Wave Breaking Dissipation in the Wave-Driven Ocean Circulation

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Abstract

If wave breaking modifies the Lagrangian fluid orbital paths by inducing an uncertainty in the path itself and this uncertainty on wave motion time scales is observable as additive noise, it is shown that within the context of a wave/current interaction model for basin and shelf scale motions it persists on long time scales.

The model of McWilliams et al. (2004) provides the general framework for the dynamics of the wave-current interactions. In addition to the deterministic part, the vortex force, which couples the total flow vorticity to the residual flow due to the waves, will have a part which is associated with the dissipative mechanism. At the same time the wave field will experience dissipation, and tracer advection is affected by the appearance of a dissipative term in the Stokes drift velocity. Consistency leads to other dynamic consequences: the boundary conditions are modified to take into account the diffusive process and proper mass/momentum balances at the surface of the ocean.

1 Introduction

The aim of this paper is to show how dissipative effects on wave scales enter the evolution equations for the general wave-current interaction model developed by McWilliams et al. (2004) (MRL04, hereon). This model has a simpler predecessor, namely the one developed by McWilliams and Restrepo (1999) (MR99). The latter model is in fact the shelf wave/current interaction model modified to account for basin scales. For expository reasons the MR99 will be employed here to show the details on how dissipation at wave scales enters the dynamics of waves and currents.

The focus in this study is on phenomena that has a spatio-temporal scale that is closer to the wave scales, rather than to currents scales. The mechanics of the derivation, however, do not preclude using the same asymptotic and
stochastic machinery to extend the variety and the spatio-temporal range of dissipative phenomena that can be handled and eventually included in a general wave/current interaction model.

Restrepo and Leaf (2002) have considered the effect of small scale dissipation on the long time residual flow due to waves in a boundary layer, namely on the Stokes drift velocity. They found that the dissipation in forced standing-wave flows affect the Lagrangian fluid orbit in such a way as to produce a mean flow resembling the deterministic one, but fluctuations in the particle path gave the residual flow structure, which allows fluid particles to move outside of the trapping cells. In the progressive-wave case the mean flow structure is steady, as it should, but will have fluctuations in the fluid paths that can stall on intermediate time scales. The implications to transport can be significant if the tracers are capable of responding to the residual flow. In that study we used more empirical means, capable of dealing with the problem in question but not generally extendable to the cases considered here, i.e., to the more complete interaction of the waves and currents.

It was Jansons and Lythe (1998) that set the author on the path exploited here (see also Vanden-Broeck (1999)). They solve a Langevin differential equation that has a progressive-wave drift term and an additive stochastic term. The conceptual leap they make is to place more prominence to the noise term than to the deterministic drift term. If there was a physical connection between their calculation and an actual physical phenomenon one would have to look at systems that are dominated by Brownian motion:
It is claimed here, however, that their framework can be adapted to the inertially-dominated wave/current problem by enforcing certain compatibility conditions and scales that lead to MR99 and MRL04.

It will be helpful to mention a particular dissipative mechanism in order to fix ideas. White-capping will be that phenomenon. White-capping is a very common sea surface event, the episodes are short lived, and random in their spatio-temporal distribution (for details and references see Hasselmann (1974) and Komen et al. (1984)). A dynamic of white-capping that has an obvious cause and effect is the dissipation it imparts on the waves and currents. The effective dissipation sometimes changes dramatically when a sudden change in wind strength and/or wind direction occurs. Whitecapping has no complete theory, and inclusion of its presence in ocean dynamics models is accomplished via parameterizations, some of which can be very sophisticated (Bauer et al. (1988), Alves and Banner (2003a), Komen et al. (1994). See also Warner and McIntyre (1999) and its references. Craig and Banner (1994) and Burchard (2001) focus on the connection between wave breaking and turbulence).

The general framework is presented in Section 2. The context will be the basin scale model in MR99. How MRL04, the more general shelf model, is modified by the presence of dissipation, is considered in Section 3. Section 4 consists of a simple computational example that will serve to illustrate the nature and significance of dissipation on the dynamics of simple waves and
currents.

The main goal of this derivation is to answer the question of what these sources of dissipation change in the wave/current picture. However, the derivational methodology also suggests a robust way to parameterize these effects in an operational model. This issue will be briefly discussed in the closing remarks, Section 5.

2 Dissipation on Basin Scales

Considered here is an oceanic region on the rotating Earth containing a stratified, incompressible fluid, whose upper free surface is at \( z = \eta(x,t) \) and whose rigid lower boundary is at \( z = -H(x) \). The vertical coordinate is aligned anti-parallel to the local gravitational force and is denoted by \( z \), \( z = 0 \) corresponds to a quiescent ocean surface; \( \hat{z} \) will denote the unit upward-pointing vector. The position vector is denoted by \((x,z)\), where the transverse or horizontal component is \( x = (x,y) \). Time is denoted by \( t \).

In MR99 it was found that the lowest order equations, time-averaged over the fast wave period scales, representing the basin scale interactions between currents and waves are as follows:

The dimensionless momentum equation, to leading order in the current velocity \( \mathbf{v}^{(0)} \), is

\[
\frac{\partial \mathbf{v}^{(0)}}{\partial T} - \mathbf{V} \times \mathbf{Z} + \nabla \Phi - b^{(0)} \hat{z} = \nu \nabla^2 \mathbf{v}^{(0)} ,
\]

where the rectified velocity is \( \mathbf{V} = \mathbf{v}^{(0)} + u^* \), and the combined local Coriolis
and vorticity is \( \mathbf{Z} = 2\mathbf{\Omega} + \omega^{(0)} \). \( \mathbf{u}^S \) is the Stokes drift velocity. The term on the right hand side is a simple description of dissipation of exclusive significance at the long time scales \( T \).

