Recap

**Explanations + Consequences ↔ Consistency**

- Observer at rest in prime-system $S'$
  \[ \text{for } x' = 0 \Rightarrow \frac{x}{t} \]
  $u$: velocity of $S'$ as observed in $S$

\[ \text{Y} \]
\[ \text{S} \]
\[ \text{S'} \]
\[ \text{x} \]

\[ x' = 0 \Rightarrow t = x' t' \]

$\dddot{t}$: time coordinate of observer in $S$

$t'$:...

$\dddot{t}'$:

- Moving with $u$

Generalization:

\[ \Sigma - t' = t \sqrt{1 - \frac{u^2}{c^2}} = \sqrt{t^2 - \frac{t^2}{c^2} - x^2} \]

\[ (ct')^2 = (ct)^2 - x^2 \]

\[ \Sigma t^2 = c dt^2 - dx^2 \]

\[ \text{Invariant for all observers} \]

We twin paradox
How do different observers see the velocity of an object?

\[
\begin{align*}
\text{in } S' & : \quad \frac{x'}{t'} = u' \\
\text{in } S & : \quad u = \frac{x}{t} = \frac{x' + vt'}{\sqrt{1 - \left( \frac{v}{c} \right)^2}} = \frac{\sqrt{1 - \left( \frac{v}{c} \right)^2}}{t' + \frac{v x'}{c^2}} \\
& = \frac{\frac{x'}{t'} + \frac{v}{c}}{t' \left( 1 + \frac{v x'/c^2}{} \right)} \\
& = \frac{u' + v}{1 + \frac{u' v}{c^2}} \\
\end{align*}
\]

\[\rightarrow u = u' + \frac{v}{c} \quad \text{C-DOCS}\]

Note: \( u \leq c \) if \( u', v \leq c \)

\[\Rightarrow \text{velocity of light is also the maximal velocity that can be observed: start with an observer at rest in a body transformer, } u'''' \leq c. \]
Relative Velocity

in Depth!

\[ dx = \gamma (dx' + v dt) \]
\[ dy = dy' \quad dz = dz' \]
\[ dt = \delta (dt' + \frac{v}{c} dx') \]

\[ \vec{u} / \vec{e}_x \]
\[ \vec{u}', \vec{u}'' \quad \text{arbitrary} \]

\[ u_x = \frac{d x'}{d t'} = \frac{\gamma}{\sqrt{\gamma^2 + \frac{v^2}{c^2}}} \]
\[ = \frac{dx' + v dt'}{dt' + \frac{v^2}{c^2} dx'} = \frac{u_x' + v}{1 + \frac{u_x v}{c^2}} \quad \checkmark \]

\[ u_y = \frac{dy'}{d t'} = \frac{d y'}{\sqrt{\gamma^2 + \frac{v^2}{c^2} dx'}} = \frac{u_y'}{\gamma} \frac{1}{1 + \frac{u_x v}{c^2}} \]

\[ u_z = \frac{d z'}{d t'} = \frac{u_z'}{\gamma} \frac{1}{1 + \frac{u_x v}{c^2}} \]

\[ \vec{u} = (u_x, u_y, u_z) \]
\[ \vec{u}^2 = u_x^2 + u_y^2 + u_z^2 \leq c^2 \frac{2}{\gamma} \]
Note: \( \vec{u} = \vec{e}_x u_x + \vec{e}_y u_y + \vec{e}_z u_z \) and
\[ 1 + \frac{u_x^2}{c^2} \equiv 1 + \frac{u^2}{c^2} \]
\[ \frac{1}{\delta^2} = 1 - \frac{\vec{u}^2}{c^2} \]

\[ u_x^2 + u_y^2 + u_z^2 = \left( \frac{1}{1 + \frac{\vec{u} \cdot \vec{u}}{c^2}} \right)^2 \cdot \left[ (\frac{u'_x + u_x}{c})^2 + \left(1 - \frac{u^2}{c^2}\right)(u_y^2 + u_z^2) \right] \]

\[ u^2 = \left( \frac{u'_x + u_x}{c} \right)^2 \frac{-\frac{1}{c^2} \vec{u}^2 \vec{u}^2 + \frac{1}{c^2} (\vec{u} \cdot \vec{u})^2}{(1 + \frac{\vec{u} \cdot \vec{u}}{c^2})^2} \]

