Wave/Current Interactions and Wave Breaking

Jorge Ramirez
Mathematics, The University of Arizona.

Joint work with:
J. Restrepo, J. McWilliams, M. Banner, B. Weir *

Uncertainty Quantification Workshop.
April 25-26/2008, Tucson, AZ.

*Work supported by NSF

Motivation
Dissipation in Wave/Current interactions.

Big goal:

✓ How does dissipation at wave scales manifest itself at longer time scales?

✓ Find the right stochastic parametrization.

Dissipative phenomenon: wave breaking.

Motivation
Dissipation in Wave/Current interactions.

Big goal:

✓ How does dissipation at wave scales manifest itself at longer time scales?

✓ Find the right stochastic parametrization.

Dissipative phenomenon: wave breaking.

Method: (McWilliams and Restrepo, 1999)

i. Separation of scales: waves and currents

ii. Random terms: uncertainty and wave breaking

iii. Average momentum equations over small scales

$\Rightarrow$ Compute expected momentum transfer

Stochastic parametrization.

Momentum equation for the mean current velocity $\boldsymbol{v}$ (large scales):

$$\frac{\partial \boldsymbol{v}}{\partial t} = \begin{array}{c}\text{large scale}\\ \text{mom. terms}\end{array} - A\nabla\cdot|\boldsymbol{b}'|^2$$

Wave breaking velocity field:

$$\boldsymbol{b}(t,\boldsymbol{x},z) = \boldsymbol{B}(t,\boldsymbol{x},z) + \boldsymbol{b}'(t,\boldsymbol{x},z), \quad \mathbb{E}\boldsymbol{b}' = \boldsymbol{0}$$

$$t \in \mathbb{R}_+, \quad \boldsymbol{x} = (x,y) \in \mathbb{R}^2, \quad z \in [-Z, \eta(t,\boldsymbol{x})]$$

Stochastic parametrization.

Momentum equation for the mean current velocity $\boldsymbol{v}$ (large scales):

$$\frac{\partial \boldsymbol{v}}{\partial t} = \begin{matrix}\text{large scale}\\ \text{mom. terms}\end{matrix} - A\nabla\cdot|\boldsymbol{b}'|^2$$

Wave breaking velocity field:

$$\boldsymbol{b}(t,\boldsymbol{x},z) = \boldsymbol{B}(t,\boldsymbol{x},z) + \boldsymbol{b}'(t,\boldsymbol{x},z), \quad \mathbb{E}\boldsymbol{b}' = \boldsymbol{0}$$

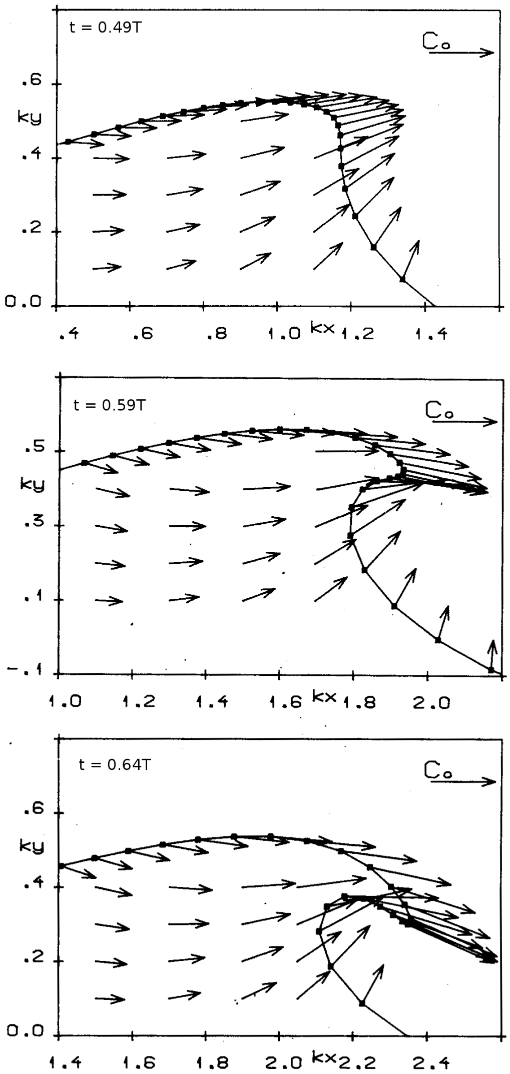
$$t \in \mathbb{R}_+, \quad \boldsymbol{x} = (x,y) \in \mathbb{R}^2, \quad z \in [-Z, \eta(t,\boldsymbol{x})]$$

Goal of parametrization:
Construct a spatio-temporal stochastic process capturing the dissipative effects of $\boldsymbol{b}$.

