Physics 371 HW #9 solutions

---

1) $$L_x = \frac{1}{2}(L_+ + L_-)$$

$$L_y = \frac{1}{2i}(L_+ - L_-)$$

$$\langle \ell m | L_\pm | \ell m \rangle = 0 \implies \langle L_x \rangle = \langle L_y \rangle = 0$$

$$\langle L_x^2 \rangle = \frac{1}{4}\langle \underbrace{\cancel{L_+^2}}_{0} + \underbrace{\cancel{L_-^2}}_{0} + L_+L_- + L_-L_+ \rangle$$

$$= \frac{1}{2}\langle \vec{L}^2 - L_z^2 \rangle = \frac{\hbar^2}{2}\left[\ell(\ell+1) - m^2\right]$$

$$\langle L_y^2 \rangle = -\frac{1}{4}\langle \underbrace{\cancel{L_+^2}}_{0} + \underbrace{\cancel{L_-^2}}_{0} - L_+L_- - L_-L_+ \rangle$$

$$= \frac{1}{4}\langle L_+L_- + L_-L_+ \rangle = \frac{\hbar^2}{2}\left[\ell(\ell+1) - m^2\right]$$

---

2) $$\Delta L_x^2 = \langle L_x^2 \rangle - \langle L_x \rangle^2 = \frac{\hbar^2}{2}\left[\ell(\ell+1) - m^2\right]$$

$$\Delta L_y^2 = \langle L_y^2 \rangle - \langle L_y \rangle^2 = \frac{\hbar^2}{2}\left[\ell(\ell+1) - m^2\right]$$

$$\Delta L_x \Delta L_y = \frac{\hbar^2}{2}\left[\ell(\ell+1) - m^2\right]$$

$$\frac{\hbar}{2}\left|\langle L_z\rangle\right| = \frac{\hbar^2}{2}|m|$$

$$\frac{\hbar^2}{2}\left[\ell(\ell+1) - m^2\right] \overset{?}{\geq} \frac{\hbar^2}{2}|m|$$

$$\ell(\ell+1) - m^2 \geq |m| \;?$$

$$\ell(\ell+1) - m^2 \geq \ell(\ell+1) - \ell^2 = \ell \geq |m| \checkmark$$

Equality holds for $m = \pm \ell$.

---

3) $6\hbar^2 = \hbar^2 \ell(\ell+1) \Rightarrow \ell = 2$

$\Rightarrow L_y = 0, \pm\hbar, \pm 2\hbar$

---

4) $\hat{L}_x \psi = \lambda_x \psi \quad \hat{L}_y \psi = \lambda_y \psi$

$$[\hat{L}_x, \hat{L}_y]\psi = i\hbar L_z \psi$$

$$(\lambda_x \lambda_y - \lambda_y \lambda_x)\psi = i\hbar L_z \psi$$

$$0 = L_z \psi$$

$$[\hat{L}_z, \hat{L}_x]\psi = i\hbar \hat{L}_y \psi = i\hbar \lambda_y \psi$$

$$\hat{L}_z \lambda_x \psi - 0 = i\hbar \lambda_y \psi$$

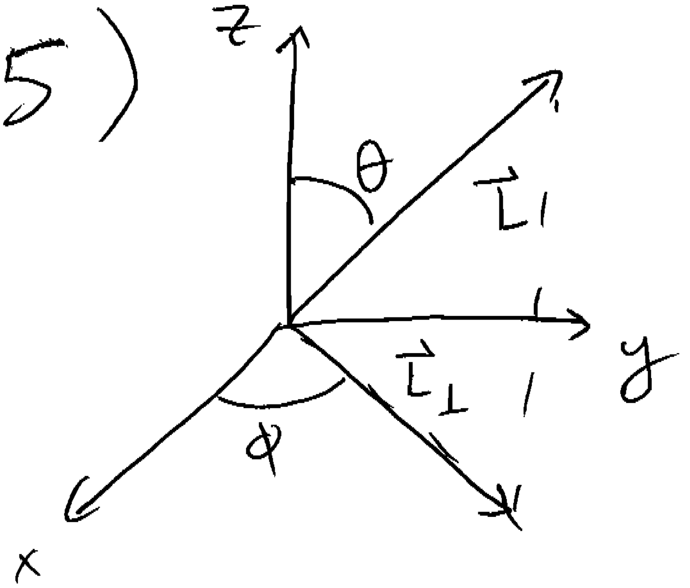
$$0 = \lambda_y \psi$$

$$[\hat{L}_y, \hat{L}_z]\psi = i\hbar \hat{L}_x \psi$$

$$0 = \lambda_x \psi \quad \Rightarrow \quad \lambda_x = \lambda_y = \lambda_z = 0$$

