Physics 371
Homework # 4 solutions

2.5) a) $A = 1/\sqrt{2}$

b) $\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_1(x)\, e^{-\frac{iE_1 t}{\hbar}} + \psi_2(x)\, e^{-\frac{iE_2 t}{\hbar}}\right)$

where $E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2} \equiv \hbar\omega n^2$

$$\rho(x,t) = |\Psi(x,t)|^2 = \frac{1}{2}\rho_1(x) + \frac{1}{2}\rho_2(x) + \psi_1(x)\psi_2(x)\cos\left(\frac{E_2 - E_1}{\hbar}t\right)$$

c) $$\langle x(t)\rangle = \int_0^L x\,\rho(x,t)\,dx$$

$$= \frac{L}{2} + \cos(3\omega t)\int_0^L x\,\psi_1(x)\psi_2(x)\,dx$$

$$= \frac{L}{2}\left(1 - \frac{32}{9\pi^2}\cos(3\omega t)\right)$$

d) $\langle p_x(t) \rangle = \int_0^L dx\, \psi^*(x,t) \frac{\hbar}{i} \frac{\partial}{\partial x} \psi(x,t)$

$$= \frac{8\hbar}{3L} \sin(3\omega t)$$

e) $\langle E \rangle = \langle \hat{H} \rangle = \int_0^L dx\, \psi^*(x,t)\, i\hbar \frac{\partial}{\partial t} \psi(x,t)$

$$= \frac{1}{2}(E_1 + E_2)$$

$$E = \begin{cases} E_1, & P = 1/2 \\ E_2, & P = 1/2 \end{cases}$$

2.6) $\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_1(x) e^{-\frac{iE_1 t}{\hbar}} + \psi_2(x) e^{-\frac{iE_2 t}{\hbar} + i\phi}\right)$

$\rho(x,t) = \frac{1}{2}\rho_1(x) + \frac{1}{2}\rho_2(x) + \psi_1(x)\psi_2(x) \cos(3\omega t - \phi)$

$\langle x(t) \rangle = \frac{L}{2}\left(1 - \frac{32}{9\pi^2} \cos(3\omega t - \phi)\right)$

2.21) a) $1 = \int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx$

$$1 = 2\int_0^{\infty} |A|^2 e^{-2ax} dx = \frac{|A|^2}{a} \Rightarrow \boxed{A = \sqrt{a}}$$

b) $\tilde{\Psi}(k) = \int_{-\infty}^{\infty} dx \; \Psi(x,0) e^{-ikx}$

$$= A \int_{-\infty}^{\infty} dx \; e^{-ikx - a|x|}$$

$$= A \int_0^{\infty} dx \; e^{-(ik+a)x} + A \int_{-\infty}^{0} dx \; e^{-(ik-a)x}$$

$$= \frac{A}{ik+a} + \frac{A}{a-ik} = \frac{A\,2a}{a^2+k^2} = \frac{2a^{3/2}}{a^2+k^2}$$

c) $\Psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{\Psi}(k) e^{ikx - i\omega_k t}$

$$\left(\hbar\omega_k = \frac{\hbar^2 k^2}{2m} \right)$$

$$= A \int_{-\infty}^{\infty} dk \; \frac{a/\pi}{k^2+a^2} e^{ikx - i\frac{\hbar k^2}{2m} t}$$

d) For large a, $\Psi(x,0)$ is sharply peaked about $x=0$, while $\tilde{\Psi}(k)$ is broad and flat; position is well defined, but momentum is ill-defined. For small a, $\Psi(x,0)$ is broad and flat, while $\tilde{\Psi}(k)$ is sharply peaked about $k=0$; position is ill-defined, but momentum is well-defined.

4) a) $j_x = \frac{1}{2m}\left(\psi_n^* \frac{\hbar}{i}\frac{\partial}{\partial x}\psi_n - \psi_n \frac{\hbar}{i}\frac{\partial}{\partial x}\psi_n^*\right)$

$$= \frac{\hbar k_n}{mL} \; ; \; j_e = q j_x = \frac{q\hbar k_n}{mL}$$

b) $V = \frac{\hbar k}{m}$; $\quad j_e = \frac{qV}{L} = I \quad$ (in 1D)

Classically, $I = \frac{\Delta Q}{\Delta t} = \frac{q}{L/V} = \frac{qV}{L}$.