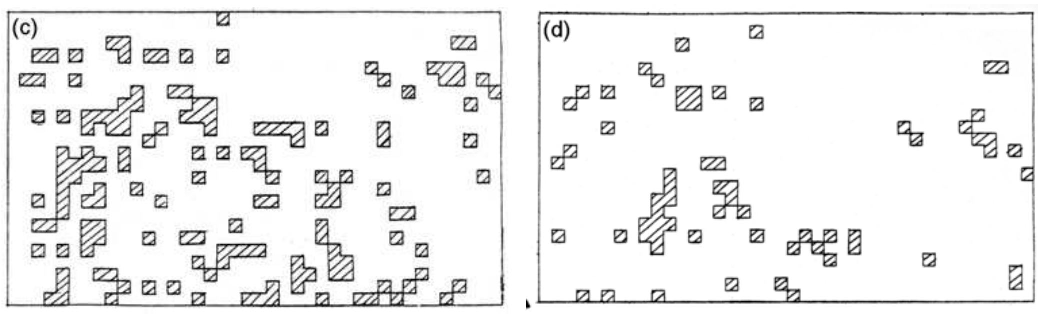
✓ Want: simple, few parameters.
✓ Assume: gravity waves in deep open ocean.

Use data to characterize the small scale (Lagrangian) dynamics of wave breaking velocity, and the large scale (Eulerian) distribution of braking events.

Deep ocean wave breaking.



Time frames of the velocity fields of a computer simulation of wave breaking in deep water (Vinje, Brevig 1981)



Sea breaking fields obtained by remote sensing under different wind regimes (Sharkov 2007).

Deep ocean wave breaking.

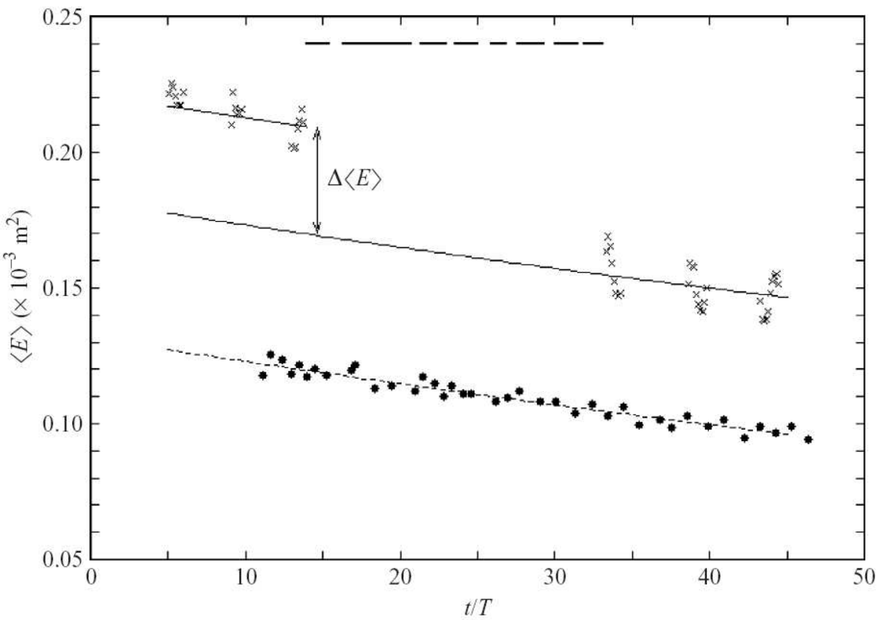
Very common, spatio-temporal random, short-lived events of enhanced turbulence. Scale: 1-5 periods.

Energy input $\mapsto$ Unstable surface $\mapsto$ Energy loss.
good parameter: energy loss of event $= E$

$\boldsymbol{b}_E(t, \boldsymbol{x}, z)$: velocity vector field during breaking.
$\Omega_E \times [-Z, \eta]$: support of $\boldsymbol{b}_E$.