The generalized geopotential function is

\[
\Phi = p^{(0)} + \frac{1}{2} \mathbf{V}^2. \tag{2}
\]

Since \( \mathbf{v}^{(0)} \) is incompressible, the elliptical problem that determines \( \Phi \) is

\[
\nabla^2 \Phi = \nabla \cdot \{ \mathbf{V} \times \mathbf{Z} + b^{(0)} \mathbf{z} + \nu \nabla^2 \mathbf{v}^{(0)} \}. \tag{3}
\]

On the free surface the pressure is equal to the atmospheric surface pressure.

In deriving (1) the Eulerian velocity \( \mathbf{q} \) had been decomposed into

\[
\mathbf{q} = \epsilon [\mathbf{u}^w, w^w](\mathbf{x}, z, t) + \epsilon \mathbf{v}(\mathbf{x}, z, t, T). \tag{4}
\]

The velocity \( \mathbf{u}^w, w^w)(\mathbf{x}, z, t) \), expressed in terms of its transverse and vertical components, is associated with the gravity wave field, the current is represented by \( \mathbf{v}(\mathbf{x}, z, t, T) \). This description of the Eulerian velocity, as will be shown shortly, will be modified by the presence of stochasticity.

In MR99 the operator \( \langle \cdot \rangle \) was an average over the faster time scale \( t \), typified by the wave period \( 2\pi/\sigma_0 \); dynamic quantities that still have time dependence after averaging will have dependence \( T = \epsilon^2 t \) – the averaging operator will be modified to handle noise later on. The small parameters \( \epsilon = k_0 a \ll 1 \), with \( k_0 \) the typical gravity wavenumber modulus, and \( a \) the typical amplitude of the waves. When the velocity is averaged over the fast
time scales the leading order velocity is an approximation of the average current

\[ \langle \mathbf{v} \rangle = \mathbf{v}^{(0)} + \mathcal{O}(\epsilon) = \langle \mathbf{q} \rangle / \epsilon^2. \]  

(5)

A compatible statement for the vorticity is

\[ \langle \mathbf{\omega} \rangle = \mathbf{\omega}^{(0)} + \mathcal{O}(\epsilon). \]  

(6)

There are also analogous expressions for the average buoyancy \( b \) and the tracer \( \Theta \), which in turn may be related to one another via an equation of state (see MR99). The relations among the non-dimensional parameters are the following:

\[ \Omega_0, \nu_0, N_0 = \mathcal{O}(\epsilon^2); \quad B_0, \tau_0, T_0 = \mathcal{O}(\epsilon^4). \]  

(7)

In order of appearance, these are the Coriolis, the long-time dissipation, the Brünt Vaïsala frequency, the buoyancy, the surface stress and the surface tracer gradient.

The dimensionless boundary conditions are

\[ w^{(0)} = \nabla \cdot \mathbf{M} \text{ at } z = 0, \]  

(8)

where \( w^{(0)} \) is the vertical component of the velocity, \( \eta^w \) is the sea elevation changes associated with the waves, and

\[ \mathbf{M} \equiv \langle \mathbf{m} \rangle, \]  

(9)

where

\[ \mathbf{m} \equiv u_w(x, 0, t)\eta^w(x, t) \]
In (8) the fact that the term $\left\langle D\eta^{(0)}/Dt \right\rangle$ is smaller by $O(\epsilon^2)$ is used. Here, $\eta^{(0)}$ is the leading order sea elevation associated with the currents. On the other hand, if it were assumed that the horizontal variation of both the currents and wave statistics were on a slow scale $X = \epsilon^2 x$, then the amplitude of $w^{(0)}$ would be smaller by $\epsilon^2$ (for 3D continuity balance), and the boundary condition (8) would be generalized by the addition of $\partial \eta^{(0)}/\partial t$ on the right-hand side. This addition would formally permit very long (i.e., shallow-water) surface gravity waves in the current dynamics.

The pressure equation is

$$p^{(0)} = \eta^{(0)} + p^a - P \text{ at } z = 0,$$

(10)

where we have assumed that the slow atmospheric pressure variations $p^a$ scale in a similar way to $p^{(0)}$, and the wave-added pressure adjustment term is

$$P \equiv \left\langle \bar{p} \right\rangle,$$

(11)

where

$$\bar{p} = \frac{\partial p^w}{\partial z}(x, 0, t)\eta^w(x, t) = \left(\frac{\partial \eta^w}{\partial t}\right)^2.$$

The slow-time surface stress condition is

$$\nu \left(\frac{\partial \eta^{(0)}}{\partial z} + S\right) = \tau \text{ at } z = 0,$$

(12)

where

$$S \equiv \left\langle \bar{s} \right\rangle$$

(13)

is the wave-added correction, with

$$\bar{s} = \frac{\partial^2 u^w(x, 0, t)}{\partial z^2} \eta^w(x, t).$$

8
The leading order tracer equation is
\begin{equation}
\frac{\partial \Theta^{(0)}}{\partial T} + \mathbf{V} \cdot \nabla \Theta^{(0)} = \kappa \nabla^2 \Theta^{(0)},
\end{equation}
the right hand side term a simple description of dissipation associated purely with the long time scales. The tracer surface boundary condition is
\begin{equation}
\kappa \frac{\partial \Theta^{(0)}}{\partial z} = T \text{ at } z = 0.
\end{equation}

