Physics 371 HW # 11. Solutions

1) $$\frac{d^2u}{dr^2} + \frac{2\mu}{\hbar^2}\left(E - V_{eff}(r)\right)u(r) = 0$$

$$u(r) = r\psi(r), \quad \Psi(r,\theta,\phi) = \psi(r)\, Y_{lm}(\theta,\phi)$$

$$V_{eff}(r) = -\frac{Ze^2}{r} + \frac{l(l+1)\hbar^2}{2\mu r^2} + \frac{C}{r^2}$$

Let $\quad s(s+1) = l(l+1) + \frac{2\mu C}{\hbar^2}$

$$s = \sqrt{\frac{1}{4} + l(l+1) + \frac{2\mu C}{\hbar^2}} - \frac{1}{2}$$

$$V_{eff}(r) = -\frac{Ze^2}{r} + \frac{s(s+1)\hbar^2}{2\mu r^2}$$

$\Rightarrow$ Just like hydrogenic atom, but with $l \rightarrow s$. Let $\lambda = \frac{1}{\hbar}\sqrt{2\mu|E|}$ and $\rho = 2\lambda r$.

$$u(\rho) = e^{-\rho/2}\rho^{s+1}\sum_{j=0}^{\infty} a_j \rho^j$$

recursion relation

$$\frac{a_{j+1}}{a_j} = \frac{j+s+1-n}{(j+1)(j+2s+2)},$$

where $n = \dfrac{\mu Z e^2}{\hbar^2 \lambda}$ (not necessarilly an integer).

Power series must terminate for finite j_{max} to satisfy B.C. as $\rho \to \infty$.

$$\Rightarrow \quad n = j_{max} + s + 1$$

$$E = -\frac{\hbar^2 \lambda^2}{2\mu} = -\frac{\mu Z^2 e^4}{2\hbar^2 n^2}$$

$$\boxed{E_{n_r l} = -\frac{\mu Z^2 e^4}{2\hbar^2}\left(n_r + \frac{1}{2} + \sqrt{\left(l+\frac{1}{2}\right)^2 + \frac{2\mu C}{\hbar^2}}\right)^{-2}}$$

$n_r = 0, 1, 2, \ldots, \infty$, $\quad l = 0, 1, 2, \ldots, \infty$

Each energy level has degeneracy $2l+1$ due to degeneracy of spherical harmonics.

2) Griffiths 4.9

$$\frac{d^2u}{dr^2} + \frac{2m}{\hbar^2}\left(E - V(r)\right) u(r) = 0$$

$$u(r) = r\,\psi(r), \quad \Psi(r,\theta,\phi) = \psi(r)\, Y_{00}$$

$$V(r) = \begin{cases} -V_0, & r \le a \quad \text{(region I)} \\ 0, & r > a \quad \text{(region II)} \end{cases}$$

For a bound state,

$$u_{I}(r) = A \sin kr + B \cos kr,$$

$$u_{II}(r) = C e^{-\kappa r},$$

where

$$E = \frac{\hbar^2 k^2}{2m} - V_0 = -\frac{\hbar^2 \kappa^2}{2m} < 0.$$

Boundary conditions:

i) $u(0) = 0 \Rightarrow B = 0$

ii) $u_{I}(a) = u_{II}(a)$

$$A \sin ka = C e^{-Ka} \qquad \frac{A}{C} = \frac{e^{-Ka}}{\sin ka}$$

iii) $$\left.\frac{d\psi_{I}}{dr}\right|_{r=a} = \left.\frac{d\psi_{II}}{dr}\right|_{r=a}$$

$$\left.\frac{d}{dr}\left(\frac{A}{r}\sin kr\right)\right|_{r=a} = \left.\frac{d}{dr}\left(\frac{C}{r}e^{-Kr}\right)\right|_{r=a}$$

$$\frac{kA\cos ka}{a} - \frac{A}{a^2}\sin ka = -\frac{KCe^{-Ka}}{a} - \frac{C}{a^2}e^{-K}$$

$$\frac{A}{C}\left(ka\cos ka - \sin ka\right) = -e^{-Ka}(Ka+1)$$

$$e^{-Ka}(ka\cot ka - 1) = -e^{-Ka}(Ka+1)$$

$$ka\cot ka = -Ka$$

$$\cot(ka) = -\frac{K}{k}$$

A bound state solution exists if $\cot(ka) = -\frac{\kappa}{k}$ has a real solution.

$$\kappa^2 = \frac{2mV_0}{\hbar^2} - k^2$$

$$-\cot(ka) = \sqrt{\frac{2mV_0}{\hbar^2 k^2} - 1}$$

Let $z = ka$, $\frac{2mV_0 a^2}{\hbar^2} = z_0^2$

$$-\cot(z) = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}$$

A real solution exists only if

$z_0 > \frac{\pi}{2}$:

$$\frac{2mV_0 a^2}{\hbar^2} > \left(\frac{\pi}{2}\right)^2$$

$$V_0 a^2 > \frac{\pi^2 \hbar^2}{8m}$$

3) Griffiths 4.17

a) $V(r) = -\frac{GMm}{r}$ so $e^2 \rightarrow GMm$.

b) $a_g = \frac{\hbar^2}{m(GMm)} = \frac{\hbar^2}{GMm^2} = 2.34 \times 10^{-138}\ m$

c) $E_n = -\frac{m}{2\hbar^2}\frac{(GMm)^2}{n^2} = -\frac{G^2M^2m^3}{2\hbar^2 n^2}$

$E_c = \frac{mv^2}{2} - \frac{GMm}{r} = -\frac{GMm}{2r}$

$\frac{\cancel{GM}m}{\cancel{2}r} = \frac{G^{\cancel{2}}M^{\cancel{2}}m^3}{\cancel{2}\hbar^2 n^2}$

