results shown in Fig. 33.6.

The rise of enhancement which we see in Fig. 33.7 as a function of the number of participants $2 < \langle N_{\rm part}\rangle < 80$ reflects on the rise of strangeness content in QGP to its chemical equilibrium abundance with an increase in volume and thus lifespan of QGP fireball. It is not surprising that the enhancement at SPS is larger than that seen at RHIC and LHC, considering that the reference yields play an important role in this comparison. Especially the high energy LHC *pp* reactions should begin to create space domains that resemble QGP but do not yet achieve the degree of chemical strangeness equilibration that would erase the enhancement effect entirely.

Detailed analysis of the RHIC and LHC *AA* particle production abundance results shows that the source of strange baryons and antibaryons is a deconfined QGP fireball which hadronizes at a common physical condition [29]. This establishes that in a large range of collision energies the final hadron abundance is sourced in the same fireball with a main and practically only difference being the volume size.

Is there a threshold in energy and size for QGP formation?

Dynamics and deconfinement

Our study of properties of hot nuclear matter assumes chemical equilibrium abundance of all strongly interacting particles, including those that are quite heavy. In the SBM approach there are very few of each kind, but there are many, many different types of particles. For each particle there is an equilibration relaxation time. The heavier the particle is, the more time is needed to produce it. Thus it is not guaranteed that the theoretical result about thermal equilibrium properties of the hot hadronic matter is a true image of the dynamical RHI collision situation.

The smaller the size of colliding nuclei, the shorter is the collision time. Thus in collisions of small size objects such as *pp* or light nuclei, one cannot presume that at relatively low collision energy a complete chemical equilibration is achieved. As nucleon number *A* increases, for large nuclei, the situation changes. However, should the hadro-chemical equilibrium be established late in the collision, the hadron dissolution into the deconfined QGP phase will have only a fleeting presence and thus leave few if any signatures. In such 'just beyond' deconfinement reactions of great importance are signatures that are based on strong interactions, as these are more likely to appear.

An important additional observation is that particle production processes are more effective with increasing collision energy. Therefore the chemical equilibration is achieved more rapidly at higher energy. This was the main reason why QGP search experiments started at the highest available energy where QGP is both more easily produced, and, in terms of more rarely produced particles, more easily detected. This said, the question about threshold of QGP production remains.

Where are thresholds of deconfinement?

The above qualitative discussion suggest that thresholds are expected as a function of collision energy and reaction volume size. The volume can be controlled by creating categories of the 'violence' of collisions which are associated with collision offset between centers of nuclei and/or value of A of the nuclei colliding. This is the participant number $\langle N_{\rm part}\rangle$.

Thus in principle for each reaction energy studied, one can explore a range of reaction volumes and compare the results by looking at observables such as strangeness. This type of data is under consideration both at the SPS at CERN (see Chapter 11) and at RHIC at BNL (see Chapter 14). A study of head-on Pb–Pb collisions as a function of energy at SPS did produce by 2010 tantalizing hints of an energy threshold to new phenomena [30] in an energy range also accessible to RHIC.

What makes the search for a threshold difficult is a likely change, as a function of both reaction energy and reaction volume, of the *probability* to enter the QGP phase. Since experiments in general 'trigger' their detector on interesting looking events in a process one would call 'maximum bias', the variation of this probability can be compensated in part by trigger procedures which are often specific to the particular approach taken by the experimental group.

In Chapter 11 discontinuities as a function of collision energy in the K^+/π^+ particle yield ratio and the inverse slope parameter of the $m_\perp$ spectra of K^-, see Fig. 11.1 are interpreted as the onset of deconfinement. We see a local maximum near to 30 A GeV, that is at 3.8+3.8 GeV, $\sqrt{s_{\rm NN}} = 7.6$ GeV collider energy collisions in both quantities. Both of these behavior 'thresholds' are to some degree mirrored in the much smaller pp reaction system. This indicates that a qualitative change in the production mechanism of strange particles occurs in a wide range of reaction volume.

An analysis of the SPS AA global particle production results shows that the fireball content in strangeness per entropy s/S nearly saturates at this $\sqrt{s_{\rm NN}} = 7.6$ GeV energy threshold [28], as is shown in Fig. 33.8. This means that both strangeness and entropy above threshold grow with energy in same manner; one can argue this signals activation of gluon and quark degrees of freedom, a point made in Chapter 11.

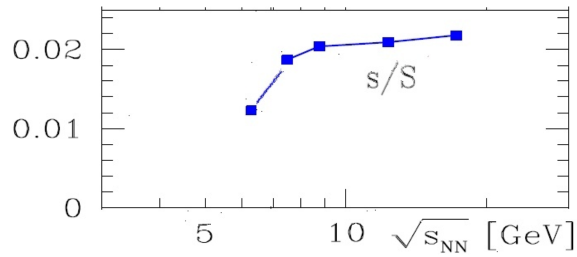


Fig. 33.8 Fireball thermal energy content divided by strange quark pair content as function of collision energy, update of results Ref. [28] .