Further details on the derivation of the above equations are found in MR99. These equations will be modified by the presence of additive noise as follows:

The Lagrangian path, described by the position vector \( \mathbf{Z}_t \equiv (x_t, z_t) \), of a fluid parcel with added stochasticity in three space dimensions and time will be assumed to be
\begin{equation}
\frac{d\mathbf{Z}_t}{dt} = \epsilon (\mathbf{u}^w, \mathbf{w}^w)(\mathbf{Z}_t, t) + \epsilon^2 \mathbf{v}(\mathbf{Z}_t, T) + B(\mathbf{Z}_t, t, T)d\mathbf{W}_t.
\end{equation}
The first two terms in the velocity correspond to the differential contribution of the deterministic part. These alone are consistent with the velocity field in MR99, see (4). The last term is the differential contribution of the velocity associated with noise, assumed attributed to the surface breaking events or other forms of dissipation at the short time scales, and modelled here as an additive vector-valued Wiener process \( \mathbf{W}_t \); the \( 3 \times 3 \) variance matrix \( B \) may have a slow and a fast temporal dependence as well as spatial dependence. The statistical description of the dissipation would be obtained from field
data. The variance above is presumed to be dependent or parameterized by the transverse component of the position vector, which is taken to be the random process; the vertical component is thus random by the coupling of vertical and transverse components. The particle path, as given by (16), is consistent with the notion that at scales much shorter than the wave period the orbital path is Brownian; at scales comparable to the wave scale it is a combination of the orbit associated with the irrotational part of the velocity and stochasticity. At much larger time scales, $T$, the velocity associated with currents enters the description of the path. The noise term, if properly chosen or derived, will conserve the Jacobian of the volumetric integrals of the velocity, albeit manifest temporal fluctuations, so that on average the incompressibility constraint is conserved. However, this term can also originate in some parameterized external force (or unresolved physics in an asymptotic setting of the equations of motion) which modifies the pressure in the flow, the gradient of which is a source of acceleration. Ultimately, a Langevin representation for the process of random fluctuations in the equations of motion, be it as additive or multiplicative, is a modelling choice with many appealing characteristics, one of which is simplicity and robustness –as compared to using a random velocity field say, a far more complex notion (Olla and Paradisi (2004) and Klyatskin and Woyczynski (1995) show typical applications of the random velocity field concept to passive scalar evolution)–. Here an ad-hoc approach is taken, making the noise and variances in the transverse and the vertical direction, two measurable degrees of freedom in the model.
Uncertainty in the sea elevation phase can be thought of accounting to uncertainty in the pressure due to wave dissipation; the uncertainty in the transverse velocity a reflection of uncertainties in the transverse component of position.

In what follows the generality of the Langevin representation will be greatly simplified to suit modelling purposes. It will be assumed that the noise enters the dynamics via surface processes, in the form of pressure fluctuations which are in turn due to external forcing, wave breaking or some other type of dissipative process, such as losses ascribed to dissipation not related to wave breaking.

The wave sea elevation $\eta^w$ is assumed to be composed of a linear superposition of individual components with different horizontal wavenumbers $k_j$, each component given by,

$$\eta^w_j = a_j \cos[k_j \cdot x_t - \sigma_j t - \sqrt{2\gamma_d} W_t + \theta_j]e^{-\gamma_d t}, \hspace{1cm} (17)$$

where $\gamma_d$ is real, non-negative and $O(\epsilon^2)$; it is the parameter associated with sea elevation dissipation due to propagation and $W_t$ is a zero-mean scalar Wiener process, and $\theta_j$ is the sea elevation phase. These gravity waves arise primarily through the interaction of the wind with the ocean surface. Their dispersion relation is given by $\sigma_j = \sqrt{k_j}$.

Consistent with (17) the Lagrangian path, to lowest orders, is

$$dx_t = \epsilon u^w dt + \sqrt{2B^h(X, T)} dW^b_t,$$

$$dz_t = \epsilon w^w dt. \hspace{1cm} (18)$$
The zero-mean components of the 2-dimensional Wiener process $W^h_t$ are assumed independent of $W_t$.

The lowest order Lagrangian path $Z_t^{(0)}$, for $t \geq 0$, is described by

$$
\begin{align*}
    x_t^{(0)} &= \sqrt{2B^h(X,T)} \cdot W^h_t, \\
    z_t^{(0)} &= z.
\end{align*}
$$

(19)

Incorporating the lowest order result, the Lagrangian path becomes

$$
\begin{align*}
    dx_t^{(1)} &= \sum_j a_j \sigma_j e^{k_j \cdot \hat{x}_t} e^{-\gamma dt} \cos(\phi_j) \left( dt - \frac{D}{\sigma_j} dw_t \right) \\
        &\quad - \frac{\gamma_d}{\sigma_j} \left( 1 + \frac{D^2}{2\gamma_d} \right) \sin(\phi_j) dt \hat{k}, \\
    dz_t^{(1)} &= \sum_j a_j \sigma_j e^{k_j \cdot \hat{x}_t} e^{-\gamma dt} \sin(\phi_j) \left( dt - \frac{D}{\sigma} dw_t \right) \\
        &\quad - \frac{\gamma_d}{\sigma} \left( 1 + \frac{D^2}{2\gamma_d} \right) \cos(\phi_j) dt,
\end{align*}
$$