$\frac{1}{r} = \frac{GMm^2}{\hbar^2 n^2}$

$n = \sqrt{\frac{Mm^2Gr}{\hbar^2}} = \sqrt{\frac{r}{a_g}} = 2.53 \times 10^{74}$

d) $$\Delta E = \frac{G^2 M^2 m^3}{2\hbar^2}\left[\frac{1}{(n-1)^2} - \frac{1}{n^2}\right]$$

$$\frac{1}{(n-1)^2} \approx \frac{1}{n^2}\left(1+\frac{2}{n}\right),$$ so

$$\Delta E \simeq \frac{G^2 M^2 m^3}{\hbar^2 n^3} = 2.09\times 10^{-41}\ \mathrm{J}$$

$$\lambda = \frac{hc}{\Delta E} = \frac{2\pi c\, \hbar^3 n^3}{G^2 M^2 m^3}$$

But $n = \left(\frac{GMm^2 r}{\hbar^2}\right)^{1/2}$, so

$$\lambda = 2\pi c\left(\frac{r^3}{GM}\right)^{1/2}$$ classical quantity; no $\hbar$'s!

Now, the period of the orbit can be determined classically from

$$\frac{mv^2}{r} = \frac{GMm}{r^2}; \quad v^2 = \frac{GM}{r}$$

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{GM}} = 2\pi\sqrt{\frac{r^3}{GM}}$$

$\lambda = cT \equiv 1$ light-year!

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4) Griffiths 4.45

a) $$P = \int |\psi|^2 d^3r = \frac{4\pi}{\pi a^3}\int_0^b e^{-2r/a} r^2\, dr$$

$$= 1 - \left(1 + \frac{2b}{a} + 2\frac{b^2}{a^2}\right)e^{-2b/a}$$

b) $$P = 1 - \left(1 + \epsilon + \frac{1}{2}\epsilon^2\right)e^{-\epsilon}$$

$$P \simeq 1 - \left(1 + \epsilon + \frac{\epsilon^2}{2}\right)\left(1 - \epsilon + \frac{\epsilon^2}{2} - \frac{\epsilon^3}{3!}\right)$$

$$P \simeq 1 - 1 - \epsilon + \epsilon + \epsilon^2 - \frac{\epsilon^2}{2} - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{6} + \frac{\epsilon^3}{2} - \frac{\epsilon^3}{2} + \cdots \simeq \frac{\epsilon^3}{6}$$

$$P \approx \frac{1}{6}\left(\frac{2b}{a}\right)^3 = \frac{4}{3}\left(\frac{b}{a}\right)^3.$$

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c) $$P \approx \frac{4}{3}\pi b^3 |\psi(0)|^2 = \frac{4}{3}\left(\frac{b}{a}\right)^3$$

d) $$P \approx \frac{4}{3}\left(\frac{10^{-15}\,m}{5\times10^{-11}\,m}\right)^3 = 1.07\times10^{-14}$$

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5) Griffiths 4.46

a) $$\psi(r) = \frac{1}{r}\rho^{l+1}e^{-\rho}\sum_{j=0}^{j_{max}} a_j\rho^j = \frac{\rho^n}{r}e^{-\rho}\sum_j a_j\rho^j$$

$$c_1 = \frac{2(n-n)}{1\cdot 2n}c_0 = 0$$

$$\Longrightarrow \psi_{n,n-1}(r) = N_n r^{n-1}e^{-r/na}$$

$$1 = \int_0^\infty |\psi(r)|^2 r^2\,dr = N_n^2\int_0^\infty r^{2n}e^{-\frac{2r}{na}}\,dr$$

$$= (N_n)^2(2n)!\left(\frac{na}{2}\right)^{2n+1}$$

$$\Rightarrow N_n = \left(\frac{2}{na}\right)^n\sqrt{\frac{2}{na\,(2n)!}}$$

b) $\langle r^l \rangle = \int_0^\infty |\psi(r)|^2 r^{l+2} \, dr$

$= N_n^2 \int_0^\infty r^{2n+l} e^{-2r/na} \, dr$

$\langle r \rangle = \left(\frac{2}{na}\right)^{2n+1} \frac{1}{(2n)!} (2n+1)! \left(\frac{na}{2}\right)^{2n+2}$

$= \left(n + \frac{1}{2}\right) na$

$\langle r^2 \rangle = \left(\frac{2}{na}\right)^{2n+1} \frac{1}{(2n)!} (2n+2)! \left(\frac{na}{2}\right)^{2n+3}$

$= \left(n + \frac{1}{2}\right)(n+1)(na)^2$

c) $(\Delta r)^2 = \langle r^2 \rangle - \langle r \rangle^2$

$= \left(\frac{n}{2} + \frac{1}{4}\right)(na)^2$

$\Delta r = na \sqrt{\frac{n + 1/2}{2}}$

$$\frac{\Delta r}{\langle r \rangle} = \frac{1}{\sqrt{2n+1}}$$

The radial probability distribution is

$$\rho(r) = r^2 |\psi(r)|^2 = N_n^2 \, r^{2n} \, e^{-2r/na}$$

Maximum occurs when

$$0 = \frac{d\rho}{dr} = N_n^2 \left[2n \, r^{2n-1} e^{-2r/na} - \frac{2r^{2n}}{na} e^{-2r/na} \right]$$

$$0 = 2n e^{-\frac{2r}{na}} r^{2n-1} \left(1 - \frac{r}{n^2 a} \right)$$

$\longrightarrow$ $r = n^2 a$ radius of nth Bohr orbit!