White Noise

A Markov Process. A sequence $\{x_n, n = 1, 2 \ldots\}$

for which $p(x_k|x_\ell) = p(x_k) \quad k > \ell$

i.e. all x_k are mutually independent. Knowing x_ℓ has no bearing on inferring x_k ... for white Gaussian noise it can be shown to be formally the derivative of Brownian motion.

Kolmogorov or Fokker Planck Equation

Return to the Itô SDE, with $\underset{\sim}{X}_t$ Markov Process

$$d\underset{\sim}{X}_t = f(\underset{\sim}{X}_t, t)\,dt + \varepsilon(\underset{\sim}{X}_t, t)\,d\underset{\sim}{W}_t \qquad t \in [t_0, T]$$

let's consider WLOG a scalar case $\underset{\sim}{X}_t = x_t$

$$(*) \quad p(x_t) = p(x,t) \quad \forall\, t \in [t_0, T]$$

and transition probability

$$p(x_t|x_\tau) = p(x|y) = p(x,t;y,\tau) \quad \text{all } t > \tau \in [t_0, T]$$

Want to know the equation obeyed by $(*)$.

Assume existence of $\frac{\partial p}{\partial t}, \frac{\partial}{\partial x}[pf(x,t)]$ and $\frac{\partial^2}{\partial x^2}[p\varepsilon^2]$

where p stands for $p(x,t)$ and $p(x,t;y,\tau)$.

In what follows $\delta(\cdot)$ stands for forward difference

$$\delta z = z_{t+\delta t} - z_t \qquad \delta t > 0$$

let $R(x)$ be > 0 twice continuously differentiable real function such that $t_1 < t_2$

$$\begin{cases} R(x) = 0 & x < t_1 \quad x > x_2 \\ R(x_1) = R(x_2) = R'(x_1) = R'(x_2) = R''(x_1) = R''(x_2) = 0 \end{cases}$$

$$\int_{x_1}^{x_2} \frac{\partial p(x,t;y,\tau)}{\partial t} \delta t + O(\delta t) \Big] R(x)\,dx$$

$$= \int_{-\infty}^{\infty} [p(x,t+\delta t;y,\tau) - p(x,t;y,\tau)] R(x)\,dx \qquad (\star)$$

Using C-K

$$p(x,t+\delta t;y,\tau) = \int_{-\infty}^{\infty} p(x,t+\delta t;z,t)\,p(z,t;y,\tau)\,dz \qquad (\ddagger)$$

using RHS of $(\ddagger)$ $(\star)$ becomes:

(**)
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,t+\delta t; z,t)\, p(z,t;y,\tau)\, R(x)\,dz\,dx - \int_{-\infty}^{\infty} p(x,t;y,\tau)\, R(x)\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(z,t+\delta t; x,t)\, p(x,t;y,\tau)\, R(z)\,dx\,dz - \int_{-\infty}^{\infty} p(x,t;y,\tau)\, R(x)\,dx$$

$$= \int_{-\infty}^{\infty} p(x,t;y,\tau)\left\{\int_{-\infty}^{\infty} p(z,t+\delta t; x,t)\, R(z)\,dz - R(x)\right\}dx$$

where we replace the dummy variable x by z & z by x

We reduce the term in braces. Expand $R(x)$ in Taylor's series

$$R(z) = R(x) + (z-x)R'(x) + \tfrac{1}{2}(z-x)^2 R''(x) + O(z-x)^2$$

then

(*)
$$\int_{-\infty}^{\infty} p(z,t+\delta t; x,t)\, R(z)\,dz - R(x) =$$

$$R(x)\int_{-\infty}^{\infty} p(z,t+\delta t; x,t)\,dz + R'(x)\int_{-\infty}^{\infty}(z-x)\, p(z,t+\delta t; x,t)\,dz$$

$$+ \tfrac{1}{2}R''(x)\int_{-\infty}^{\infty}(z-x)^2 p(z,t+\delta t; x,t)\,dz$$

$$+ \int O(z-x)^2 p(z,t+\delta t; x,t)\,dz - R(x)$$

Note $\displaystyle\int_{-\infty}^{\infty} p(z,t+\delta t; x,t)\,dz = 1$

In (*) z is a realization of $X_{t+\delta t}$ and x a realization of x_t $\therefore$ $z - x = \delta x$ is a realization of δx_t