---

5)



$$\tan\theta = \frac{|\vec{L}_\perp|}{|L_z|}$$

$$\vec{L}_\perp^2 = \vec{L}^2 - L_z^2$$

$$= \hbar^2[\ell(\ell+1) - m^2]$$

$$\frac{|\vec{L}_\perp|}{|L_z|} = \sqrt{\frac{\ell(\ell+1) - m^2}{m^2}} = \sqrt{\frac{\ell(\ell+1)}{m^2} - 1}$$

$$\tan\theta \geq \sqrt{\frac{\ell(\ell+1)}{\ell^2} - 1} = \sqrt{\frac{1}{\ell}} \quad \text{Q.E.D.}$$

6) Griffiths 4.1

(a) $[x,y] = xy - yx = 0$, etc., so $\boxed{[r_i, r_j] = 0.}$

$$[p_x, p_y]f = -\hbar^2\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) = 0, \text{ etc.}$$

so $\boxed{[p_i, p_j] = 0.}$

$[x, p_x] = i\hbar = [y, p_y] = [z, p_z]$, but

$$[x, p_y]f = -i\hbar\left(x\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}(xf)\right) = 0, \text{ etc.},$$

so $\boxed{[r_i, p_j] = i\hbar\,\delta_{ij}.}$

(b) $$\frac{d}{dt}\langle \vec{r} \rangle = \frac{1}{i\hbar}\langle [\vec{r}, \hat{H}] \rangle = \frac{1}{2i\hbar m}\langle [\vec{r}, \vec{p}^{\,2}] \rangle$$

$[x, \vec{p}^{\,2}] = 2i\hbar p_x$, etc., so $[\vec{r}, \vec{p}^{\,2}] = 2i\hbar \vec{p}$

$$\boxed{\frac{d}{dt}\langle \vec{r} \rangle = \frac{1}{m}\langle \vec{p} \rangle}$$

$$\frac{d}{dt}\langle \vec{p} \rangle = \frac{1}{i\hbar}\langle [\vec{p}, \hat{H}] \rangle = \frac{1}{i\hbar}\langle [\vec{p}, V(\vec{r})] \rangle$$

$$[\vec{p}, V(\vec{r})] f(\vec{r}) = \frac{\hbar}{i} \nabla (Vf) - \frac{\hbar}{i} V \nabla f$$
$$= (-i\hbar \nabla V) f$$

$$\Rightarrow \quad [\vec{p}, V(\vec{r})] = -i\hbar \nabla V$$

$$\boxed{\frac{d}{dt} \langle \vec{p} \rangle = \langle -\nabla V \rangle}$$

(c) $\Delta r_i \, \Delta p_j \geq \frac{1}{2} |\langle [r_i, p_j] \rangle| = \frac{\hbar}{2} \delta_{ij}$

