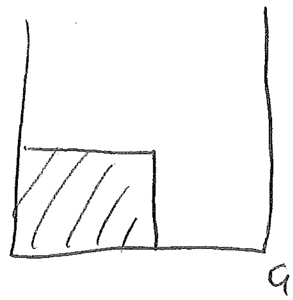
Phys 371 HW6 Solutions

2.8)



a) $$\Psi(x,0) = \begin{cases} \sqrt{\frac{2}{a}}, & 0 < x \le \frac{a}{2} \\ 0, & x > \frac{a}{2} \end{cases}$$

b) $$E = \frac{\pi^2 \hbar^2}{2ma^2} \qquad E_n = \frac{\pi^2\hbar^2}{2ma^2} n^2,$$

$$n = 1, 2, 3, \ldots, \infty$$

$$E = E_1 \qquad \psi_1(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$$

$$P(E_1) = |c_1|^2, \text{ where } \Psi(x,0) = \sum_n c_n \psi_n(x)$$

$$\langle n|\Psi\rangle = \sum_{n'} c_n \underbrace{\langle n|n'\rangle}_{\delta_{nn'}} = c_n$$

$$c_1 = \langle 1|\Psi\rangle = \int_0^{a/2} dx \, \frac{2}{a} \sin\left(\frac{\pi x}{a}\right)$$

$$= \frac{2}{a}\,\frac{a}{\pi}\left[-\cos\left(\frac{\pi x}{a}\right)\right]_0^{a/2}$$

$$= \frac{2}{\pi} \; ; \quad P(E_1) = \frac{4}{\pi^2}$$

2.38) $\psi(x,0) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$

$$\psi_n(x) = \sqrt{\frac{2}{2a}} \sin\left(\frac{\pi n x}{2a}\right), \quad n = 1, 2, 3, \dots \infty$$

$$= \sqrt{\frac{1}{a}} \sin\left(\frac{n\pi x}{2a}\right)$$

$$E_n = \frac{\pi \hbar^2 n^2}{8 m a^2}, \quad n = 1, 2, 3, \dots, \infty$$

$$\psi(x,0) = \sum_n c_n \psi_n(x)$$

$$c_n = \langle n | \psi \rangle = \int_0^a dx \, \frac{\sqrt{2}}{a} \sin\left(\frac{n\pi x}{2a}\right) \sin\left(\frac{\pi x}{a}\right)$$

identity: $\sin\alpha \sin\beta = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$

$$c_n = \frac{1}{\sqrt{2}\,a} \int_0^a dx \left\{ \cos\left[\left(\frac{n}{2}-1\right)\frac{\pi x}{a}\right] - \cos\left[\left(\frac{n}{2}+1\right)\frac{\pi x}{a}\right] \right\}$$

$$= \frac{1}{\sqrt{2}\,\cancel{a}} \left\{ \frac{\sin\left(\frac{n}{2}-1\right)\pi}{\left(\frac{n}{2}-1\right)\frac{\pi}{\cancel{a}}} - \frac{\sin\left(\frac{n}{2}+1\right)\pi}{\left(\frac{n}{2}+1\right)\frac{\pi}{\cancel{a}}} \right\}$$

$$= \frac{\sin\left(\frac{n}{2}+1\right)\pi}{\sqrt{2}\,\pi} \left(\frac{1}{\frac{n}{2}-1} - \frac{1}{\frac{n}{2}+1} \right)$$

$$C_n = \frac{4\sqrt{2}}{\pi} \frac{\sin\left[\left(\frac{n}{2}+1\right)\pi\right]}{n^2-4} = \begin{cases} 0, & n \text{ even} \\ \pm \frac{4\sqrt{2}}{\pi(n^2-4)}, & n \text{ odd} \\ \text{indeterminate}, & n=2 \end{cases}$$