Write: $\Omega_E = [0, \tau_E] \times \tilde{\Omega}_E$

$$\therefore\ E = \rho g \int_{\tilde{\Omega}_E} \eta^2(\tau_E, \boldsymbol{x}) - \eta^2(0, \boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

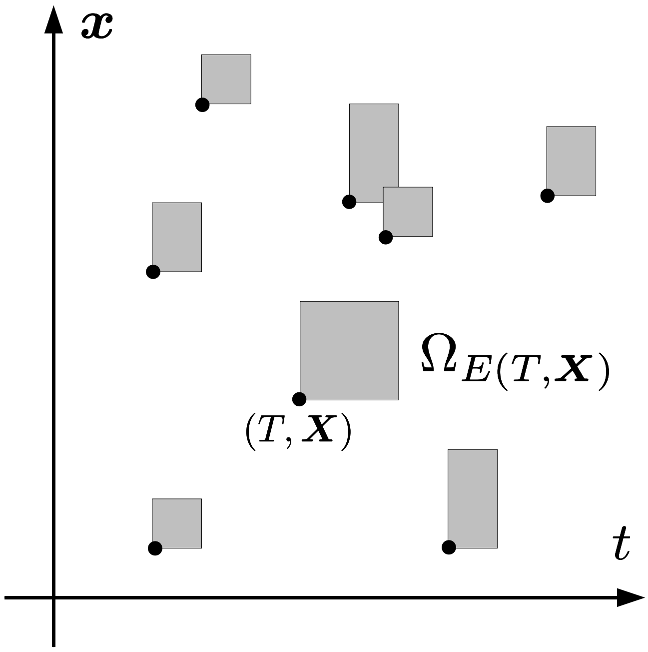


Experimental measurements of energy in waves during breaking ($\times$) and on the onset of breaking ($\star$) (Banner, Peirson 2007).

Spatio-temporal marked Poisson process

Location of wave breaking events:
$\Phi = \{(t, \boldsymbol{X})\}$ a **Poisson process** on $\mathbb{R}_+ \times \mathbb{R}^2$ with intensity measure $\lambda(\,\mathrm{d}t, \mathrm{d}(x, y))$,

$$\mathbb{E}\#\{(t, \boldsymbol{X}) \in \Phi \cap \Gamma\} = \lambda(\Gamma), \quad \Gamma \subseteq \mathbb{R}_+ \times \mathbb{R}^2$$



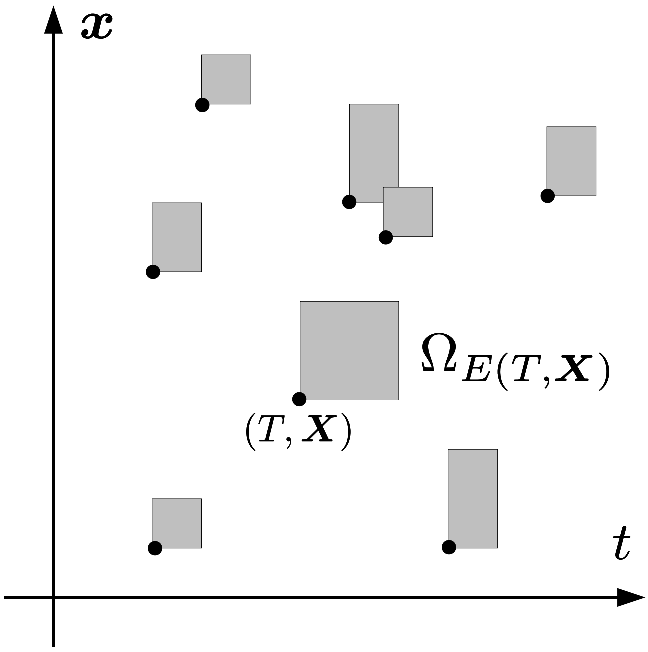
Mark Φ with energy loss: $E(t, \boldsymbol{X})$,

$$\mathbb{P}(E(T, \boldsymbol{X}) \in \mathrm{d}E \,\big|\, (T, \boldsymbol{X}) \in \Phi) = p_{T,\boldsymbol{X}}(E)\,\mathrm{d}E$$