(20)

where

$$
\phi_j = [k_j \cdot \hat{x} + Dw_t - \sigma_j t],
$$

(21)

and

$$
D = \sqrt{2k^2 B^h(X,T) + 2\gamma_d(X,T)}.
$$

(22)

Here $w_t$ is the composite Wiener process $w_t$. Strictly speaking the asymptotic procedure makes the two Wiener processes lose independence from each other. From a practical standpoint this might be a moot point, as it might prove difficult to measure independently the parameters for the two stochastic processes. The spatial dependence of the variance has been taken to be
at much larger scales than the typical wavelength of the gravity waves and thus approximately independent of the stochastic process $x_t$. The dynamics of these waves, to leading order, are not influenced by the stratification; however, there is a wave-correlated component of the buoyancy and tracer fields, $b^w$ and $\Theta^w$, due to the stratification – there is an implied relationship between the buoyancy and tracers, as described in MR99. The order (1) buoyancy balance and its resulting wave solution is

$$b_j^w = -\sum_j a_j e^{kz} \hat{N}^2(z) e^{-\gamma d t} [\cos(\phi_j) dt - \frac{D}{\sigma_j} dw_t]$$

$$- \frac{\gamma_d}{\sigma_j} \left( 1 + \frac{D^2}{2\gamma_d} \right) \sin(\phi_j) dt [cos(\phi_j) dt - \frac{D}{\sigma_j} dw_t],$$

with analogous relations for $\Theta^w$. $\hat{N}$ is the Brünt-Väisälä frequency. The pressure fluctuations are trivially related to the sea elevation and thus will have a stochastic component.

The vector and scalar field variables are decomposed into mean and fluctuating components. The averaging operator used in MR99 and MRL04 is now more general: the average of a quantity $r$, say, will be defined by

$$\langle En(r(\cdot, T)) \rangle = \lim_{T_p \gg T^*} \frac{1}{T_p} \int_0^{T_p} \int_{-\infty}^{\infty} r(\cdot, x, T, t') \Pi(x) dx dt',$$

where $T_p \gg T^*$ is meant to convey that $T_p$ should be sufficiently long compared to the time scale related to the variance of the noise and the length of the waves, i.e., approximately $T_p >> \lambda/kD_f$, where $D_f$ is an estimate of the size of the variance. $\Pi$ is the probability measure, which in the concrete examples to be shown later is taken to be Gaussian. The new $t$-average still
yields a quantity that varies at time scales typical of the longer wind and current variability, however, its interpretation is different.

Restating (20) the Lagrangian path at order (1) is given by

\[
d\mathbf{Z}^{(1)}_t = (u^w, w^w)(\mathbf{Z}^{(0)}_t, t) \, dt, \tag{25}
\]

which leads to

\[
\mathbf{Z}^{(1)}_t = \int_0^t (u^w, w^w)(\mathbf{Z}^{(0)}_s, s) \, ds. \tag{26}
\]

At the next order

\[
\frac{d\mathbf{Z}^{(2)}_t}{dt} = \int_0^t (u^w, w^w)(\mathbf{Z}^{(0)}_s, s) \, ds \cdot \nabla (u^w, w^w)(\mathbf{Z}^{(0)}_t, t) + \mathbf{v}. \tag{27}
\]

The first term is recognized, after averaging, as the Stokes drift velocity

\[
\mathbf{u}^s = \left\langle \frac{En}{\sigma} \left( \int_0^t (u^w, w^w)(\mathbf{Z}^{(0)}_s, s) \, ds \cdot \nabla (u^w, w^w)(\mathbf{Z}^{(0)}_t, t) \right) \right\rangle, \tag{28}
\]

with a dimensionalizing scale of \( \epsilon^2 \sigma_0 / k_0 \).

### 2.1 Simple Example, Basin Scale Case

Consider the case of a monochromatic wave, as given by (17). It will be assumed in this example that the dynamics have a single transverse direction and depth dependence only. We will simplify (20) further by assuming that the order (1) particle paths are given by

\[
d\mathbf{x}^{(1)}_t = a \sigma e^{kz} e^{-\gamma_d t} \left[ \cos(\phi) \, dt - \frac{\gamma_d}{\sigma} \left( 1 + \frac{D^2}{2 \gamma_d} \right) \sin(\phi) \, dt \right] \hat{k},
\]

\[
dz^{(1)}_t = a \sigma e^{kz} e^{-\gamma_d t} \left[ \sin(\phi) \, dt - \frac{\gamma_d}{\sigma} \left( 1 + \frac{D^2}{2 \gamma_d} \right) \cos(\phi) \, dt \right]. \tag{29}
\]
Explicit calculation of the transverse and vertical components of the Stokes drift velocity yields

\[ u^S(z, T) = a^2 k \sigma e^{2kz} \left[ \hat{k} \left( 1 + \frac{1}{\sigma^2} \left( \gamma_d^2 - \frac{D^4}{4} \right) \right) , -\frac{2 \gamma_d}{\sigma} \right] D. \]  

(30)

Here,

\[ D = \frac{e^{-2\gamma_0 T}}{1 + \frac{1}{\sigma^2} \left[ 1 + \left( \gamma_d - \frac{D^2}{2} \right)^2 \right]}. \]

