Recap 1

1. $c = c' \quad \text{in vacuum}
   \Rightarrow \quad \text{Light clock}
   \Rightarrow \quad \text{Time dilation } \Delta t = \frac{t}{\sqrt{1 - \frac{v^2}{c^2}}}
   \Rightarrow \quad \text{Lorentz contraction } l' = \frac{l}{\sqrt{1 - \frac{v^2}{c^2}}}

2. Michelson-Morley Experiment

- Measures optical paths parallel & perpendicular, arbitrary to velocity $v$.

$\Rightarrow$ No fringe shift under rotation

$\Rightarrow$ No ether wind

$\Rightarrow$ Ether unobservable
Important difference between time dilation ↔ Lorentz contract.

A lorentz contracted object when brought to rest FORGETS its history, \( \ell \rightarrow \text{back to } \ell \).

A twin coming back from a trip has an AGE that depends on HISTORY of expedition.

We cannot 'undo' time dilation, though a clock that is brought to rest ticks again at same rate as before the trip.

REASON: NO TIME MACHINE FOR DIFFERENT...
2.4 Coordinate transformation

=> observe from system at velocity \( v_1 \) or \( v_2 \) with respect to lab where experiment is done. (usually \( v_1 = 0, \ v_2 = v \))

Be able to predict what observer '2' sees if observation of '1' is known. Remind yourself of:

- Inertial Frames
- Principle of Relativity (Newtonian)
- Galilean invariance

If this appears to be foreign language, this is the wrong class for you.

Galileo transformation:

\[
\begin{align*}
x' &= x - v_x t \\
y' &= y \\
z' &= z \\
t' &= t
\end{align*}
\]
"Coordinate" Systems

(2.9) Space-Time Diagram

Future: what \textit{event} can influence

Past: can influence \textit{event}

World lines if you are at rest

Future, Past defined by:
\[ c^2t^2 - x^2 > 0 \]

\[ c^2t^2 > 1x^2 \]

\[ 1x' = ct' \] (no visual of z)
LORENTZ COORDINATE TRANSFORMATION

Has nothing to do with changes in a body or space-time. The question is e.g.: What are the space-time coordinates of an event for observers at different 

Once we figure this out, maybe we can predict outcome of measurement (Lorentz contraction, time dilation etc.) but this is another step

\[
\begin{align*}
\begin{align*}
    z' &= z \\
    y' &= y \\
    x' &= a_{11} x - \frac{a_{12} t}{t'} \\
    t' &= a_{21} x + a_{22} t
\end{align*}
\text{Galileo}
\end{align*}
\]

Experiment on which all observers must agree

\[
\frac{dx'}{dt'} = c' = c = \frac{dx}{dt} \quad \text{light cons}
\]
6) Let prime system move with velocity \( v \) with respect to unprimed system. Observer at rest in prime system observes

\[ \begin{align*}
\dot{\mathbf{a}} &= \frac{\partial \mathbf{a}}{\partial x} + \mathbf{v} \times \frac{\partial \mathbf{a}}{\partial t} \\
\dot{\mathbf{v}} &= \frac{\partial \mathbf{v}}{\partial x} + \mathbf{v} \times \frac{\partial \mathbf{v}}{\partial t} \\
\dot{\mathbf{a}} &= \frac{\partial \mathbf{a}}{\partial x} + \mathbf{v} \times \frac{\partial \mathbf{a}}{\partial t} \\
\end{align*} \]

\( x = a(x-v) \quad t = a(t-v) \]

\( x' = a(x-v) \\
\Rightarrow x = a(x-v) \]
So we have
\[
\begin{align*}
x' &= a_{11}(v^2)(x - vt) \\
x &= a_{11}(v^2)(x' + vt')
\end{align*}
\]
This must also be true along the light cone.

\[
x'_t = ct'_t, \quad x_t = ct_t \quad \leftarrow \text{light cone}
\]

Same \(c\)!

\[
x'_t = c t'_t = a_{11}(v^2) t (c - u)
\]
\[
x = c t = a_{11}(v^2) t' (c + u)
\]
two relations between \(t, t'\)

\[
ct'_t = a_{11} \frac{ct}{c^2} (c - u)(c + u)
\]
\[
\Rightarrow a_{11}^2 = \frac{1}{1 - \frac{2u}{c^2}}
\]

still to do

\[
\begin{align*}
a_{11} &= \gamma \\
a_{12} &= \nu \gamma
\end{align*}
\]

procedure: use light cone to replace \(t', t\) by \(x', x\)

\[
\begin{align*}
t' &= a_{22}(v)t - a_{21}(v)x \\
t &= a_{22}(-v)t' - a_{21}(-v)x'
\end{align*}
\]
\[ t' = \gamma (t - \frac{v}{c^2} x) \]
\[ x' = x - v t \]
\[ y' = y \]
\[ z' = z \]