Also
\[ \frac{\vec{u}^2}{c^2} = 1 - (1 - \frac{\vec{u}'^2}{c^2})(1 - \frac{\vec{u}^2}{c^2}) \frac{1}{(1 + \frac{\vec{u} \cdot \vec{u}}{c^2})^2} < 1! \]

(NOT ON GRE)

Verify nominator only:
\[-(1 + \frac{u'_x u_x}{c^2})^2 \cdot \frac{-\frac{1}{c^2} \vec{u}^2 \vec{u}^2 + \frac{1}{c^2} (\vec{u} \cdot \vec{u})^2}{(1 + \frac{\vec{u} \cdot \vec{u}}{c^2})^2} \frac{-\frac{1}{c^2} \vec{u}^2 \vec{u}^2 + \frac{1}{c^2} (\vec{u} \cdot \vec{u})^2}{(1 + \frac{\vec{u} \cdot \vec{u}}{c^2})^2} \]

\[ = \left( \frac{u'_x + u_x}{c} \right)^2 - \frac{u'_x u_x}{c^2} + \left( \frac{u'_y u_y}{c} \right)^2 + \left( \frac{u'_z u_z}{c} \right)^2 \]

\[ = \left( \frac{u'_x + u_x}{c} \right)^2 - \frac{u'_x u_x}{c^2} + \left( \frac{u'_y u_y}{c} \right)^2 + \left( \frac{u'_z u_z}{c} \right)^2 \]
**LT & Rotation (Student Q)**

What is the relative velocity?

**Problem:** When \( v \ll c \), this is a very strongly accelerated system and cannot be treated as a body in uniform motion. General relativistic solutions for rotating 'black' holes are available (Kerr solution).

For our class

\[
V_{\text{rel}} = \frac{v - (-v)}{1 - \frac{v(-v)}{c^2}} = \frac{2v}{1 + \frac{v^2}{c^2}} < c
\]
Definition \[ I = A^2_0 - \vec{A}^2 \]

4 vector

take a regular vector \( \vec{A} = (A_x, A_y, A_z) \)
complement it with \( A_0 \)
show that the 4-set \( (A_0, \vec{A}) \equiv A^\mu \)

\[
\begin{align*}
A_0' &= \gamma (A_0 - \frac{\gamma v_x}{c} A_x) \\
A_x' &= \gamma (A_x - v_x A_0) \\
A_y' &= A_y \\
A_z' &= A_z \\
x^\mu &= (ct, x_1, x_2, x_3) \\
\frac{dx^\mu}{d\tau} &= u^\mu = (c \frac{dt}{d\tau}, \frac{dx_1}{d\tau}, \frac{dx_2}{d\tau}, \frac{dx_3}{d\tau}) \\
&= \delta (c, v_\mu) \\
\mathbb{E} (u^\mu)^2 &= c^2 v^2 (1 - \frac{v^2}{c^2}) = c^2
\end{align*}
\]
**Accelerated Rocket**

Recall:

\[ u = \frac{x}{t} \]

\[ u' = \frac{x'}{t'} = \frac{x + vt}{1 - \frac{v^2}{c^2}} \]

in the rocket rest frame a decent acceleration \( g \) is very non-relativistic.

