PHYS 371 HW 3

1) $\int \psi^*\psi\, dx = \int_{-\infty}^{\infty} A^2 e^{-\frac{(x-x_0)^2}{2\sigma_x^2}} dx = A^2\cdot\sqrt{2\sigma_x^2\pi} = 1 \Rightarrow A = (2\pi\sigma_x^2)^{-1/4}$ $\quad \left(\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\pi/a}\right)$

2) $$\tilde{\psi}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(x)e^{-ikx}dx = \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{i(k_0-k)x} e^{-\frac{(x-x_0)^2}{4\sigma_x^2}}dx = \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{4\sigma_x^2}[x^2-2x_0x+x_0^2-4\sigma_x^2 i(k_0-k)x]}dx$$

$$= \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{4\sigma_x^2}\{x-[x_0+2\sigma_x^2 i(k_0-k)]\}^2}\cdot e^{-\frac{4\sigma_x^4(k_0-k)^2}{4\sigma_x^2}}\cdot e^{\frac{4\sigma_x^2 i\cdot x_0(k_0-k)}{4\sigma_x^2}}dx$$

$$= \frac{A}{2\pi}\cdot\sqrt{\pi\cdot 4\sigma_x^2}\cdot e^{-\sigma_x^2(k_0-k)^2} e^{i(k_0-k)x_0} = \frac{\sigma_x^{1/2}}{2^{1/4}\cdot\pi^{3/4}} e^{-\sigma_x^2(k_0-k)^2} e^{i(k_0-k)x_0}$$

3) $$\langle x\rangle = A^2\int_{-\infty}^{\infty} x e^{-\frac{(x-x_0)^2}{2\sigma_x^2}}dx \overset{x-x_0=t}{=\!=} A^2\int_{-\infty}^{\infty}(t+x_0)e^{-\frac{t^2}{2\sigma_x^2}}dt = \underbrace{A^2\int_{-\infty}^{\infty}\cancel{t e^{-t^2/2\sigma_x^2}}dt}_{\text{(odd function)}} + A^2\int_{-\infty}^{\infty} x_0 e^{-t^2/2\sigma_x^2}dx = x_0$$

$$\langle x^2\rangle = A^2\int_{-\infty}^{\infty} x^2 e^{-\frac{(x-x_0)^2}{2\sigma_x^2}}dx = A^2\int_{-\infty}^{\infty}(t+x_0)^2 e^{-\frac{t^2}{2\sigma_x^2}}dt = A^2\int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{2\sigma_x^2}}dt + A^2\int_{-\infty}^{\infty}\cancel{2x_0 t e^{-\frac{t^2}{2\sigma_x^2}}}dt + A^2\int_{-\infty}^{\infty} x_0^2 e^{-\frac{t^2}{2\sigma_x^2}}dt$$

using $\int_{-\infty}^{\infty} dx\cdot x^{2n}\cdot e^{-x^2} = \frac{1\cdot 3\cdot 5\cdot\cdots(2n-1)\sqrt{\pi}}{2^n}$, we get $A^2\int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{2\sigma_x^2}}dt = \sigma_x^2$

$\langle x^2\rangle = \sigma_x^2 + x_0^2 \qquad \Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sigma_x$

4) $-i\hbar\frac{\partial}{\partial x}\psi = -i\hbar\psi\left(ik_0 - \frac{x-x_0}{2\sigma_x^2}\right)$

$$\langle p\rangle = \int\psi^*\hat{p}\psi\,dx = -i\hbar A^2\int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{2\sigma_x^2}}\left(ik_0-\frac{x-x_0}{2\sigma_x^2}\right)dx = \hbar k_0 A^2\int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma_x^2}}dt + \frac{i\hbar A^2}{2\sigma_x^2}\int_{-\infty}^{\infty}\cancel{t e^{-\frac{t^2}{2\sigma_x^2}}}dt = \hbar k_0$$

$$-\hbar^2\frac{\partial^2}{\partial x^2}\psi = -\hbar^2\frac{\partial}{\partial x}\left[\psi\left(ik_0-\frac{x-x_0}{2\sigma_x^2}\right)\right] = -\hbar^2\left(ik_0-\frac{x-x_0}{2\sigma_x^2}\right)\frac{\partial}{\partial x}\psi + \hbar^2\psi\cdot\frac{1}{2\sigma_x^2} = -\hbar^2\left(ik_0-\frac{x-x_0}{2\sigma_x^2}\right)^2\psi + \frac{\hbar^2}{2\sigma_x^2}\psi$$