**Must Know**

\[ \gamma = \frac{1}{\sqrt{1 - (\frac{v}{c})^2}} \]
\[ \beta = \frac{v}{c} \]

Note: \[ c \ell' = \ell \] \( \ell \) dim. length

\[ x' = x (\ell - \beta \ell) \]
\[ y' = y \]
\[ z' = z \]

\[ \text{side note: } \cosh y = \gamma^2 \quad \sinh y = \beta \gamma \]

\( \ell' = \cosh y \ell - \sinh y x \)
\[ x' = -\sinh y \ell + \cosh y x \]
\[ y' = y \]
\[ z' = z \]

shows symmetry between time & space

\[ \int \text{ hyperb.} \]
In normal rotations length to a coordinate point invariant

\[ c^2 x^2 = c^2 t^2 - x^2 \]

Similarly, LT leaves the length invariant.

\[ c^2 x'^2 = \frac{c^2 x^2}{1 - \frac{x^2}{c^2}} \]

\[ = \gamma^2 \left((ct)^2 + (\beta x)^2 - 2ct\beta x\right) \]

\[ - \gamma^2 \left(\beta^2 (ct)^2 + x^2 - 2\beta ct\beta x\right) \]

\[ = \gamma^2 (1 - \beta^2)((ct)^2 - x^2) \]

\[ = (ct)^2 - x^2 \quad \text{(QED)} \]
Lorentz Transformation

Is a coordinate transformation expressing what different observers will measure at the coordinate of an event \( S' \)

\[
x' = \gamma (x - vt) \\
y' = y \\
z' = z \\
t' = \gamma (t - \frac{vx}{c^2})
\]

by symmetry,

\( x \leftrightarrow x', \ t \leftrightarrow t', \ v \leftrightarrow v' \)

\[
x = \gamma (x' + vt') \\
y = y' \\
z = z' \\
t = \gamma (t' + \frac{v}{c^2} x')
\]

Proportional to

\[
ct \equiv \sqrt{(ct)^2 - x^2} = ct \sqrt{1 - \left(\frac{v}{c}\right)^2} = ct \sqrt{1 - \frac{v^2}{c^2}}
\]

Increment

\[
dct = dt \sqrt{1 - \frac{v^2}{c^2}} \text{ momentaneous velocity}
\]
AN INVARIANT or better

LORENTZ - INVARIANT

IS A QUANTITY ALL INERTIAL OBSERVERS MEASURE/Observe
to have the same value

CONSIDER PROPER TIME

\[ c \tau' = \sqrt{c^2 t'^2 - x'^2} \]

\[ t' = \frac{t - ux}{c^2} \]

\[ x' = \frac{x - ut}{c^2} \]

\[ c \tau' = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \sqrt{t^2 c^2 - x^2} = c \tau \]

PROPER TIME IS A \( L \)-INVARIANT

Proper time of a photon = 0

we used \( c^2 t^2 - x^2 = 0 = c^2 t'^2 - x'^2 \)
AN INVARIANT or better

LORENTZ - INVARIANT

IS A QUANTITY ALL INERTIAL OBEYSERS MEASURE/ OBSERVE
TO HAVE THE SAME VALUE

CONSIDER PROPER TIME

\[ c \Delta t' = \sqrt{c^2 t'^2 - x'^2} \]

\( t' = \gamma (t - \frac{u}{c^2} x) \)
\( x' = \gamma (x - ut) \)

\[ c \Delta t' = \gamma \sqrt{(c t - \frac{u}{c} x)^2 - (x - ut)^2} \]

\[ = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \sqrt{t^2 c^2 - x^2} = c \Delta t \]

PROPER TIME IS A L - INVARIANT!