It is only the accumulation of long-term acceleration that requires relativistic physics.
$u(t) = a'dt'$

$\frac{du}{dt} = a'dt'$

$\frac{1}{u^2/c^2} = a'dt' = \frac{1}{u^2/c^2} - \frac{1}{u^2/c^2}$

$\frac{d}{dt}(C(t) - \frac{1}{u^2/c^2}) = \frac{d}{dt}(C(t) - \frac{1}{u^2/c^2})$

$[\text{propagating molecule - rocket}]
\begin{align*}
\frac{du}{dt} &= \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{d}{dt}(\mu + \eta) \\
\frac{d}{dt}(\mu + \eta) &= \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \\
\frac{1}{u^2/c^2} &= \frac{1}{u^2/c^2}$
\end{align*}$

$u = u(t)$

Gentle acceleration $\rightarrow$ small in rocket frames

Earth's observer, rocket rests, velocity $u$!
Hyperbolic Functions 2

consider

\[ u = \tanh x \]

\[ du = dx \cdot \frac{d}{dx} \tanh x = dx \cdot \frac{d}{dx} \frac{\sinh x}{\cosh x} \]

\[ \frac{d}{dx} \frac{\sinh x}{\cosh x} = \frac{\cosh x}{\cosh x} - \frac{\sinh x}{\cosh^2 x} \]

\[ = 1 - \left( \frac{\sinh x}{\cosh x} \right)^2 \]

\[ du = dx \cdot (1-u^2) \]

or

\[ \frac{du}{1-u^2} = dx \cdot u \to \frac{dv}{\left( \frac{v}{u} \right)^2} = cdx \]

solution of Riccati equation

\[ u(x) = c \tanh \left[ \sqrt{\frac{2}{c}} \int a' dt \right] \]
In fact that is a total differential of a known function

\[ \int_0^u \frac{du}{1-v^2/c^2} = \int_0^t \alpha dt' \]

\[ \mu = C \tanh \left( \frac{S(t) \alpha}{C} \right) \rightarrow ctanh \frac{\alpha t}{C} \]

'typical' Numbers (reference)

\[ \tau = 1 y \rightarrow \pi \times 10^7 s \]

for \( a = g = 9.8 \text{ m/s}^2 \)

\[ \frac{g \tau}{c} = \frac{9.8 \text{ m/s}^2 \times 3.1 \times 10^7 \text{ s}}{3 \times 10^8 \text{ m/s}} = \frac{1}{11} \]

After 3y acceleration, \( \mu = C \tanh 3 = 0.9851C \)

Now 20y in rocket: How long on earth, how far travelled??
\[ \frac{dt}{dt'} = \gamma \quad \gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \]

\[ = \frac{1}{\sqrt{1 - \tanh^2 \frac{a't'}{c}}} \]

\[ = \cosh \frac{a't'}{c} dt' \]

\[ dt = \cosh \left( \frac{a't'}{c} \right) dt' \]

\[ \leftrightarrow \quad t = \frac{c}{a'} \sinh \frac{a't'}{c} \]

"Amusing!" travel 20y under acceleration, \( t' = 20 \)
on Earth: \( t = \frac{4y}{c} \sinh 20 \)
\[ = 0.24 \times 10^9 \text{y} \]

**DISTANCE TRAVELLED**

\[ S = \int u dt = \int u \left( \frac{dt}{dt'} \right) dt' \]

\[ = \int c \tanh \frac{a't'}{c} \cdot \cosh \frac{a't'}{c} dt' \]

\[ = \int c \sinh \frac{a't'}{c} \cdot \cosh \frac{a't'}{c} dt' = c \frac{c}{a'} \left( \cosh \frac{a't'}{c} - 1 \right) \]

\[ = c \frac{c}{a'} \left( \frac{1 + \frac{a't'}{c}}{2} - 1 \right) \to c \frac{c}{a'} \]

\[ \approx 10.19 \text{y} \]
A spaceship is sent to study a planet 10 light years away. It is required that the spaceship crew returns within 21 years spaceship time back to Earth (including 1 year study period).

NOTE: In your solution please do identify clearly the frame of reference in which you consider physical quantities.