For $n=2$

$$C_2 = \int_0^a dx \frac{\sqrt{2}}{a} \sin^2\left(\frac{\pi x}{a}\right) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

a) $\mathcal{P}(E_2) = \frac{1}{2}$ is the most probable result

$$E_2 = \frac{\pi^2 \hbar^2}{2ma^2}$$

b) $\mathcal{P}(E_1) = \frac{32}{\pi^2 9} = 0.360$

c) $\langle E \rangle = \int dx\, \psi^* \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\psi(x)$

$$= \frac{2}{a}\int_0^a dx \sin\left(\frac{\pi x}{a}\right) \frac{\hbar^2 \pi^2}{2ma^2} \sin\left(\frac{\pi x}{a}\right)$$

$$= \frac{\hbar^2\pi^2}{2ma^2} \frac{2}{a} a \frac{1}{2} = \frac{\hbar^2\pi^2}{2ma^2}$$

2.39) $\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-i\frac{E_n t}{\hbar}}$

$$E_n = \frac{\pi^2 \hbar^2 n^2}{2ma^2}$$

$$\frac{E_n T}{\hbar} = \frac{\pi^{\cancel{2}} \cancel{\hbar^2} n^2}{\cancel{2} \cancel{m} \cancel{a^2}} \frac{4 \cancel{m} \cancel{a^2}}{\cancel{\pi} \cancel{\hbar^2}} = 2\pi n^2$$

$$e^{-i\frac{E_n T}{\hbar}} = e^{-i 2\pi n^2} = 1$$

a) $\therefore \quad \Psi(x,T) = \Psi(x,0)$

b) $T_{cl} = \frac{2a}{V(E)} \qquad V(E) = \sqrt{\frac{2E}{m}}$

$$T_{cl}(E) = 2a\sqrt{\frac{m}{2E}}$$

c) $T_{cl}(E) = \frac{4ma^2}{\pi\hbar}$

$$E = \frac{\pi^2\hbar^2}{8ma^2} = \frac{E_1}{4}$$

3.7) a) $\hat{Q} f = q f$, $\hat{Q} g = q g$

let $h(x) = a f(x) + b g(x)$

$$\hat{Q} h(x) = \hat{Q} (a f(x) + b g(x)) = a \hat{Q} f + b \hat{Q} g$$

$$= q (a f + b g) = q h(x) \checkmark$$

b) $\frac{d^2 f}{dx^2} = e^x = f$ $\qquad$ $\frac{d^2 g}{dx^2} = e^{-x} = g$ $\checkmark$

$\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$

$= \frac{f - g}{2}$ and $= \frac{f + g}{2}$

are orthogonal on the interval $(-1, 1)$.

3.26) $\hat{Q}^+ = - \hat{Q}$ (a)

$$\langle \hat{Q} \rangle = \langle \psi | \hat{Q} \psi \rangle$$

$$\langle \hat{Q} \rangle^* = \langle \hat{Q} \psi | \psi \rangle \equiv \langle \hat{Q}^+ \rangle = - \langle \hat{Q} \rangle$$

$$\Rightarrow \langle \hat{Q} \rangle \in \mathbb{I}$$

3.26(b)

$$[\hat{A},\hat{B}]^{+} = (\hat{A}\hat{B} - \hat{B}\hat{A})^{+}$$
$$= \hat{B}^{+}\hat{A}^{+} - \hat{A}^{+}\hat{B}^{+}$$
$$= \hat{B}\hat{A} - \hat{A}\hat{B} \quad \text{if } \hat{A},\hat{B} \text{ are Hermitian}$$
$$= -[\hat{A},\hat{B}] \quad \text{Q.E.D.}$$

Also true for anti-hermitian operators.