$$\langle p^2\rangle = \int\psi^*\hat{p}^2\psi\,dx = A^2\int_{-\infty}^{\infty}\left[\hbar^2k_0^2 e^{-\frac{(x-x_0)^2}{2\sigma_x^2}} + \cancel{\frac{i\hbar^2k_0(x-x_0)}{\sigma_x^2}e^{-\frac{(x-x_0)^2}{2\sigma_x^2}}} - \hbar^2\frac{(x-x_0)^2}{4\sigma_x^4}e^{-\frac{(x-x_0)^2}{2\sigma_x^2}} + \frac{\hbar^2}{2\sigma_x^2}e^{-\frac{(x-x_0)^2}{2\sigma_x^2}}\right]dx$$

$$A^2\int_{-\infty}^{\infty}\hbar^2\frac{(x-x_0)^2}{4\sigma_x^4}e^{-\frac{(x-x_0)^2}{2\sigma_x^2}}dx = A^2\hbar^2\int_{-\infty}^{\infty}\frac{t^2}{4\sigma_x^4}e^{-\frac{t^2}{2\sigma_x^2}}dt = \frac{A^2\hbar^2}{\sqrt{2}\sigma_x}\int_{-\infty}^{\infty}\left(\frac{t}{\sqrt{2}\sigma_x}\right)^2 e^{-\left(\frac{t}{\sqrt{2}\sigma_x}\right)^2}d\left(\frac{t}{\sqrt{2}\sigma_x}\right) = \frac{A^2\hbar^2}{\sqrt{2}\sigma_x}\cdot\frac{\sqrt{\pi}}{2} = \frac{\hbar^2}{4\sigma_x^2}$$

$\langle p^2\rangle = \hbar^2k_0^2 - \frac{\hbar^2}{4\sigma_x^2} + \frac{\hbar^2}{2\sigma_x^2} = \hbar^2k_0^2 + \frac{\hbar^2}{4\sigma_x^2} \qquad \Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \frac{\hbar}{2\sigma_x}$

5) $\frac{\partial}{\partial x}\psi = \left(ik_0 - \frac{x-x_0}{2\sigma_x^2}\right)\psi \quad , \quad \frac{\partial}{\partial x}\psi^* = \left(-ik_0 - \frac{x-x_0}{2\sigma_x^2}\right)\psi^*$

$$j = -\frac{i\hbar}{2m}\left(\psi^*\frac{\partial}{\partial x}\psi - \psi\frac{\partial}{\partial x}\psi^*\right) = -\frac{i\hbar}{2m}\left[\psi^*\left(ik_0-\frac{x-x_0}{2\sigma_x^2}\right)\psi - \psi\left(-ik_0-\frac{x-x_0}{2\sigma_x^2}\right)\psi^*\right] = -\frac{i\hbar}{2m}\cdot 2ik_0\psi^*\psi = \frac{\hbar k_0}{m}|\psi|^2$$

6) $E_1 = \frac{\pi^2\hbar^2}{2mL^2} = \frac{p^2}{2m} \Rightarrow p = \frac{\pi\hbar}{L} \sim \Delta p \quad , \quad \Delta x \sim L \Rightarrow \Delta p\cdot\Delta x = \pi\hbar > \frac{\hbar}{2}$

if $E_1 = 0$, $\Delta p = 0 \Rightarrow \Delta x \to \infty$, not satisfied

7) $i\hbar\frac{\partial}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_1 + iV_2\right)\psi \quad , \quad -i\hbar\frac{\partial}{\partial t}\psi^* = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_1 - iV_2\right)\psi^*$

$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\frac{\partial}{\partial t}(\psi^*\psi) = i\hbar\psi\frac{\partial}{\partial t}\psi^* + i\hbar\psi^*\frac{\partial}{\partial t}\psi$

$$= -\psi\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_1 - iV_2\right)\psi^* + \psi^*\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_1 + iV_2\right)\psi = -\frac{\hbar^2}{2m}\left(\psi^*\frac{\partial^2}{\partial x^2}\psi - \psi\frac{\partial^2}{\partial x^2}\psi^*\right) + 2iV_2\psi^*\psi$$

the additional term is $2iV_2\psi^*\psi = 2iV_2\rho$

if we integrate the equation above, the probability is no longer conserved

such potentials are useful for particle creation or annihilation