It is noted that the exponential loss term was made to survive the time averaging, presumed appropriate because \( \gamma_d t = \gamma_0 T \). When \( \gamma_d \) and \( D \) go to zero we obtain the familiar deterministic result for the Stokes drift velocity, with the vertical component identically zero. As shown by Jansons and Lythe (1998), it is also the case here that bi-directional wave fields can cancel out the deterministic portion of the Stokes drift velocity and leave a non-zero diffusive part. A similar calculation yields

\[ M = \langle En(\hat{m}) \rangle = \frac{1}{2} a^2 \sigma \left[ 1 + \frac{1}{\sigma^2} \left( \gamma_d - \frac{D^2}{2} \right)^2 - \frac{D^4}{\sigma^2} \right] \hat{D} k, \]

\[ S = k^2 M, \]

(31)

and

\[ P = \langle En(\hat{p}) \rangle = \frac{1}{2} \sigma^2 a^2 e^{-2\gamma_0 T} \left[ 1 + \frac{1}{\sigma^2} \left( \frac{D^2}{2} + \gamma_d \right)^2 \right], \]

\[ N = \langle En(\hat{n}) \rangle = \sigma^2 a^2 e^{2kz} e^{-2\gamma_0 T} \left[ 1 + \frac{1}{\sigma^2} \left( \frac{D^2}{2} + \gamma_d \right)^2 \right], \]  

(32)

with

\[ \hat{n} = \frac{1}{2} (u^w)^2 + (w^w)^2. \]

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Quantities dependent on the Stokes drift velocity will also be affected by the presence of dissipation, namely, the tracer equation and the geopotential function $\Phi$.

3 The Shallow Water Case

With the aim of making brief how the shelf case, its deterministic formulation appearing in MRL04 and LRM05, is modified by the presence of dissipation we will make frequent reference to these two papers and modify the notation slightly to make the comparison easier. The main difference in the scaling, between the basin case and the shelf scale case, is that in the latter there are three time scales: the fast wave scale, $t$, the intermediate long-wave scale $\tau = \epsilon^2 t$, and the current scale $T = \epsilon^4 t$. Associated with these time scales is a short and long spatial scale, the latter being $X = \epsilon^2 x$.

In MRL04/LRM05 two averaging operators were defined, namely the average over the fast scales and the average over intermediate–long-wave–scales. The fast-time average is identical to (24), however the symbol used in MRL04 for this average is the overbar. The average over intermediate scales was denoted by angled brackets. Both of these are modified, as in (24), by the need to ensemble average. Fluctuations at the long-wave time scale were denoted $(\cdot)^{\dagger} \equiv \overline{\cdot} - \langle \cdot \rangle$.

Adapting to the notation of LRM05 the particle path is described as

$$dZ_t = \epsilon U dt + B(Z_t, t, \tau, T)dW_t,$$

(33)
where
\[ U = (u_0, w_0) + \epsilon(q^l w + v) + \epsilon^2 u^{lw} + \epsilon^3 (w^l w + w^c). \] (34)

The first term is associated with the linearized wave field. \( q^l w \) is the long-
wave velocity, higher order corrections to the linearized wave velocity carry
the superscript \( w^v \). The current velocity is \((v, w^c)\).

The \( j^{th} \) wave spectral component of the leading order velocity is
\[
(u_0)_j = \frac{a_j k_j \cosh Z_j}{\sigma_j \tanh \mu \cosh \mathcal{H}_j} \left[ \cos(\phi_j) dt - \frac{\gamma d}{\sigma_j} \left( 1 - \frac{D^2}{2\gamma d} \right) \sin(\phi_j) dt \right],
\]
\[
w_0 = a_j k_j \cosh Z_j \tanh Z_j \frac{\sigma_j \tanh \mu \cosh \mathcal{H}_j}{\sigma_j \tanh \mu \cosh \mathcal{H}_j} \left[ \sin(\phi_j) dt - \frac{\gamma d}{\sigma_j} \left( 1 - \frac{D^2}{2\gamma d} \right) \cos(\phi_j) dt \right].
\] (35)

The associated sea elevation is
\[ \eta_j^w = a_j e^{-\gamma d T} \cos(\phi_j), \] (36)

where \( \phi_j \) as given by (21), \( \mathcal{H}_j = k_j \mu H \) and \( Z_j = k_j(z + \mu H); \mu = k_0 H_0, \) is a
scaling parameter, of order 1 in the shelf case, which arises naturally in the
nondimensionalization (see MRL04). \( \gamma d = O(\epsilon^4) \) now due to a change of time
scales relevant to shelf dynamics. The equation for the complex amplitude
for the waves is
\[
\frac{dA_j}{d\tau} = C_{g,j} \cdot \nabla_X A_j + \frac{1}{2} A_j \nabla_X \cdot C_{g,j} + \frac{1}{2} i M_j |A_j|^2 A_j
+ \frac{ikA_j}{\sinh[2\mathcal{H}_j]} (\sigma_j Z + 2 \int_{-\mu H}^{0} \cosh[2Z_j] \mathcal{V}_j(z) dz),
\] (37)
where \( C_{g,j} \) is the group velocity; \( Z = \eta^w + \zeta \) is the long-time sea elevation, and \( V_j = k_j \cdot (q^w + v) \). The functions \( M_j \) and \( Z \), which have a bearing on the phase but not on the amplitude of the complex \( A_j \) (described in MRL04) are modified by the presence of dissipation, however, their explicit calculation is omitted here. The wave dispersion relation is

\[
\sigma^2 = \frac{k}{\tanh |\mu|} \tanh H_j.
\]

The evolution equations for the large spatio-temporal scale dynamics of the wave number and frequency will remain unchanged with dissipation.

