Physics 371 HW # 12

Solutions

1) $$\frac{d\langle\vec{L}\rangle}{dt} = \frac{1}{i\hbar}\langle[\vec{L},\hat{H}]\rangle + \cancelto{0}{\left\langle\frac{\partial\vec{L}}{\partial t}\right\rangle}$$

$$= \frac{1}{i\hbar}\left\langle\left[\vec{L}, -\frac{q}{2Mc}\vec{L}\cdot\vec{B}\right]\right\rangle$$

$$= \frac{iq}{2M\hbar c}\langle[\vec{L},\vec{L}\cdot\vec{B}]\rangle$$

$$\frac{d\langle L_x\rangle}{dt} = \frac{iq}{2M\hbar c}\langle[L_x, L_xB_x + L_yB_y + L_zB_z]\rangle$$

$$= \frac{iq}{2M\hbar c}\langle B_y[L_x,L_y] + B_z[L_x,L_z]\rangle$$

$$= \frac{iq}{2M\hbar c}\langle i\hbar L_zB_y - i\hbar L_yB_z\rangle$$

$$= \frac{q}{2Mc}\langle L_yB_z - L_zB_y\rangle$$

$$= \langle\vec{\mu}\times\vec{B}\rangle_x$$

Similarly, $\frac{d\langle L_y\rangle}{dt} = \frac{q}{2Mc}\langle L_z B_x - L_x B_z\rangle$ /2

$$= \langle \vec{\mu} \times \vec{B}\rangle_y$$

$$\frac{d\langle L_z\rangle}{dt} = \frac{q}{2Mc}\langle L_x B_y - L_y B_x\rangle = \langle \vec{\mu} \times \vec{B}\rangle_z$$

$$\Rightarrow \frac{d\langle \vec{L}\rangle}{dt} = \langle \vec{\mu} \times \vec{B}\rangle = -\langle \vec{B} \times \vec{\mu}\rangle$$

$$= -\vec{B} \times \langle \vec{\mu}\rangle$$ (since $\vec{B}$ is a constant vector)

$$= -\frac{q\vec{B}}{2Mc} \times \langle \vec{L}\rangle = \vec{\Omega} \times \langle \vec{L}\rangle$$

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2) Griffiths 4.32

a) $$C_+^{(x)} = \chi_+^{(x)\dagger}\chi = \frac{1}{\sqrt{2}}(1 \;\; 1)\begin{pmatrix}\cos\frac{\alpha}{2}\, e^{i\gamma B_0 t/2} \\ \sin\frac{\alpha}{2}\, e^{-i\gamma B_0 t/2}\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\left[\cos\frac{\alpha}{2}\, e^{i\gamma B_0 t/2} + \sin\frac{\alpha}{2}\, e^{-i\gamma B_0 t/2}\right]$$

$$P_+^{(x)}(t) = |C_+^{(x)}|^2 = \frac{1}{2}\left[1 + \sin\alpha\cos(\gamma B_0 t)\right]$$

b) $$P_+^{(y)}(t) = \frac{1}{2}\left[1 - \sin\alpha\sin(\gamma B_0 t)\right]$$

c) $$P_+^{(z)} = \cos^2\frac{\alpha}{2}$$

3) Griffiths 4.49

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a) $1 = |A|^2(1+4+4) = 9|A|^2$ $\boxed{A = 1/3}$

b) $P(S_z = \frac{\hbar}{2}) = \frac{1+4}{9} = \frac{5}{9}$

$P(S_z = -\frac{\hbar}{2}) = \frac{4}{9}$

$\langle S_z \rangle = \frac{\hbar}{2}\left[\frac{5}{9} - \frac{4}{9}\right] = \frac{\hbar}{18}$

c) $P(S_x = \frac{\hbar}{2}) = \frac{13}{18}$, $P(S_x = -\frac{\hbar}{2}) = \frac{5}{18}$

$\langle S_x \rangle = \frac{\hbar}{2}\left(\frac{13}{18} - \frac{5}{18}\right) = \frac{2\hbar}{9}$

d) $P(S_y = \frac{\hbar}{2}) = \frac{17}{18}$, $P(S_y = -\frac{\hbar}{2}) = \frac{1}{18}$

$\langle S_y \rangle = \frac{\hbar}{2}\left(\frac{17}{18} - \frac{1}{18}\right) = \frac{4\hbar}{9}$

4) Griffiths 4.58

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix}, \text{ with } |a|^2 + |b|^2 = 1$$

$$\langle S_z \rangle = \frac{\hbar}{2}\left(|a|^2 - |b|^2\right)$$

$$\langle S_x \rangle = \hbar \, \mathrm{Re}(ab^*), \quad \langle S_y \rangle = -\hbar \, \mathrm{Im}(ab^*)$$

$$\langle S_x^2 \rangle = \langle S_y^2 \rangle = \frac{\hbar^2}{4}$$

Let $a > 0$ be a real #.

$$b = |b| e^{-i\theta}$$

$$\langle S_x \rangle = \hbar |a||b| \cos\theta$$

$$\langle S_y \rangle = -\hbar |a||b| \sin\theta$$

$$(\Delta S_x)^2 = \frac{\hbar^2}{4} - \hbar^2 |a|^2 |b|^2 \cos^2\theta$$

$$(\Delta S_y)^2 = \frac{\hbar^2}{4} - \hbar^2 |a|^2 |b|^2 \sin^2\theta$$

When is $(\Delta S_x)^2(\Delta S_y)^2 = \frac{\hbar^2}{4}\langle S_z\rangle^2$?

$$\left(\frac{\hbar^2}{\cancel{4}}\right)^2\left(1-4|a|^2|b|^2\cos^2\theta\right)\left(1-4|a|^2|b|^2\sin^2\theta\right)$$

$$=\left(\frac{\hbar^2}{\cancel{4}}\right)^2\left(|a|^2-|b|^2\right)^2$$

$$1-4|a|^2|b|^2(\cos^2\theta+\sin^2\theta)+16|a|^4|b|^4\sin^2\theta\cos^2\theta$$
$$=|a|^4+|b|^4-2|a|^2|b|^2$$

$$1-4|a|^2|b|^2+4|a|^4|b|^4\sin^2 2\theta$$
$$=|a|^4+|b|^4-2|a|^2|b|^2$$

$$1+4|a|^4|b|^4\sin^2(2\theta)=|a|^4+2|a|^2|b|^2+|b|^4$$
$$1+4|a|^4|b|^4\sin^2(2\theta)=\left(|a|^2+|b|^2\right)^2=1$$

$$\Longrightarrow \sin^2 2\theta = 0, \quad \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

$b\in\mathbb{R}$ or $b\in\mathbb{I}$.