---

7) Griffiths 4.19

$L_x = y p_z - z p_y$

$L_y = z p_x - x p_z$

$L_z = x p_y - y p_x$

(a) $[L_z, x]$

$\quad \parallel$

$[x p_y - y p_x, x] = -y [p_x, x]$

$= i\hbar y$ ✓

$[L_z, y] = [x p_y - y p_x, y] = x [p_y, y] = -i\hbar x$ ✓

$[L_z, z] = [x p_y - y p_x, z] = 0$ ✓

$[L_z, p_x] = [x p_y - y p_x, p_x] = [x, p_x] p_y = i\hbar p_y$ ✓

$$[L_z, p_y] = [xp_y - yp_x, p_y] = -[y, p_y]p_x = -i\hbar p_x$$

$$[L_z, p_z] = [xp_y - yp_x, p_z] = 0$$

(b) $$[L_z, L_x] = [xp_y - yp_x, yp_z - zp_y]$$

$$= [xp_y, yp_z] + [yp_x, zp_y]$$

$$= x[p_y, y]p_z + z[y, p_y]p_x$$

$$= -i\hbar xp_z + i\hbar zp_x = i\hbar(zp_x - xp_z)$$

$$= i\hbar L_y$$

(c) $$[L_z, r^2] = [L_z, x^2] + [L_z, y^2] + 0$$

$$= x[L_z, x] + [L_z, x]x + y[L_z, y] + [L_z, y]y$$

$$= 2i\hbar xy - 2i\hbar xy = 0$$

$$[L_z, p^2] = [L_z, p_x^2] + [L_z, p_y^2]$$

$$= 2i\hbar p_x p_y - 2i\hbar p_x p_y = 0$$

(d) $$[\vec{L}, \hat{H}] = \underbrace{[\vec{L}, p^2]}_{=0}\frac{1}{2m} + [\vec{L}, V(r)]$$

$$[L_x, V(r)] = [yP_z - zP_y, V(r)]$$

$$= y[P_z, V(r)] - z[P_y, V(r)]$$

$$[P_z, V(r)]f(\vec{r}) = \left(\frac{\hbar}{i}\frac{\partial V}{\partial z}\right) f(\vec{r})$$

$$[P_y, V(r)] = \frac{\hbar}{i}\frac{\partial V}{\partial y}$$

$$r = \sqrt{x^2+y^2+z^2} \qquad r^2 = x^2+y^2+z^2$$

$$2r\,dr = 2x\,dx + 2y\,dy + 2z\,dz$$

$$\frac{\partial}{\partial y}V(r) = \frac{\partial V}{\partial r}\frac{\partial r}{\partial y} = \frac{y}{r}\frac{dV}{dr}$$

$$\frac{\partial}{\partial z}V(r) = \frac{z}{r}\frac{dV}{dr}$$

$$\Rightarrow [L_x, V(r)] = \frac{\hbar}{i}\frac{yz}{r}\frac{dV}{dr} - \frac{\hbar}{i}\frac{zy}{r}\frac{dV}{dr} = 0$$

similarly, $[L_y, V(r)] = 0$, $[L_z, V(r)] = 0$.

$$\Rightarrow \left[\vec{L}, \frac{\vec{P}^2}{2m} + V(r)\right] = 0$$

8) Griffiths 4.21

(a) $$L_+L_- f = -\hbar^2 e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left[e^{-i\phi}\left(\frac{\partial f}{\partial\theta} - i\cot\theta\frac{\partial f}{\partial\phi}\right)\right]$$

$$= -\hbar^2 e^{i\phi}\left\{e^{-i\phi}\left[\frac{\partial^2 f}{\partial\theta^2} - i\left(-\csc^2\theta\frac{\partial f}{\partial\phi} + \cot\theta\frac{\partial^2 f}{\partial\theta\partial\phi}\right)\right]\right.$$

$$+ i\cot\theta\left[-ie^{-i\phi}\left(\frac{\partial f}{\partial\theta} - i\cot\theta\frac{\partial f}{\partial\phi}\right)\right.$$

$$\left.\left. + e^{-i\phi}\left(\frac{\partial^2 f}{\partial\phi\partial\theta} - i\cot\theta\frac{\partial^2 f}{\partial\phi^2}\right)\right]\right\}$$

$$= -\hbar^2\left(\frac{\partial^2 f}{\partial\theta^2} + i\csc^2\theta\frac{\partial f}{\partial\phi} - i\cot\theta\frac{\partial^2 f}{\partial\theta\partial\phi}\right.$$

$$+ \cot\theta\frac{\partial f}{\partial\theta} - i\cot^2\theta\frac{\partial f}{\partial\phi} + i\cot\theta\frac{\partial^2 f}{\partial\phi\partial\theta}$$

$$\left. + \cot^2\theta\frac{\partial^2 f}{\partial\phi^2}\right)$$

$$= -\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2}\right.$$

$$\left. + i\left(\csc^2\theta - \cot^2\theta\right)\frac{\partial}{\partial\phi}\right]f$$

so $$L_+L_- = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right).$$

Q.E.D

(b) $L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$, $\vec{L}^2 = L_+ L_- + L_z^2 - \hbar L_z$,

so $\vec{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right)$

$$-\hbar^2\frac{\partial^2}{\partial\phi^2} - \hbar\left(\frac{\hbar}{i}\right)\frac{\partial}{\partial\phi}$$

$$= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

Q.E.D.