This shows that while the formation of QGP has been clearly achieved, the work on the characterization of the relation of phase diagram and experimental conditions has just begun. Two new accelerator complexes (FAIR at GSI and NICA at Dubna) should improve experimental access to these questions in the future. This effort continues today the tradition begun 50 years ago, when Hagedorn Temperature was invented. We can toast, see Fig. 33.9, to 50 more years of transforming advances in the study of 'hot' strong interactions.

Fig. 33.9 1978: Rolf Hagedorn (on right) toasts to work accomplished. *Photo: JR*

33.5 Conclusions

This report barely touches the surface of the physics program that has emerged in the past 17 years of hard work. By showing a few qualitative and quantitative pictures I have aimed to illustrate how the interest in *Melting Hadrons and Boiling Quarks* morphed into a comprehensive experimental program addressing strangeness observable of QGP. In a nutshell, the theoretical and experimental highlights are:

- (Multi) strangeness enhancement from QGP fireball formed in *AA* collisions is natural and was predicted.
- All SPS, RHIC and LHC data clearly shows it consistent with predictions.
- At LHC energy the particle multiplicity and thus space-time volume of the reaction increases strongly; therefore even *pp* data show gradual approach to strangeness equilibration.
- For large *AA* colliding nuclei the onset of new physics with collision energy occurs early, permitting an intense experimental exploration of the physical properties of the deconfined state in the coming decade.

Acknowledgements I thank Berndt Müller for constructive criticism.

References

1. K. G. Wilson: Confinement of Quarks, Phys. Rev. D **10**, 2445 (1974).
2. T. D. Lee: Is The Physical Vacuum A Medium?, September 28, 1979 lecture at Columbia University. preprint CU-TP-170 (1979); In Lee, T.D.: Selected papers, vol. 2, 213-225. (Trans. N.Y. Acad. Sci., Ser. II 40 (1980) 111-123).
3. A. Rothkopf, T. Hatsuda and S. Sasaki: Complex Heavy-Quark Potential at Finite Temperature from Lattice QCD, Phys. Rev. Lett. **108**, 162001 (2012)
4. A. Bazavov and P. Petreczky: On static quark anti-quark potential at non-zero temperature, Nucl. Phys. A **904-905**, 599c (2013)
5. O. K. Kalashnikov and V. V. Klimov: Phase Transition in Quark - Gluon Plasma, Phys. Lett. B **88**, 328 (1979). preprint A-79-80 ; LEBEDEV-INST-P-80 1979. - 8 p. Time stamp CERN Library 31 July 1979
6. S. A. Chin: Transition to Hot Quark Matter in Relativistic Heavy Ion Collision, Phys. Lett. B **78**, 552 (1978)
7. E. V. Shuryak: Quark-Gluon Plasma and Hadronic Production of Leptons, Photons and Psions, Phys. Lett. B **78**, 150 (1978) [Sov. J. Nucl. Phys. **28**, 408 (1978)] [Yad. Fiz. **28**, 796 (1978)].
8. S. Durr, Z. Fodor, J. Frison, C. Hoelbling, R. Hoffmann, S. D. Katz, S. Krieg and T. Kurth *et al.*: Ab-Initio Determination of Light Hadron Masses, Science **322**, 1224 (2008);
9. A. S. Kronfeld: Lattice Gauge Theory and the Origin of Mass, in *100 Years of Subatomic Physics*, pp. 493-518, edts E.M. Henley and S.D. Ellis, World Scientific (Singapore 2013).
10. G. F. Chapline and A. K. Kerman: On the Possibility of Making Quark Matter in Nuclear Collisions, M I T Cambridge - CTP-695 (78,REC.APR) 10p, available as: http://inspirehep.net/record/134446/files/CTP-695.pdf

