28. Consider the initial value problem

\[ y' = ty(4 - y)/(1 + t), \quad y(0) = y_0 > 0. \]

(a) Determine how the solution behaves as \( t \to \infty \).
(b) If \( y_0 = 2 \), find the time \( T \) at which the solution first reaches the value 3.99.
(c) Find the range of initial values for which the solution lies in the interval 3.99 < \( y < 4.01 \) by the time \( t = 2 \).

29. Solve the equation

\[ \frac{dy}{dx} = \frac{ay + b}{cy + d}, \]

where \( a, b, c, \) and \( d \) are constants.

**Homogeneous Equations.** If the right side of the equation \( dy/dx = f(x, y) \) can be expressed as a function of the ratio \( y/x \) only, then the equation is said to be homogeneous. Such equations can always be transformed into separable equations by a change of the dependent variable. Problem 30 illustrates how to solve first order homogeneous equations.

30. Consider the equation

\[ \frac{dy}{dx} = \frac{y - 4x}{x - y}. \]  

(a) Show that Eq. (i) can be rewritten as

\[ \frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)}, \]

thus Eq. (i) is homogeneous.
(b) Introduce a new dependent variable \( v \) so that \( v = y/x \), or \( y = xv(x) \). Express \( dy/dx \) in terms of \( x, v, \) and \( dv/dx \).
(c) Replace \( y \) and \( dy/dx \) in Eq. (ii) by the expressions from part (b) that involve \( v \) and \( dv/dx \). Show that the resulting differential equation is

\[ v + x \frac{dv}{dx} = \frac{v - 4}{1 - v}, \]

or

\[ x \frac{dv}{dx} = \frac{v^2 - 4}{1 - v}. \]  

Observe that Eq. (iii) is separable.
(d) Solve Eq. (iii), obtaining \( v \) implicitly in terms of \( x \).
(e) Find the solution of Eq. (i) by replacing \( v \) by \( y/x \) in the solution in part (d).
(f) Draw a direction field and some integral curves for Eq. (i). Recall that the right side of Eq. (i) actually depends only on the ratio \( y/x \). This means that integral curves have the same slope at all points on any given straight line through the origin, although the slope changes from one line to another. Therefore the direction field and the integral curves are symmetric with respect to the origin. Is this symmetry property evident from your plot?

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3The word "homogeneous" has different meanings in different mathematical contexts. The homogeneous equations considered here have nothing to do with the homogeneous equations that will occur in Chapter 3 and elsewhere.