(Proper time of a photon = 0)

We used \( c^2 t^2 - x^2 = 0 = c^2 t'^2 - x'^2 \)
\[ ct' = \gamma (ct - \gamma x) \]

\[ x' = \gamma (x - \beta ct) \]

\[ \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} ; \quad \beta = \frac{v}{c} \]

\[ ct' = \cosh \gamma \, ct \quad - \quad \sinh \gamma \, x \]

\[ x' = \cosh \gamma \, x \quad - \quad \sinh \gamma \, ct \]

\[ \cosh \gamma = \gamma, \quad \sinh \gamma = \beta \gamma \quad \text{for} \quad \beta \ll 1 \]

\[ \cosh^2 \gamma - \sinh^2 \gamma = \gamma^2 (1 - \beta^2) = 1 \]

\[ \cosh \gamma = \frac{1}{2} (e^\gamma + e^{-\gamma}) \quad \sinh \gamma = \frac{1}{2} (e^\gamma - e^{-\gamma}) \]

\[ x' = \cosh \gamma \, x - \sinh \gamma \, y \]

\[ y' = \cosh \gamma \, y + \sinh \gamma \, x \]

\[ v^2 = 1 \]

\[ y = \frac{1}{2} \ln \frac{1 + \beta}{1 - \beta} \]
Obviously, train is contracted!

Frame of reference selected by measurement prescription:
\[ t_1 - t_2 = 0 \text{ for the observer.} \]

Two events:
\[ x_1' - x_2' \]

Principle of Relativity is maintained

Note: Observer in train is not \( t_1' - t_2' = 0 \)
A physical body is measured to be Lorentz-contracted by an observer whose set-up is such that measurement of ends of a 'stick' is at equal time.

Note: This is what we mean by measurement, all this just explains how it can be that twin on Earth claims his brother/sister is contr. and vice versa....
EXTENDED BODIES

Objective: Show that length of a (rigid) body is shorter since measured at equal time "Lorentz-contraction"

Expl: TO MEASURE SIZE WE STUDY BY DEFAULT AT LEAST TWO SPACE-TIME EVENTS, THUS WE MUST BY CONSIDERATION OF THE MEASUREMENT PROCESS DETERMINE IN WHICH FRAME THE TIME DIFF. = 0
\( t_1, x_1 \quad t_2, x_2 \) in rest frame

\( t'_2 = \gamma(t_2 - \frac{v}{c^2} x_2) \)

\( \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \)

\( x'_i = \gamma(x_i + vt) \)

\[ x'_1 - x'_2 = \gamma(x_1 - x_2) - \gamma v (t_1 - t_2) \]

Observer has \( t_1 - t_2 = 0 \)

\[ \implies x'_1 = \gamma(x_1 + vt_1) \]

\( x'_i = \gamma(x'_i + vt'_i) \)

\( x_1 - x_2 = \gamma(x'_1 - x'_2) + \gamma v (t'_1 - t'_2) \)

If observer were in the moving body, his \( t'_1 - t'_2 = 0 \) and thus measurement results reverse, body left on 'earth' appears shorter.

Length measurement made at equal time shorter.
Question

Can sequence of events be reversed for different observers??

Answer

Look at:

\[ t' = \gamma (t - \frac{v}{c^2} x) \]

\[ \text{time in } S' = \gamma t (1 - \frac{V}{c^2} \frac{x}{l}) \]

\[ = \gamma t (1 - \frac{v/u}{c^2}) \quad u: \text{velocity of body in } S \]

\[ v: \text{velocity of } S' \text{ in } S \]

\[ \Rightarrow \text{as long as } |V|/|u| < c \]

sign of \( t' \) is same as sign of \( t \)

n.b. all this can be done with \( u, v \) not co-linear
→ Recall

\[ u = u' + v. \]

→ How do different observers see velocity of a moving object?

→ Use Lorentz transform to find:

\[ t' = \gamma (t - \frac{v}{c^2} x), \quad v \parallel x, \]

\[ x' = \gamma (x - vt) \]

\text{Inverse:} \quad t = \gamma (t' + \frac{v^2}{c^2} x')

\[ x = \gamma (x' + vt') \]
\[ u \equiv \frac{x}{t} = \frac{x(x' + vt')}{x(t' + \frac{v^2}{c^2}x')} = \frac{t'(\frac{x'}{c} + v)}{t'(1 + \frac{vx'}{c^2})} \]

\[ u' \equiv \frac{x'}{t'} \]

\[ \text{RESULT:} \quad u = \frac{u' + v}{1 + \frac{u'v}{c^2}} \]

\[ \text{for } u'v \ll c : u \approx u' + v \]

\[ \text{MAXIMAL VELOCITY:} \]

1. \( u \leq c \) if \( u'v \leq c \)

2. Start with observer at rest in a body, do success.\[ \text{transform} \] to faster inertial frames \[ \ldots \]

\[ u_{\text{final}} \leq c. \]
New Space-Time

World line of a body with $x = Ut$

Passing event $x = t = 0$

Future

Past

Acausal

Light 'cone'

Cone in 2d