11. S. A. Chin and A. K. Kerman: Possible Longlived Hyperstrange Multi-Quark Droplets, Phys. Rev. Lett. **43**, 1292 (1979).
12. R. Hagedorn, I. Montvay and J. Rafelski: Thermodynamics Of Nuclear Matter From The Statistical Bootstrap Model, CERN-TH-2605, in 'Erice Hadronic Matter 1978', ed. by N. Cabibbo and L. Sertorio, Plenum Press (New York 1980), p. 49
13. P. Koch and J. Rafelski: Why the Hadronic Gas Description of Hadronic Reactions Works: The Example of Strange Hadrons, South Afr. J. Phys. **9**, 8 (1986).
14. K. Geiger: Thermalization in ultrarelativistic nuclear collisions. 1. Parton kinetics and quark gluon plasma formation, Phys. Rev. D **46**, 4965 (1992).
15. K. Geiger: Thermalization in ultrarelativistic nuclear collisions. 2. Entropy production and energy densities at RHIC and LHC, Phys. Rev. D **46**, 4986 (1992).
16. A. Kurkela and E. Lu: Approach to Equilibrium in Weakly Coupled Non-Abelian Plasmas, Phys. Rev. Lett. **113**, no. 18, 182301 (2014)
17. J. Rafelski and B. Muller: Strangeness production in the Quark-Gluon Plasma, Phys. Rev. Lett. **48**, 1066 (1982) [Erratum-ibid. **56**, 2334 (1986)]
18. J. Rafelski: Strangeness In Quark - Gluon Plasma, lecture at: *Quark Matter Formation And Heavy Ion Collisions*, Bielefeld, May 1982; due to mail mishap ultimately published in: South Afr. J. Phys. **6**, 37 (1983); see Chapters 30 and 31 in this volume
19. L. Van Hove: Quark Matter Formation And Heavy Ion Collisions: The Theoretical Situation, Preprint CERN-TH-3360 19 July 1982, in *'Multiparticle Dynamics 1982* Volendam 6-11 June 1982; pp 244-266, W. Kittel, W. Metzger, A. Stergiou edts. (Singapore, World Scientific, 1983)
20. F. Antinori: The heavy ion physics programme at the CERN OMEGA spectrometer, In *Symposium on the CERN Omega Spectrometer: 25 Years of Physics* 19 Mar 1997. Geneva, Switzerland, Proceedings edited by M. Jacob and E. Quercigh, CERN-Yellow Report 97-02, pp.43-49(Geneva, Switzerland 1997)
21. P. Koch, B. Muller and J. Rafelski: Strangeness in Relativistic Heavy Ion Collisions, Phys. Rept. **142**, 167 (1986)
22. E. Andersen *et al.* [WA97 Collaboration], Strangeness enhancement at mid-rapidity in Pb Pb collisions at 158-A-GeV/c, Phys. Lett. B **449**, 401 (1999).
23. B. B. Abelev *et al.* [ALICE Collaboration]: Multi-strange baryon production at mid-rapidity in Pb-Pb collisions at $\sqrt{s_{NN}}$ = 2.76 TeV, Phys. Lett. B **728**, 216 (2014) [Erratum-ibid. B **734**, 409 (2014)]
24. J. Rafelski: Strange anti-baryons from quark - gluon plasma, Phys. Lett. B **262**, 333 (1991)
25. J. Rafelski, J. Letessier and A. Tounsi: Strange particles from dense hadronic matter, Acta Phys. Polon. B **27**, 1037 (1996)
26. J. Letessier and J. Rafelski: Chemical nonequilibrium and deconfinement in 200-A/GeV sulphur induced reactions, Phys. Rev. C **59**, 947 (1999)
27. J. Letessier and J. Rafelski: Observing quark gluon plasma with strange hadrons, Int. J. Mod. Phys. E **9**, 107 (2000)
28. J. Letessier and J. Rafelski: Hadron production and phase changes in relativistic heavy ion collisions, Eur. Phys. J. A **35**, 221 (2008)
29. M. Petran and J. Rafelski: Universal hadronization condition in heavy ion collisions at $\sqrt{s_{\mathrm{NN}}} =$ 62 GeV and at $\sqrt{s_{\mathrm{NN}}} = 2.76$ TeV, Phys. Rev. C **88**, no. 2, 021901 (2013)
30. M. Gazdzicki, M. Gorenstein and P. Seyboth: Onset of deconfinement in nucleus-nucleus collisions: Review for pedestrians and experts, Acta Phys. Polon. B **42**, 307 (2011)

Back Page

This book retraces how, from humble beginnings at CERN in 1964, the idea of limiting temperature (Hagedorn) galvanized many minds and, through Hagedorns pioneering work on multi-hadron production phenomena and subsequent developments, culminated in the discovery of the quark-gluon plasma announced, also at CERN, in February 2000.

Following the foreword by Herwig Schopper the Director General of CERN at the key historical juncture (1981-1988) the first part is a tribute to Rolf Hagedorn (1919-2003) and includes contributions by contemporary friends and colleagues, and those who were most touched by Hagedorn today: Tamás Biró, Igor Dremin, Torleif Ericson, Marek Gaździcki, Mark Gorenstein, Hans Gutbrod, Maurice Jacob, István Montvay, Berndt Müller, Grażyna Odyniec, Emanuele Quercigh, Krzysztof Redlich, Helmut Satz, Luigi Sertorio, Ludwik Turko, and Gabriele Veneziano.

The second and third parts describe the conceptual developments that within 15 years after discovery of the Hagedorn temperature led to its recognition as the melting point of hadrons into boiling quarks, and, respectively, to the rise of the experimental relativistic heavy ion collision program with the development of strangeness as QGP observable. These parts contain previously unpublished material authored by Hagedorn and Rafelski from the years 1964-1983: conference retrospectives, research notes, workshop reports, in some instances abbreviated to avoid duplication of material, and rounded off with the editors explanatory notes.

This volume was edited by Johann Rafelski, Hagedorns postdoctoral assistant and later collaborator at CERN. Rafelski is a theoretical physicist working at The University of Arizona in Tucson, USA. Born in 1950 in Krakow, Poland, he received his Ph.D. with Walter Greiner in Frankfurt, Germany in 1973. He has contributed greatly to the establishment of the quark-gluon plasma research field, and is well known in the field for proposing and developing the strangeness quark flavor as the signature of quark deconfinement.