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a) Draw the space-time diagram (qualitatively) with spaceship world-line, marking the coordinates as defined in the problem. Identify how the distance to the planet is related to time of travel in Earth's rest frame.

b) Find the cruising velocity (ignoring the periods of acceleration and deceleration) that our spaceship captain must choose to come back on time.

\[
\frac{d}{dt} = 10 \text{cy} \\
\text{proper time in ship: } \tau = t \left(1 - \frac{v^2}{c^2}\right) + \text{study period} = 10y + 10y + 1y \\
\tau_0 = \text{allowance to get there} \\
\text{solution: } v = \frac{c}{\sqrt{1 + \left(\frac{c}{10y}\right)^2}} = \frac{c}{\sqrt{2}} = 0.71c
\]
c) How much time will have passed on Earth by the time the spaceship returns? How much younger/older is captain's twin waiting for his return on Earth?

\[ t_{\text{earth}} = \frac{2ct}{V} + \text{study} = \frac{20cy}{c/\sqrt{2}} + 1y \]

\[ \sqrt{t_c} = (20\sqrt{2} + 1)y = 29.28y \]

Twin on Earth aged 29.28y thus traveling captain is 8.28y younger upon return.

d) By how much is the spaceship Lorentz-contracted during its cruise period?

\[ \gamma = \frac{1}{\sqrt{1-(\frac{\dot{x}}{c})^2}} = \frac{1}{\sqrt{1-\frac{1}{2}}} = \sqrt{2} = 1.41 \]

Lorentz-contraction factor is 1.41, that is ship is 0.71% of its length.
TWIN PARADOX?

Proper time, the age of each material body is objectively defined and can be measured by arbitrary observer.

TWIN PARADOX involves gently accelerated bodies.

- No paradox at all! Just a bit of misunderstanding.
  Age depends on history.

\[
\text{AGE} = \int_{t_0}^{t} dt
\]

Lightcone

CASE 1

Body at rest

\[
\int d\tau = \int dt = t - t_0
\]

\[
\tau = \tau_0
\]
CASE 2

Body moving with $x/t = u < c$

\[
\int_1^2 dt' = \int_1^2 dt \sqrt{1 - \frac{1}{c^2} (\frac{x}{t'})^2}
\]

\[
= \sqrt{1 - \frac{u^2}{c^2}} (t_2 - t_1)
\]

\[\equiv \tau_2 < \tau_0\]

CASE 3

'Instantly' Reversing Twin

\[
\int_1^2 dt' = \int_1^T dt' + \int_T^2 dt'
\]

\[
= \sqrt{1 - \frac{u'^2}{c^2}} (T - t_1) + \sqrt{1 - \frac{u^2}{c^2}} (t_2 - T)
\]

\[
= \sqrt{1 - \frac{u^2}{c^2}} (t_2 - t_1)
\]

\[\equiv \tau_1 < \tau_0\]
CASE 4 - TRIPLETS...

TWIN #1: \[ \Sigma_1 = \int d\tau_1 = \int_1^T + \int_{T_1}^{T_1'} \frac{S}{T_1} + S_{T_1'}^2 \]

\[ = \sqrt{1 - \frac{V}{c^2} (T_1' - t_1)} + (t_2 - T_1') \]

TWIN #2: \[ \Sigma_2 = \int d\tau_2 = \sqrt{1 - \frac{V}{c^2} (t_2 - t_1)} \]

\[ \Sigma_2 < \Sigma_1 < \Sigma_0 \]

most mobile 'twin' stays youngest
Gently accelerate, deaccelerate.

\[ S_{dt} = \int \sqrt{1 - \frac{\mu^2}{c^2}} \, dt \]

MIT
**Names**

- Distance \( S^2 = (t_1 - t_2)^2 c^2 - (x_1 - x_2)^2 \) **positive**
  
  i.e. \( (t_1 - t_2)c' > (x_1 - x_2)^2 \)
  
  or \( (t_1 - t_2)c > |x_1 - x_2| \)

- Timelike

- Spacelike \( S^2 = (t_1 - t_2)c^2 - (x_1 - x_2)^2 \) **negative**
  
  \( \Rightarrow \) outside of lightcone

**Sign Preserved**

For all 'proper' Lorentz Observers