### 3.1 Long-Wave Dynamics

The unforced conservative equations for the long-wave component of the flow are

\[
\begin{align*}
\frac{\partial q^w}{\partial \tau} + \nabla_X p^w &= -\nabla_X En(\bar{n})^\dagger \\
\frac{\partial p^w}{\partial z} &= -\frac{\partial}{\partial z} En(\bar{n})^\dagger \\
\nabla_X \cdot q^w + \frac{\partial w^w}{\partial z} &= 0 \\
w^w(-\mu H) + q^w(-\mu H) \cdot \nabla_X [\mu H] &= 0 \\
w^w(0) - \frac{\partial \eta^w}{\partial \tau} &= \nabla_X \cdot En(\bar{m})^\dagger \\
p^w(0) - \frac{1}{\tanh \mu} \eta^w &= -En(\bar{p})^\dagger.
\end{align*}
\]

The fast-wave dynamics appear in these equations, imparting momentum and mass flux into the long-wave dynamics. Hence, even in the absence of forcing, the long waves can respond dynamically due to the waves (e.g.
set-up/set-down). The quasi-static pressure and sea level are, respectively, 
\( \hat{p}^{lw} = -(E_n(\tilde{n}(z)))^\dagger \) and \( \hat{\eta}^{lw} = \tanh[\mu](-(E_n(\tilde{n}(0)))^\dagger + E_n(\tilde{p}))^\dagger \) (compare to MRL04, Eq. (6.3)).

Explicit calculation of the dissipative expressions is possible using a particularly simple form for the noise. To simplify, it is assumed that the wave field is monochromatic and that spatial dependence is in transverse and depth dimensions only. Assuming that the variance is of the form (22) the following are modified by the presence of surface dissipation:

\[
M = \left( \frac{1}{2} a^2 k \sigma \left[ 1 + \frac{1}{\sigma^2} \left( \gamma_d - \frac{D^2}{2} \right)^2 - \frac{D^4}{\sigma^2} \right] D \right)^\dagger \tag{39}
\]

\[
S = k^2 M \tag{40}
\]

\[
P = \left( \frac{1}{2} a^2 k \tanh[H] \frac{\sinh[H]}{\sinh[2H]} e^{-2\gamma_0 T} \left[ 1 + \frac{1}{\sigma^2} \left( \gamma_d + \frac{D^2}{2} \right)^2 \right] \right)^\dagger \tag{41}
\]

\[
N = \left( \frac{1}{2} a^2 k \cosh[2Z] \frac{\cosh[2Z]}{\cosh[2H]} \sinh[2H] e^{-2\gamma_0 T} \left[ 1 + \frac{1}{\sigma^2} \left( \gamma_d + \frac{D^2}{2} \right)^2 \right] \right)^\dagger \tag{42}
\]

### 3.2 Dissipation at Current Scales

Following the same strategy and assumptions as in the immediately prior subsection, explicit forms of the averaged quantities can be computed. The quantities \( M, S, P, \) and \( N \) that appear above have the same form however the averages are now understood as being taken over the longer time scale \( T \), \( i.e. \) the average \( \langle \cdot \rangle \) is used.

The Stokes drift velocity is

\[
v^{St} = \left\langle \frac{a^2 \sigma}{2 \sinh^2[H]} k \left[ \cosh[2Z] + \frac{1}{\sigma^2} \left( 2\gamma^2_d + \left[ \gamma_d - \frac{D^2}{2} \right]^2 \right) \right] D \right\rangle, \tag{43}
\]
and

\[ w^{St} = -\left\langle \frac{a^2 \sigma}{\sinh^2[H]} k \frac{\gamma_d}{\sigma} D \right\rangle. \]

When dissipative effects are present under exceptional circumstances it would be true that

\[ w^{St} = -\nabla_N \int_{-\mu H}^{z} v^{St} dz', \]

which is always the case when the dissipation is zero. The mass balance has to take into account a momentum flux due to the waves. The boundary condition at \( z = 0 \), as given by MRL04 Eq. (9.12), is now

\[ w^c(0) = \nabla \cdot T^{St}, \tag{44} \]

where \( T^{St} = \langle En(m) \rangle \).

The static sea-level and pressure fields, given by Eq. (9.8), MRL04, are

\[ \hat{\zeta} = -\frac{\tanh[\mu]}{\epsilon} P_0 - N(0) + P \]

\[ \hat{p}(z) = -\frac{1}{\epsilon} P_0 - N. \tag{45} \]

The dynamic boundary conditions, given by (9.9)-(9.11), MRL04,

\[ \frac{1}{\tanh[\mu]} \zeta^c - P^c(0) = \hat{\zeta} \frac{\partial \hat{p}}{\partial z}(0) + \left\langle \eta^{lw} \frac{\partial p^{lw}}{\partial z}(0) \right\rangle + \frac{1}{\tanh[\mu]} P_0, \tag{46} \]

where \( P_0 \) is given in MRL04 Eq. (9.11) and LRM05 Eq. (3.7). It is convenient to redefine the vertical coordinate range to be \( -H(x) \leq z \leq \zeta + \hat{\zeta} \) rewriting the momentum equations with the quasi-static component of the pressure and sea elevation removed (see MRL04 and LRM05); by doing so the calculation of \( P_0 \) is not required. The Bernoulli head \( K \) (See LRM05 Eq. (3.15)) with the
quasi-static components removed simplifies greatly as well. The modification due to the presence of dissipation is that the Bernoulli head would appear multiplied by
\[ e^{-2\gamma_0 T} \left[ 1 + \frac{1}{\sigma^2} \left( \gamma_d + \frac{D^2}{2} \right)^2 \right]. \]
The momentum equations (Eq. (9.15) MRL04) also have a vortex force term \( J \), and an adjustment to the quasi-static pressure \( K \). These change owing to the new dissipative Stokes drift velocity.

