Physics 371 HW #10
Solutions

1) $$L_x = \begin{pmatrix} 0 & \hbar/\sqrt{2} & 0 \\ \frac{\hbar}{\sqrt{2}} & 0 & \hbar/\sqrt{2} \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{pmatrix}$$ for $l=1$

$$L_y = \begin{pmatrix} 0 & -i\hbar/\sqrt{2} & 0 \\ i\hbar/\sqrt{2} & 0 & -i\hbar/\sqrt{2} \\ 0 & i\hbar/\sqrt{2} & 0 \end{pmatrix}$$ for $l=1$

$$0 = \begin{vmatrix} -\lambda_x & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & -\lambda_x & \hbar/\sqrt{2} \\ 0 & \frac{\hbar}{\sqrt{2}} & -\lambda_x \end{vmatrix} = -\lambda_x\left(\lambda_x^2 - \frac{\hbar^2}{2}\right) + \left(\frac{\hbar}{\sqrt{2}}\right)^2 \lambda_x$$

$$0 = -\lambda_x^3 + \hbar^2 \lambda_x = \lambda_x\left(\hbar^2 - \lambda_x^2\right)$$

$$\Longrightarrow \lambda_x = 0, \hbar, -\hbar$$

$$0 = \begin{vmatrix} -\lambda_y & -i\hbar/\sqrt{2} & 0 \\ i\hbar/\sqrt{2} & -\lambda_y & -i\hbar/\sqrt{2} \\ 0 & i\hbar/\sqrt{2} & -\lambda_y \end{vmatrix} = -\lambda_y\left(\lambda_y^2 - \frac{\hbar^2}{2}\right) + \left(\frac{i\hbar}{\sqrt{2}}\right)^2 (-\lambda_y)$$

$$0 = -\lambda_y\left(\lambda_y^2 - \frac{\hbar^2}{2}\right) + \frac{\lambda_y^2}{2}$$

$$0 = \lambda_y\left(\hbar^2 - \lambda_y^2\right) \Rightarrow \lambda_y = 0, \pm\hbar$$

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2) $L_x = 0$

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow b = 0,\ c = -a$$

$$\boxed{\psi_{L_x=0} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$$

$L_x = \hbar$

$$\hbar\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \hbar\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \frac{1}{\sqrt{2}}b = a,\quad \frac{1}{\sqrt{2}}(a+c) = b,\quad \frac{1}{\sqrt{2}}b = c$$

$$\Rightarrow a = c = b/\sqrt{2}$$

$$\psi_{L_x=\hbar} = A\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$1 = A^2(1+2+1) = 4A^2$$

$$A = \frac{1}{2}$$

$$\psi_{L_x=\hbar} = \begin{pmatrix} \frac{1}{2} \\ 1/\sqrt{2} \\ \frac{1}{2} \end{pmatrix}$$

i) $\underline{L_x = 0}$ $\quad \langle L_z \rangle = \frac{\hbar}{2}(1 \ 0 \ -1)\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$= \frac{\hbar}{2}(1 \ 0 \ -1)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\langle L_z^2 \rangle = \frac{\hbar^2}{2}(1 \ 0 \ -1)\begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} =$$

$$= \frac{\hbar^2}{2}(1 \ 0 - 1)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \hbar^2$$

ii) $\underline{L_x = \hbar}$ $\quad \langle L_z \rangle = \frac{\hbar}{4}(1 \ \sqrt{2} \ 1)\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$

$$= \frac{\hbar}{4}(1 \ \sqrt{2} \ 1)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0$$

$$\langle L_z^2 \rangle = \frac{\hbar^2}{4} \begin{pmatrix} 1 & \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$= \frac{\hbar^2}{4} \begin{pmatrix} 1 & \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{\hbar^2}{2}$$