Spatio-temporal marked Poisson process

Location of wave breaking events:
$\Phi = \{(t, \boldsymbol{X})\}$ a **Poisson process** on $\mathbb{R}_+ \times \mathbb{R}^2$ with intensity measure $\lambda(\mathrm{d}t, \mathrm{d}(x, y))$,

$$\mathbb{E}\#\{(t, \boldsymbol{X}) \in \Phi \cap \Gamma\} = \lambda(\Gamma), \quad \Gamma \subseteq \mathbb{R}_+ \times \mathbb{R}^2$$



Mark Φ with energy loss: $E(t, \boldsymbol{X})$,

$$\mathbb{P}\left(E(T, \boldsymbol{X}) \in \mathrm{d}E \,\middle|\, (T, \boldsymbol{X}) \in \Phi\right) = p_{T,\boldsymbol{X}}(E)\,\mathrm{d}E$$

Breaking velocity field: $t \geqslant 0$, $\boldsymbol{x} \in \mathbb{R}^2$,

$$\boldsymbol{b}(t, \boldsymbol{x}, z) = \sum_{(T,\boldsymbol{X}) \in \Phi} \boldsymbol{b}_{E(T,\boldsymbol{X})}(t - T, \boldsymbol{x} - \boldsymbol{X}, z)$$

$$\boldsymbol{b}(t,\boldsymbol{x},z) = \sum_{(T,\boldsymbol{X})\in\Phi} \boldsymbol{b}_{E(T,\boldsymbol{X})}(t-T,\boldsymbol{x}-\boldsymbol{X},z)$$

Let F be a function on the local vector field $\boldsymbol{b}_E$,

$$\bar{F}(t,\boldsymbol{x},z) := \mathbb{E}\Big\{\sum_{\Phi} F(\boldsymbol{b}_{E(T,\boldsymbol{X})})\Big\}$$

$$= \int_0^\infty \int_{(t,\boldsymbol{x})-\Omega_E} F(\boldsymbol{b}_E(t-T,\boldsymbol{x}-\boldsymbol{X},z))\, p_{T,\boldsymbol{X}}(E)\, \lambda(\mathrm{d}T,\mathrm{d}\boldsymbol{X})\,\mathrm{d}E$$

$$\boldsymbol{b}(t,\boldsymbol{x},z) = \sum_{(T,\boldsymbol{X})\in\Phi} \boldsymbol{b}_{E(T,\boldsymbol{X})}(t-T,\boldsymbol{x}-\boldsymbol{X},z)$$

Let F be a function on the local vector field $\boldsymbol{b}_E$,

$$\bar{F}(t,\boldsymbol{x},z) := \mathbb{E}\Big\{\sum_{\Phi} F(\boldsymbol{b}_{E(T,\boldsymbol{X})})\Big\}$$

$$= \int_0^\infty \int_{(t,\boldsymbol{x})-\Omega_E} F(\boldsymbol{b}_E(t-T,\boldsymbol{x}-\boldsymbol{X},z))\, p_{T,\boldsymbol{X}}(E)\,\lambda(\mathrm{d}T,\mathrm{d}\boldsymbol{X})\,\mathrm{d}E$$

In particular, if:

$$p_{T,\boldsymbol{X}}(E) \equiv p(E), \qquad \lambda(\mathrm{d}T,\mathrm{d}\boldsymbol{X}) \equiv \lambda\,\mathrm{d}T\mathrm{d}\boldsymbol{X},$$

$$\bar{F}(t,\boldsymbol{x},z) = \lambda \int_0^\infty \int_{\Omega_E} F(\boldsymbol{b}_E(T,\boldsymbol{X},z))\,\mathrm{d}T\,\mathrm{d}\boldsymbol{X} p(E)\,\mathrm{d}E$$

Examples:

$$\boldsymbol{B} = \lambda \int_0^\infty \left[\int_{\Omega_E}\int_{-H}^{\eta} \boldsymbol{b}_E(T,\boldsymbol{X},z)\,\mathrm{d}z\,\mathrm{d}T\,\mathrm{d}\boldsymbol{X}\right] p(E)\,\mathrm{d}E$$

$$\mathbb{E}|\boldsymbol{b}|^2 = \lambda \int_0^\infty \|\boldsymbol{b}_E\|_{L^2}^2 p(E)\,\mathrm{d}E$$