The tracer equation and buoyancy, as given by Eqs. (10.7) and (11.12), MRL04, include the Stokes drift velocity and thus will be affected by dissipation as well. The tracer evolution equation is also modified by changes in \( e^2 \). (see Eqs. (10.8), (10.9), MRL04):
\[ e^2 = a^2 e^{-2\gamma_0 T} \sinh^2[Z] \sinh^2[H]. \]
The gradient of the buoyancy modifies the long-time vorticity as well as the vertical momentum (see Eqs. (11.12)-(11.13), MRL04) and thus dissipation also modifies the momentum balance due to the stratification.

4 A Numerical Illustration

In MRL04 an example calculation was used to show that the interaction between waves and currents is significant. Insights into this example appear in LRM05 as well. In that example we chose for illustration to calculate the interaction of waves and currents on a broad shelf region with a gentle bottom slope up toward the west and a circular depression (Figure 1a. See
MRL04 for full details). The primary wave was specified to be incident from the deeper region to the east, and propagated westward through the domain en route to a coastline farther west. The currents were dominated by a cyclonic vortex, initially centered over the bottom depression (Figure 1b). We examined both wave and current solutions, but their mutual interaction was artificially constrained to simplify things: the wave field was in steady-state balance (on the $\tau$ scale) with the initial vortex and the vortex evolution (on the $T$ scale) was calculated with the wave field frozen in its initial state, rather than co-evolving with the currents.

Here the effect of dissipation on current scales is shown and compared to the case without dissipation, the MRL04 case. In the baseline case for the wave evolution, the horizontal domain was a square with a span of $L = 56$ km. The resting depth decreases from 25 m in the east to 20 m in the west, and the superimposed depression was 2 m deep with a Gaussian decay
on a spatial scale of 7 km and a center in the northeast quadrant (Figure 1a). The incident wave was uniform along the eastern boundary. It had an amplitude of $a = 1.5 \text{ m}$, slow phase of $\theta = 0$, wavelength of $2\pi/k = 160 \text{ m}$, and propagation direction to the southwest. The associated wave period was $2\pi/\sigma = 11.5 \text{ s}$, phase speed was $13.6 \text{ m s}^{-1}$, and group velocity is $|C_g| = 10.5 \text{ m s}^{-1}$. A cyclonic current vortex was centered over the bottom depression (Figure 1b). It had a Gaussian shape for $\chi^c(X)$ with a peak amplitude of $10^{-4} \text{ s}^{-1}$. The widths of both the depression and vortex were 7 km. The associated velocity field had a maximum speed of about $0.16 \text{ m s}^{-1}$. The initial cyclonic vortex (Figure 1a) had a peak amplitude of $10^{-4} \text{ s}^{-1}$ that was equal to $f$; hence the initial vortex Rossby number was one.

The calculations in MRL04 and those presented here made use of depth-averaged and the suppression of any $z-$dependence of all dynamical quantities. In the calculations the vertical component of the Stokes drift was suppressed and the drift velocity actually is used in its depth-averaged form. In MRL04 the vertical component of the Stokes drift was diagnosed from the transverse one and thus of little interest. As was derived in this study if dissipative effects are important the vertical component of the Stokes drift is not simply given by an integral of the transverse component. In this two-dimensional calculation the vertical component of the Stokes drift is artificially suppressed, however.

Figure 2 reproduces the vorticity and velocity fields, after 4 days of evolution, when no dissipation is present. Figure 3 shows the Stokes drift and
Figure 2: (a) vorticity $\chi^c(X) \ [s^{-1}]$ and (b) velocity $v(X) \ [m \ s^{-1}]$ (both as vector and contoured speed) for the currents in the baseline case at a time of $T = 4.0$ days.

For simplicity the wave dissipation $\gamma_d$ has been set to zero and there is no time dependence in the variance and thus none in the dissipation. The spatial dependence enters the dissipation function via $D(X)$. The dissipation function produces a localized region of intense dissipation. This patch is stationary and present during the full 4-day simulation. The dissipation is maximally 30% higher in the epicenter of the localized region, than in locations far away from it.

The resulting vorticity and velocity, with dissipation present, are shown...
Figure 3: (a) Stokes drift $v^{St}(X) [\text{m s}^{-1}]$ and (b) the combined velocity $v + v^{St}(X) [\text{m s}^{-1}]$ (both as vectors and contoured speeds) in the baseline case at a time of $T = 4.0$ days.

Figure 4: Dissipation as a function of space.
in Figure 5. The depth-averaged transverse Stokes drift, and the combined Stokes drift and current velocities, appear in Figure 6.

The figures make it clear that dissipation could play a significant role in the dynamics of the wave/current system. In fact, since the dissipation can have a long-time temporal dependence it is possible the collision of these waves and temporally-variable white-capping events with the currents can produce some fairly complex resulting spatio-temporal hydrodynamics.