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3) $$S_z = \hbar \begin{pmatrix} 3/2 & & & 0 \\ & 1/2 & & \\ & & -1/2 & \\ 0 & & & -3/2 \end{pmatrix}$$

$$\langle s'm' | S_\pm | s\, m \rangle = \hbar \sqrt{s(s+1) - m(m\pm1)}\; \delta_{ss'}\, \delta_{m', m\pm1}$$

$$S_+ = \begin{pmatrix} 0 & \sqrt{3}\hbar & 0 & 0 \\ 0 & 0 & 2\hbar & 0 \\ 0 & 0 & 0 & \sqrt{3}\hbar \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad S_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3}\hbar & 0 & 0 & 0 \\ 0 & 2\hbar & 0 & 0 \\ 0 & 0 & \sqrt{3}\hbar & 0 \end{pmatrix}$$

$$S_x = \frac{1}{2}\left(S_+ + S_-\right) = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2}\hbar & 0 & 0 \\ \frac{\sqrt{3}}{2}\hbar & 0 & \hbar & 0 \\ 0 & \hbar & 0 & \frac{\sqrt{3}}{2}\hbar \\ 0 & 0 & \frac{\sqrt{3}\hbar}{2} & 0 \end{pmatrix}$$

$$S_y = -\frac{i}{2}\left(S_+ - S_-\right) = \begin{pmatrix} 0 & -i\frac{\sqrt{3}}{2}\hbar & 0 & 0 \\ \frac{i\sqrt{3}\hbar}{2} & 0 & -i\hbar & 0 \\ 0 & i\hbar & 0 & -i\frac{\sqrt{3}}{2}\hbar \\ 0 & 0 & i\frac{\sqrt{3}}{2}\hbar & 0 \end{pmatrix}$$

$$S_x^2 + S_y^2 + S_z^2 = \begin{pmatrix} \frac{15}{4}\hbar^2 & & & \\ & \frac{15}{4}\hbar^2 & & 0 \\ & 0 & \frac{15}{4}\hbar^2 & \\ & & & \frac{15}{4}\hbar^2 \end{pmatrix}$$

$$= \vec{S}^2$$

4) $$E_{\vec{n}} = \frac{\hbar^2 \pi^2 \vec{n}^2}{2m L^2}$$

$$\min E_{\vec{n}} = E_{1,1,1} = \frac{3\hbar^2\pi^2}{2mL^2} = \frac{3\hbar^2\pi^2}{2m} V^{-2/3}$$

$$P = -\frac{\partial E_0}{\partial V} = \frac{2}{3}\,\frac{3\hbar^2\pi^2}{2m} V^{-5/3}$$

$$= \frac{2}{3}\frac{E}{V} \qquad \frac{pV}{E} = \frac{2}{3}$$

Ideal gas: $pV = N k_B T$

$$E = \frac{3}{2} N k_B T$$

$$\frac{pV}{E} = \frac{2}{3} \implies \text{same ratio} \checkmark$$

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5) Griffiths, 4.25 $\qquad r_c = \dfrac{e^2}{mc^2}$ (c.g.s.)

$$I_{sphere} = \frac{2}{5} m r_c^2 \qquad I\omega = \frac{\hbar}{2}$$

$$\omega = \frac{\hbar}{2I}, \qquad V = \omega r = \frac{\hbar r_c}{2I}$$

$$V_{equator} = \frac{5\hbar r_c}{4 m r_c^2} = \frac{5\hbar}{4 m r_c} = \frac{5\hbar c^2}{4 e^2}$$

$$V_{equator} = \left(\frac{5}{4}\frac{\hbar c}{e^2}\right) c$$

$$\frac{e^2}{\hbar c} \simeq \frac{1}{137} = \text{fine-structure constant}$$

$$V_{equator} \simeq 171\, c$$

violates special relativity!

Spin cannot be thought of as physical rotation — it is a purely quantum mechanical degree of freedom.

6) Griffith 4.27

(a) $1 = |\chi|^2 = \chi^\dagger \chi = |A|^2 \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix}$

$= |A|^2 (9 + 16) = |5A|^2$

$\boxed{A = \frac{1}{5}}$

(b) $\langle S_x \rangle = \frac{\hbar}{2} \langle \sigma_x \rangle = \frac{\hbar}{2} \chi^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \chi$

$= \frac{1}{25} \frac{\hbar}{2} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix}$

$= \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 3i \end{pmatrix} = 0$

$\langle S_y \rangle = \frac{\hbar}{2} \chi^\dagger \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \chi = \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix}$

$= \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} -4i \\ -3 \end{pmatrix} = \frac{\hbar}{50} (-12 - 12)$