Estimating λ, $p(E)$ - Gaussian surfaces

$$\lambda(\Gamma) = \mathbb{E}\#\{(t, \boldsymbol{X}) \in \Phi \cap \Gamma\}$$

Assume a homogenous Gaussian random surface:

$$\eta(t, \boldsymbol{x}) = Re \int_{\mathbb{R}^2_+} \exp\{\boldsymbol{\lambda} \cdot \boldsymbol{x} + \omega t\}\, \mathbb{W}(\,\mathrm{d}\boldsymbol{\lambda}), \quad \omega = \sqrt{g|\boldsymbol{\lambda}|},$$

where $\mathbb{W}$ is a complex valued noise with spectral density $2R$, $\mathbb{E}|\mathbb{W}(A)|^2 = \int_A 2R(\boldsymbol{\lambda})\,\mathrm{d}\boldsymbol{\lambda}$, $A \subseteq \mathbb{R}^2_+$.

Covariance function:

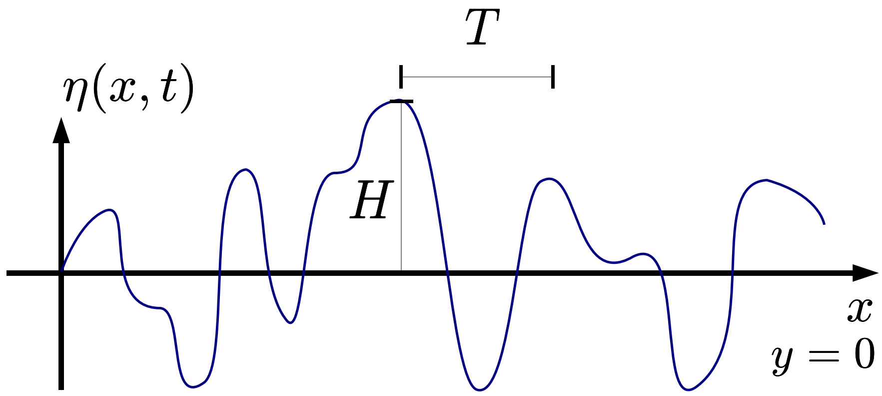
$$\begin{aligned}\mathbb{E}\left[\eta(t, \boldsymbol{x})\eta(t + \Delta t, \boldsymbol{x} + \Delta\boldsymbol{x})\right] &= \int_{\mathbb{R}^2_+} 2\cos(\boldsymbol{\lambda} \cdot \boldsymbol{x} + \omega t) R(\boldsymbol{\lambda})\,\mathrm{d}\boldsymbol{\lambda} \\ &= C(\Delta t, \Delta \boldsymbol{x})\end{aligned}$$

$$\eta(t, \boldsymbol{x}) \sim \text{Gaussian}(0, C(0, \mathbf{0})).$$

Smoothness of $\eta \leftrightarrow$ smoothness of C

Wave breaking criteria - an example

Critical steepness: $H \geqslant \alpha g T^2$, $\alpha \sim 0.002$

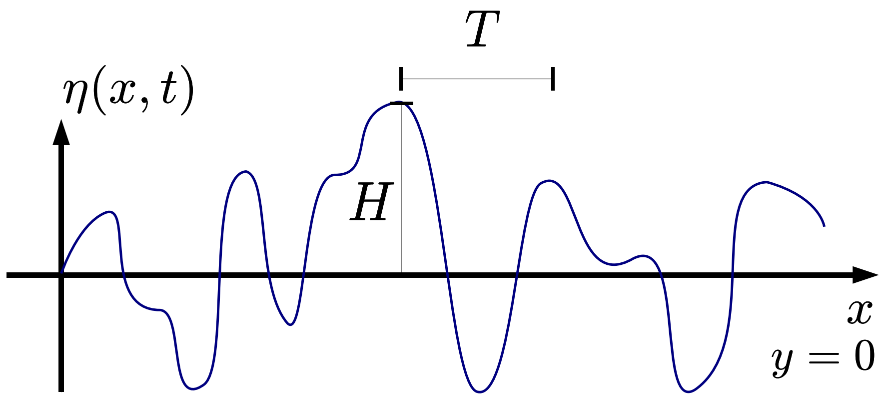


Find probability of breaking:

$\mathbb{P}(H \geqslant \alpha g T^2)$ — **What?**

Wave breaking criteria - an example

Critical steepness: $H \geqslant \alpha g T^2$, $\alpha \sim 0.002$



Find probability of breaking:

$$\mathbb{P}(H \geqslant \alpha g T^2) \text{ — } \textbf{What?}$$

Fix t, $y = 0$, let

$$N_X^m := \#_{|x| \leqslant X}\{\eta \text{ has a local max at } x\}$$

$$N_X^m(A) := \#_{|x| \leqslant X}\{\eta \text{ has a local max satisfying } A\}$$

$$\mathbb{P}(H \geqslant \alpha g T^2) := \lim_{X \to \infty} \frac{\mathbb{E} N_X^m(H \geqslant \alpha g T^2)}{\mathbb{E} N_X^m}$$

An useful tool: Rice's formula

Let $f : \mathbb{R} \to \mathbb{R}$ be a **nice** stochastic process. Number of down-crossings of level u:

$$N_X^{u-} := \#_{|x| \leqslant X}\{f(x) = u, f'(x) < 0\}$$

$$\begin{aligned}\therefore \mathbb{E} N_X^{u-} &= \mathbb{E} \int_{-X}^{X} \delta_u(f(x))\, \mathbb{1}_{\mathbb{R}_-}(f'(x))\, f'(x)\, \mathrm{d}x \\ &= \int_{-X}^{X} \int_{-\infty}^{0} z\, P_{f,f'}(x; u, z)\, \mathrm{d}z\, \mathrm{d}x\end{aligned}$$

$P_{f,f'}(x; \cdot, \cdot)$ = joint density of $(f(x), f'(x))$.

An useful tool: Rice's formula

Let $f : \mathbb{R} \to \mathbb{R}$ be a **nice** stochastic process. Number of down-crossings of level u:

$$N_X^{u-} := \#_{|x| \leqslant X} \{ f(x) = u, f'(x) < 0 \}$$

$$\begin{aligned} \therefore \mathbb{E} N_X^{u-} &= \mathbb{E} \int_{-X}^{X} \delta_u(f(x))\, \mathbb{1}_{\mathbb{R}_-}(f'(x))\, f'(x)\, \mathrm{d}x \\ &= \int_{-X}^{X} \int_{-\infty}^{0} z\, P_{f,f'}(x; u, z)\, \mathrm{d}z\, \mathrm{d}x \end{aligned}$$

$P_{f,f'}(x; \cdot, \cdot)$ = joint density of $(f(x), f'(x))$.

Apply for $f = \eta_x(t, \cdot)$,

$$\mathbb{E} N_X^m = 2X \int_{-\infty}^{0} z\, P_{\eta_x, \eta_{xx}}(0, z)\, \mathrm{d}z$$

Approximate: $\eta_{xx} = (-\frac{2\pi}{T})^2 \eta$ (narrow spectrum)

$$\mathbb{E} N_X^m (H \geqslant \alpha g T^2) \approx 2X \int_{-\infty}^{-\frac{\alpha g}{4\pi^2}} z\, P_{\eta_x, \eta_{xx}}(0, z)\, \mathrm{d}z$$

Thank you.

Some references:

McWilliams, J.C., and J.M. Restrepo. The wave-driven ocean circulation. J. Phys. Oceanogr. 29 (1999), 2523—2540.

Banner, M. L., W. L. Peirson. Wave breaking onset and strength for two-dimensional deep-water wave groups. J. Fluid Mech. 585 (2007), 93–115.

Sharkov, E. Breaking ocean waves. Springer, 2007.

Vinje, T., P. Brevig. Breaking waves on finite water depths: a numerical study. Ship Research Institute of Norway, Report R, 1981

Ochi, M. V. Ocean waves: the stochastic approach. Cambridge, 2007.

Longuet-Higgins, M. S., D. G. Dommermuth. Crest instabilities of gravity waves. III. Nonlinear development and breaking. J. Fluid Mech. 336 (1997), 33–50.

Adler, R. J., T. E. Jonathan E. Random fields and geometry. Springer Monographs in Mathematics. Springer, 2007.