3.27) $\psi_1 = \dfrac{3\phi_1 + 4\phi_2}{5}$, $\psi_2 = \dfrac{4\phi_1 - 3\phi_2}{5}$

a) $\psi = e^{i\alpha}\psi_1$, $\alpha \in \mathbb{R}$

b) b_1 or b_2 with probabilities

$$P(b_1) = \left|\frac{3}{5}\right|^2 = \frac{9}{25}, \quad P(b_2) = \left|\frac{4}{5}\right|^2 = \frac{16}{25}$$

c) $\langle\phi_1|\psi_1\rangle = \frac{3}{5}$ $\langle\phi_1|\psi_2\rangle = \frac{4}{5}$

$\langle\phi_2|\psi_1\rangle = \frac{4}{5}$ $\langle\phi_2|\psi_2\rangle = -\frac{3}{5}$

$$P(a_1) = P(b_1)\,|\langle\psi_1|\phi_1\rangle|^2 + P(b_2)\,|\langle\psi_1|\phi_2\rangle|^2$$

$$\mathcal{P}(q_1) = \frac{9}{25}\frac{9}{25} + \frac{16}{25}\frac{16}{25}$$

$$= 0.5392$$

7) a) $[A, B+C] = A(B+C) - (B+C)A$

$= AB + AC - BA - CA$

$= AB - BA + AC - CA = [A,B] + [A,C]$

b) $[A, BC] = ABC - BCA$

$B[A,C] + [A,B]C = \cancel{BAC} - BCA + ABC - \cancel{BAC}$

$= [A, BC]$ ✓

c) $[f(x), p_x]\psi(x)$

$= \frac{\hbar}{i}\left(f\frac{\partial\psi}{\partial x} - \frac{\partial}{\partial x}(f\psi)\right)$

$= \frac{\hbar}{i}\left(\cancel{f\frac{\partial\psi}{\partial x}} - \frac{\partial f}{\partial x}\psi - \cancel{f\frac{\partial\psi}{\partial x}}\right)$

$= i\hbar\frac{\partial f}{\partial x}\psi(x) \qquad \forall\ \psi(x)$

$\therefore \quad [f(x), p_x] = i\hbar\frac{\partial f}{\partial x}$

8) $$\Delta p_x \, \Delta E \geq \frac{1}{2} \left| \langle [\hat{p}_x , \hat{H}] \rangle \right|$$

$$[\hat{p}_x , \hat{H}] = [\hat{p}_x , V(x)]$$

$$[\hat{p}_x , V(x)] \psi(x) = \frac{\hbar}{i} \left[\frac{d}{dx}(V\psi) - V \frac{d\psi}{dx} \right]$$

$$= \left(\frac{\hbar}{i} \frac{dV}{dx} \right) \psi$$

$$\Rightarrow \quad [\hat{p}_x , V(x)] = \frac{\hbar}{i} \frac{dV}{dx}$$

$$\Rightarrow \quad \Delta p_x \, \Delta E \geq \frac{\hbar}{2} \left| \left\langle \frac{dV}{dx} \right\rangle \right| .$$

For a stationary state, $\Delta E = 0$.

$\Rightarrow \quad \left\langle \frac{dV}{dx} \right\rangle = 0$ (average force on particle vanishes).

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9) $$\frac{d}{dt}\langle x\rangle = \frac{1}{i\hbar}\langle[\hat{x}, \hat{H}]\rangle$$

$$[\hat{x}, \hat{H}] = \frac{1}{2m}[\hat{x}, \hat{p}_x^2] = \frac{\hat{p}_x}{2m}[\hat{x}, \hat{p}_x] + [\hat{x}, \hat{p}_x]\frac{\hat{p}_x}{2m}$$

$$= i\hbar\frac{\hat{p}_x}{m}$$

$$\Rightarrow m\frac{d\langle x\rangle}{dt} = \langle\hat{p}_x\rangle$$

$$m\frac{d^2\langle x\rangle}{dt^2} = \frac{d\langle\hat{p}_x\rangle}{dt} = \frac{1}{i\hbar}\langle[\hat{p}_x, \hat{H}]\rangle$$

$$[\hat{p}_x, \hat{H}] = [\hat{p}_x, V(x)] = \frac{\hbar}{i}\frac{\partial V}{\partial x}$$

$$\Rightarrow m\frac{d^2\langle x\rangle}{dt^2} = \left\langle -\frac{\partial V}{\partial x}\right\rangle$$