1. Recast the following initial/boundary value problem as a discrete estimation problem. First, cast the problem in terms of (center) finite differences in space and then use explicit Euler in time. This explicit numerical method is not ideally suited for this PDE, but the point here is to have some practice re-casting problems so that they are suitable for estimation. Assume that $u = u(x,t)$, $t \geq 0$, $x \in [0,1]$, and $f = f(x,t)$ is a given forcing function.

   The problem is

$$\begin{aligned} u_t &= \nu u_{xx} + f(x,t) \\ u(x,0) &= U_0 \\ u(0,t) &= g(t) \\ u(1,t) &= 0. \end{aligned}$$

   You do not have to solve the system.

2. This is a classic problem in Kalman Filter theory. For details, see M. Athans, R. P. Wishner, A. B. Bertolini, "Suboptimal state estimation for continuous nonlinear systems from discrete noisy measurements," *IEEE Transactions on Automatic Control*, AC-13(5), pp504-514, 1968. The problem is concerned with tracking a body falling through the atmosphere. The model for the dynamics is given by

$$\begin{aligned} \dot{x}_1(t) &= -x_2(t) \\ \dot{x}_2(t) &= -e^{-\gamma x_1(t)} x_2^2(t) x_3(t) \\ \dot{x}_3(t) &= 0. \end{aligned}$$

   The gravitational force is assumed negligible here when compared to air drag. The last entry in the state vector is called the *ballistic coefficient*. Measurements from the radar position occur at δ second intervals, $t_m = (0, 1, 2, ...)\delta$, and are given by

$$y(t_m) = r(t_m) + w(t_m) = \sqrt{M^2 + (x_1(t_m) - H)^2}.$$

   Here, r is the radar range. The time interval should be $[0, 60]$ seconds. Make δ commensurate with the time stepping of the explicit Euler. You will need to take about 6000 Euler steps.

The model parameters are

$$\begin{aligned} M &= 100,000 \quad \text{ft} \\ H &= 100,000 \quad \text{ft} \\ \gamma &= 5 \times 10^{-5} \\ E[w^2(t_m)] = 10^4 \quad \text{ft}^2. \end{aligned}$$

The measurement error is assumed to be Gaussian with zero mean. The initial state of the system is

$$\begin{aligned} x_1(0) &= 300,000 \quad \text{ft} \\ x_2(0) &= 20,000 \quad \text{ft/s} \\ x_3(0) &= 10^{-3}. \end{aligned}$$

Use an explicit Euler to solve for the dynamic equations (you can also use a Runge-Kutta 4 with just a little more effort). You need a "guess" for the initialization of the state estimates. Take them to be

$$\begin{aligned} \hat{x}_1(0) &= x_1(0) \\ \hat{x}_2(0) &= x_2(0) \\ \hat{x}_3(0) &= 3x_3(0), \end{aligned}$$

and the covariance matrix is taken as the diagonal matrix

$$\hat{P}(0) = [10^6, 4 \times 10^6, 10^{-4}]^T I_{3\times 3}.$$

Vary the data measurement frequency and

(a) Plot the "true trakectory" as a function of time for the position, velocity, and ballistic coefficient.

(b) plot the error, as a function of time, between the "true trajectory" and the filter solution for the estimated position, velocity, and ballistic coefficient.

(c) (Optional), recast the estimation problem using the extended Kalman filter.

(d) (Very Optional), You might want to try on your own to implement a filter/smoother using the RTS algorithm.

3. Consider the estimation problem for the position $x(t)$ of a particle in a double-well potential subject to Gaussian noise,

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + \kappa\eta(t) \\ x(0) &= 0, \end{aligned}$$

where $< \eta(t) >= 0$, and $< \eta(t)\eta(t') >= \delta(t-t')$. The force

$$f(x) = -\frac{dU}{dx},$$

where the potential $U = x^4 - 2x^2$. This is system has 2 stable critical points at $x = \pm 1$, and an unstable critical point at $x = 0$. The noise strength κ should be fixed at 0.5 for specificity. The particle has been observed at $t_m = 0, 1, 2, .., 10$, and these measurements are given by

$$y(t_m) = x(t_m) + r\gamma(t_m),$$

where γ is a delta-correlated zero mean Gaussian noise process representing the measurement error.

(a) Ignore the measurements for now: show that the equilibrium statistical distribution $P_s(x) \propto \exp(-2U(x)/\kappa)$.

(b) Set up but do not solve the EKF estimation problem for the particle position.

(c) Instead of actually solving the system instead write down some of the conclusions reached by R. N. Miller, M. Ghil, and F. Gauthiez in "Advanced data assimilation in strongly nonlinear systems,", *Journal of Atmospheric Science*, **51**, pp 1037-1056, 1994.

(d) (Optional), set up the RTS/EKF estimation problem for the particle position.