Physics 371 HW #8

solutions

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1) $[b, b^+] = [a+\beta, a^+ + \beta] = [a, a^+] = 1$

$$H = E_0 (b^+ - \beta)(b - \beta) + E_1 (b^+ + b - 2\beta)$$

$$= E_0 b^+ b + (E_1 - \beta E_0)(b^+ + b)$$

$$+ \beta^2 E_0 - 2\beta E_1$$

let $\beta = E_1/E_0 \Rightarrow$ eliminate linear term in $b + b^+$

$$H = E_0 b^+ b - E_1^2/E_0$$

$$\Rightarrow E_n = E_0 n - E_1^2/E_0, \quad n = 0, 1, 2, \ldots, \infty$$

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2) a) $N^+ = (c^+ c)^+ = c^+ c = N$ ✓

b) $N^2 = c^+ c c^+ c = c^+(1 - c^+ c) c = c^+ c$

(since $c^2 = 0$) $= N$

c) Suppose $\hat{N} \Psi_n = n \Psi_n \qquad \hat{N}^2 \Psi_n = n^2 \Psi_n$

$\Rightarrow n^2 = n \Rightarrow n = 0, 1$

2 d) $\hat{N}(c^+\Psi_0) = c^+cc^+\Psi_0$

$= c^+(1-c^+c)\Psi_0 = c^+\Psi_0$

$\Rightarrow c^+\Psi_0 = \alpha\Psi_1$

$\langle c^+\Psi_0 | c^+\Psi_0\rangle = |\alpha|^2$

$\qquad \parallel$

$\langle\Psi_0|cc^+|\Psi_0\rangle = \langle\Psi_0|\,1-c^+c\,|\Psi_0\rangle = \langle\Psi_0|\Psi_0\rangle =$

$\Rightarrow \alpha = 1 \qquad c^+\Psi_0 = \Psi_1$ ✓

$c^+\Psi_1 = (c^+)^2\Psi_0 \longrightarrow 0$ ✓

But $(c^+)^2 = (c^2)^+ = 0$

$cc^+\Psi_0 = c\Psi_1$

$(1-c^+c)\Psi_0 = c\Psi_1 \longrightarrow \Psi_0 = c\Psi_1$

$c\Psi_0 = c^2\Psi_1 = 0\cdot\Psi_1 = 0$ ✓

Phys 371 HW 9 Solutions

2.12) $x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^{+})$, $P_x = i\sqrt{\frac{m\hbar\omega}{2}}\,(a^{+} - a)$

$$\langle n | x | n \rangle = 0 = \langle n | P_x | n \rangle$$

$$\langle n | x^2 | n \rangle = \frac{\hbar}{2m\omega} \langle n | a^2 + (a^{+})^2 + a^{+}a + aa^{+} | n \rangle$$

$$aa^{+} - a^{+}a = 1 \qquad aa^{+} = a^{+}a + 1$$

$$\langle n | x^2 | n \rangle = \frac{\hbar}{2m\omega} \langle n | 2a^{+}a + 1 | n \rangle = \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right)$$

$$\langle n | P_x^2 | n \rangle = -\frac{m\hbar\omega}{2} \langle n | (a^{+})^2 + a^2 - a^{+}a - aa^{+} | n \rangle$$

$$= \frac{m\hbar\omega}{2} \langle n | a^{+}a + aa^{+} | n \rangle = m\hbar\omega\left(n + \frac{1}{2}\right)$$

$$\langle n | \hat{T} | n \rangle = \frac{m\omega^2}{2} \langle n | x^2 | n \rangle = \frac{\hbar\omega}{2}\left(n + \frac{1}{2}\right)$$

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right)$$

$$(\Delta P_x)^2 = \langle P_x^2 \rangle - \langle P_x \rangle^2 = m\hbar\omega\left(n + \frac{1}{2}\right)$$

$$(\Delta x)^2 (\Delta P_x)^2 = \hbar^2\left(n + \frac{1}{2}\right)^2 \geq \left(\frac{\hbar}{2}\right)^2 \quad \checkmark$$

3.1) a) Let f and g be square-integ.

$\alpha \in \mathbb{C}$.

i) $\int dx\, |\alpha f(x)|^2 = |\alpha|^2 \int dx\, |f(x)|^2 < \infty$

ii) $\int dx\, |f+g|^2 = \int dx\, |f|^2 + \int dx\, |g|^2$

$+ \int dx\, f^* g + \int dx\, f g^* < \infty$

Schwarz inequality:

$$\left| \int f^* g\, dx \right| \le \sqrt{\int |f|^2 dx \int |g|^2 dx} < \infty$$

Q.E.D.

The set of all normalized functions is not a vector space.

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b) $\langle f|g\rangle \equiv \int_a^b dx\, f^*(x) g(x)$

i) $\langle g|f\rangle = \langle f|g\rangle^*$ ✓

ii) $\langle f|f\rangle = \int_a^b dx\, |f(x)|^2 \geq 0$

and $\langle f|f\rangle = 0$ only if $f(x) = 0$. ✓

iii) $\langle\alpha|\left(b|\beta\rangle + c|\gamma\rangle\right) = b\langle\alpha|\beta\rangle + c\langle\alpha|\gamma\rangle$

$$\int dx\, \alpha^*(x)\left(b\,\beta(x) + c\,\gamma(x)\right)$$

$$= b\int dx\, \alpha^*\beta + c\int dx\, \alpha^*\gamma$$

$$= b\langle\alpha|\beta\rangle + c\langle\alpha|\gamma\rangle \checkmark$$

3.6) $\hat{Q} = \frac{d^2}{d\phi^2}$, $f(\phi+2\pi) = f(\phi)$

$$\langle f|\hat{Q}g\rangle = \int_0^{2\pi} d\phi\, f^*(\phi)\frac{d^2g}{d\phi^2}$$

$$= \cancelto{0}{f^*(\phi)\frac{dg}{d\phi}\Big|_0^{2\pi}} - \int_0^{2\pi} d\phi\, \frac{df^*}{d\phi}\frac{dg}{d\phi}$$

$$= -\cancelto{0}{\frac{df^*}{d\phi}g(\phi)\Big|_0^{2\pi}} + \int_0^{2\pi} d\phi\, \frac{d^2f^*}{d\phi^2}g(\phi)$$

$$= \langle\hat{Q}f|g\rangle \implies \hat{Q} = \hat{Q}^\dagger$$

$$\hat{Q} f(\phi) = q f(\phi)$$

$$f''(\phi) = q f(\phi)$$

Ansatz $$f(\phi) = \frac{1}{\sqrt{2\pi}} e^{i\lambda\phi}$$

$$f''(\phi) = -\lambda^2 f(\phi)$$

$$f(\phi + 2\pi) = f(\phi)\, e^{i 2\pi\lambda} = f(\phi)$$

$$e^{i 2\pi \lambda} = 1 \ , \quad \lambda \in \mathbb{Z}$$

$$\Rightarrow \quad f_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$$

eigenvalue $-m^2$.

Spectrum

$0, -1, -4, -9, \ldots$

degeneracy

$1, 2, 2, 2, \ldots$