5 Concluding Remarks

This study suggested an answer to the question of how dissipation at time scales of the waves manifests itself at longer time scales within the wave/current interaction model in MRL04 and LRM05. An example of the type of dissipation we have in mind is whitecapping. Parameterized wave attenuation
Figure 6: (a) Stokes drift $v^{St}(X)$ [m s$^{-1}$] and (b) the combined velocity $v + v^{St}(X)$ [m s$^{-1}$] (both as vectors and contoured speeds) in the baseline case at a time of $T = 4.0$ days. With dissipation. See Figure 3 for comparison.

was also included in the phenomenology. The approach used here is purely ad-hoc and as such assumes that the dissipation parameters are obtained from data, for example, as there is no dynamic for the noise.

The basic idea was to assume that breaking lead to fluctuations in particle paths at time scales smaller than the wave time scale as well as uncertainties in the phase. The e-folding distance for the attenuation of the gravity waves was made large in accordance with what is known about the propagation of these waves. How the Lagrangian paths were modified due to the presence of dissipation does not lead to a purely random velocity field, but rather, a velocity which has a stochastic component.

Dissipation modifies such quantities as the Stokes drift velocity, the pressure, and the vortex force, and thus also modifies the radiation stresses.

Dissipation also modifies how boundary conditions at the air-sea inter-
face are applied. A sequel to this paper will develop further the stochastic parametrization and with it, compute in detail the dynamic variables beyond the order at which was done here. With this more complete model we will then investigate in detail how radiation stresses change in the presence of dissipation. A full derivation of the radiation stresses for the shelf setting due to wave/current interactions appears in Lane et al. (2005). Although most of the quantities that contribute to the radiation stress have been examined here, a derivation and the full impact of wave dissipation on these quantities has yet to be done.

It should not be too difficult to propose a crude slow-time evolution equation for the wave dissipation variance and at some point in the near future such idea will be pursued, leading to an evolutionary model for the interaction of waves, currents, and dissipative effects at all scales. A truly useful evolution equation for the dissipation variance would include a parametric or real dependence on the wind, since the wind has such a strong effect on the wave climatology.

Sullivan et al. (2004) exchanged phenomenological understanding of wave breaking in their study of the boundary layer and free surface flows for a better parametrization of dissipation. Their technique is stochastic parametrization, in the same spirit as is done here. Their choice of this technique allowed them to consider the boundary layer and the free surface in a practical way and at large spatio-temporal scales. The exchange, it seems, paid off.

There is nothing new with regard to representing noise and dynamics as
a Langevin system. However, in doing so, we depart in a fundamental way in the manner in which wave breaking is commonly represented in ocean dynamics: a technique that has a long and distinguished history is spectral (Alves and Banner (2003b) is a recent article on this).

The Langevin model has been chosen with forethought: Firstly, a great deal is known about the Langevin representation, and it is very flexible and robust. More importantly, it is relatively easier to construct the mean and the variance of the stochastic term as a function of space and time from radio or acoustic scatter or photographs of the ocean surface. If not easier, at the very least we exploit here an important strength of these technologies: to deliver large quantities of “global” data over large swaths of ocean and this is done so very economically. The other reason for liking this parametrization is that it lends itself to a variety of data assimilation schemes (see Wunsch (1996), Eyink et al. (2004), Kim et al. (2003), Alexander et al. (2005)) and thus it is possible to conceive of using near-optimal model/data combinations. Thirdly, the spectral approach to wave breaking can be used to estimate the parameters needed in the Langevin approach (see Ding and Farmer (1994) and references contained therein on wave breaking statistics. See also Wu (1979)). This last issue is not as straightforward as it seems: a better way to connect to the spectral approach would be to first develop more general multi-chromatic Langevin equation for the waves and currents.

Perhaps surprising at first glance is that the noise does not enter as a small perturbation to the deterministic dynamics; the strong noise limit would seem
more appropriate in low Reynolds flows. However, the wave/current interaction model is fundamentally a time scale ordering in which it is reasonable to assume that fluctuations induced by whitecapping occur at the very fastest of time scales.

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References


**Figure Captions**

1. (a) Bottom depth $H(X)$ [m]. (b) Initial vortex velocity $v(X, 0)$ [m s$^{-1}$].

2. (a) Vorticity $\chi^c(X)$ [s$^{-1}$] and (b) velocity $v(X)$ [m s$^{-1}$] (both as vector and
contoured speed) for the currents in the baseline case at a time of \( T = 4.0 \) days.

3. (a) Stokes drift \( \mathbf{v}^{St}(X) \) [m s\(^{-1}\)] and (b) the combined velocity \( \mathbf{v} + \mathbf{v}^{St}(X) \) [m s\(^{-1}\)] (both as vectors and contoured speeds) in the baseline case at a time of \( T = 4.0 \) days.

4. Dissipation as a function of space.

5. (a) Vorticity \( \chi^c(X) \) [s\(^{-1}\)] and (b) velocity \( \mathbf{v}(X) \) [m s\(^{-1}\)] (both as vector and contoured speed) for the currents in the baseline case at a time of \( T = 4.0 \) days. With dissipation. See Figure 2 for comparison.

6. (a) Stokes drift \( \mathbf{v}^{St}(X) \) [m s\(^{-1}\)] and (b) the combined velocity \( \mathbf{v} + \mathbf{v}^{St}(X) \) [m s\(^{-1}\)] (both as vectors and contoured speeds) in the baseline case at a time of \( T = 4.0 \) days. With dissipation. See Figure 3 for comparison.