$= -\frac{12}{25} \hbar$

$$\langle S_z \rangle = \frac{\hbar}{50}(-3i \;\; 4)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

$$= \frac{\hbar}{50}(-3i \;\; 4)\begin{pmatrix} 3i \\ -4 \end{pmatrix} = \frac{\hbar}{50}(9-16)$$

$$= -\frac{7}{50}\hbar$$

(c) $\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = \left(\frac{\hbar}{2}\right)^2$

so $\Delta S_x = \sqrt{\left(\frac{\hbar}{2}\right)^2 - 0} = \frac{\hbar}{2}$

$$\Delta S_y = \sqrt{\left(\frac{\hbar}{2}\right)^2 - \left(\frac{12}{25}\hbar\right)^2} = \sqrt{\frac{49}{2500}\hbar^2} = \frac{7}{50}\hbar$$

$$\Delta S_z = \sqrt{\left(\frac{\hbar}{2}\right)^2 - \left(\frac{7}{50}\hbar\right)^2} = \frac{12}{25}\hbar$$

(d) $\Delta S_x \Delta S_y = \frac{7}{100}\hbar^2$

$\frac{\hbar}{2}|\langle S_z \rangle| = \frac{7}{100}\hbar^2 \quad \Rightarrow \Delta S_x \Delta S_y = \frac{\hbar}{2}|\langle S_z \rangle|$

$$\Delta S_y \Delta S_z = \frac{7 \cdot 12 \hbar^2}{2 \cdot (25)^2} = \frac{42}{625} \hbar^2$$

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$$\langle S_x \rangle = 0 \implies \Delta S_y \Delta S_z > \frac{\hbar}{2} |\langle S_x \rangle|$$

$$\Delta S_z \Delta S_x = \frac{12}{25} \frac{1}{2} \hbar^2 = \frac{6}{25} \hbar^2$$

$$\frac{\hbar}{2} |\langle S_y \rangle| = \frac{\hbar}{2} \frac{12}{25} \hbar = \frac{6}{25} \hbar^2$$

$$\implies \Delta S_z \Delta S_x = \frac{\hbar}{2} |\langle S_y \rangle|$$

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7) Griffiths 4.30

$$S_r = \frac{\hbar}{2} \hat{r} \cdot \vec{\sigma} = \frac{\hbar}{2} \left[\sin\theta\cos\phi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin\theta\sin\phi \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$$

$$= \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

$$\hat{r}\cdot\vec{\sigma}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta - \lambda & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta - \lambda \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad 0 = \begin{vmatrix} \cos\theta - \lambda & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta - \lambda \end{vmatrix}$$

$$0 = \lambda^2 - \cos^2\theta - \sin^2\theta = \lambda^2 - 1$$

$\lambda = \pm 1 \quad \Rightarrow$ $\boxed{\text{eigenvalues of } S_r \text{ are } \pm\hbar/2}$

$$\hat{r}\cdot\vec{\sigma}\begin{pmatrix} a \\ b \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta \mp 1 & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \mp 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{\cos\theta \pm 1}{\sin\theta \, e^{i\phi}}$$

$$\chi_{S_r = \pm \frac{\hbar}{2}} = A \begin{pmatrix} \cos\theta \pm 1 \\ \sin\theta \, e^{i\phi} \end{pmatrix}$$

normalization

$$1 = \chi^{\dagger}\chi = |A|^2 \left[(\cos\theta \pm 1)^2 + \sin^2\theta \right]$$

$$1 = |A|^2 (2 \pm 2\cos\theta)$$

$$A = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1 \pm \cos\theta}}$$

$$\chi_{S_r = \pm \hbar/2} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm\sqrt{1 \pm \cos\theta} \\ \dfrac{\sin\theta \, e^{i\phi}}{\sqrt{1 \pm \cos\theta}} \end{pmatrix}$$

But $\sin^2 \frac{\theta}{2} = \frac{1}{2}\left(1 - \cos\theta\right)$,

$$\cos^2 \frac{\theta}{2} = \frac{1}{2}\left(1 + \cos\theta\right)$$

$$\Rightarrow \quad \chi_{S_r = \hbar/2} = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \end{pmatrix}$$

$$\chi_{S_r = -\hbar/2} = \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2)\, e^{i\phi} \end{pmatrix}$$