$$p(z,t+\delta t;x,t) = P_{x_{t+\delta t}|x_t}(z|x) = P_{x_{t+\delta t}|x_t}(x+\delta x|x)$$

$$= P_{\delta x_t|x_t}(\delta x|x)$$

$$\int_{-\infty}^{\infty} (z-x)^k p(z,t+\delta t;x,t)\,dz = E(\delta x_t^k | x_t) \quad k=1,2\ldots$$

but since $dx_t = f(x_t,t)dt + \varepsilon\, dW_t$ and

$$E(dW_t^n) = \begin{cases} 0 & \text{all odd } n \geq 1 \\ 1\cdot 3\cdot 5\cdots(n-1)(\sigma\,\delta t^{1/2})^n & \text{all even } n \geq 2 \end{cases}$$

then
$$\left.\begin{aligned} E(\delta x_t|x_t) &= f(x,t)\,\delta t \\ E(\delta x_t^2|x_t) &= \varepsilon^2(x,t)\,\delta t + O(\delta t) \\ E(\delta x_t^k|x_t) &= O(\delta t) \quad k>2 \end{aligned}\right\} (**)$$

Combining (**) with then (*) becomes

$$\int_{-\infty}^{\infty} p(z,t+\delta t;x,t)R(z)\,dz - R(x)$$

$$= R'(x)f(x,t)\delta t + \tfrac{1}{2}R''(x)\,\varepsilon^2(x,t)\,\delta t + O(\delta t)$$

In view of (**) then (*) becomes

$$\int_{x_1}^{x_2} \left[\frac{\partial p}{\partial t}(x,t;y,\tau)\,\delta t + o(\delta t) \right] R(x)\,dx$$

$$= \int_{-\infty}^{\infty} p(x,t;y,\tau)\,R'(x) f(x,t)\,\delta t + \tfrac{1}{2} R''(x)\,\varepsilon^2(x,t)\,\delta t + o(\delta t)]\,dx$$

& dividing by δt and $\delta t \to 0$ we get

$$\int_{x_1}^{x_2} \frac{\partial p}{\partial t}(x,t;y,\tau)\,R(x)\,dx = \int_{x_1}^{x_2} p(x,t;y,\tau)\,f(x,t)\,R'(x)\,dx$$

$$+ \frac{1}{2} \int_{x_1}^{x_2} p(x,t;y,\tau)\,\varepsilon^2(x,t)\,R''(x)\,dx$$

evaluating integrals on RHS by parts & using the properties of $R(x)$

$$\int_{x_1}^{x_2} \left[\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}(pf) - \frac{1}{2}\frac{\partial^2 (p\varepsilon^2)}{\partial x^2} \right] R(x)\,dx = 0$$

since $R(x)$ arbitrary

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}(pf) - \frac{1}{2}\frac{\partial^2 (p\varepsilon^2)}{\partial x^2} = 0$$

if x is multidimensional random variable

$$\frac{\partial p}{\partial t} + \nabla\cdot(p\underset{\sim}{f}) - \frac{1}{2}\nabla^2(p\varepsilon^2) = 0$$

Fokker-Planck Equation

or Forward Kolmogorov Eq

It describes the evolution of the transition probability density of the Markov process generated by

$$dX_t = f(X_t, t)\delta t + \varepsilon(X_t, t)\, dW_t$$

A Markov process ~~whose diffusion~~ whose transition probability satisfy FPE is called a "diffusion process".

For well-posedness you need I.C. & decay properties:

The I.C. is

$$\lim_{t \to \tau} P_{x_t|x_\tau}(x|y) = \delta(x-y)$$

And the B.C. at $x = \pm\infty$ are

$$P_{x_t|x_\tau}(\infty|y) = P_{x_t|x_\tau}(-\infty|y) = 0.$$

<u>Note</u>: there's a Kolmogorov Backward Eq which is the formal adjoint of the Forward Eq

$$-\frac{\partial p}{\partial \tau}(x,t;y,\tau) = f(y,\tau)\frac{\partial p}{\partial y}(x,t;y,\tau)$$

$$+ \frac{1}{2}\varepsilon^2(y,\tau)\frac{\partial^2}{\partial y^2}p(x,t;y,\tau)$$