Physics 371 Lecture 3

Wave mechanics

Planck: $E = h\nu = \hbar\omega$

$(\omega = 2\pi\nu)$
"angular frequency"

de Broglie: $p = \dfrac{h}{\lambda} = \hbar k$

$\left(k = \dfrac{2\pi}{\lambda}\right)$
"wave number"

In terms of their complementary wave/particle properties, the photon and material particles, such as the electron, are quite similar. Thus, we will endeavor to write down a wave equation for electrons, similar to the E+M wave equation:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}(\vec{r},t) = 0,$$

in free space. Here $\vec{A}$ is the vector potential.

A plane-wave solution has the form:

$$\vec{A}(\vec{r},t) = \vec{\epsilon}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

Plugging this into the wave

equation gives

$$\left(-\vec{k}^2 + \frac{\omega^2}{c^2}\right)\vec{E} = 0$$

or $\omega = c|\vec{k}|$. For a free particle of mass m, one has (neglecting relativistic effects)

$$E = \frac{\vec{p}^{\,2}}{2m}$$

$$\hbar\omega = \frac{\hbar^2 \vec{k}^2}{2m}.$$

A plane wave

$$\psi(\vec{r},t) = A e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

for such a particle must

- satisfy a different type of wave equation. Notice that

$$\nabla \psi = i\vec{k}\,\psi$$

$$\frac{\partial \psi}{\partial t} = -i\omega\,\psi$$

- $\vec{p}\,\psi = \hbar \vec{k}\,\psi = \frac{\hbar}{i}\nabla \psi$

$$E\psi = \hbar\omega\psi = i\hbar \frac{\partial \psi}{\partial t}$$

$$\left(E - \frac{p^2}{2m}\right)\psi = 0$$

- $\left[i\hbar \frac{\partial}{\partial t} - \left(\frac{\hbar}{i}\nabla\right)^2 \Big/ 2m\right]\psi = 0$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \psi$$

In general, for a particle in an external potential $V(\vec{r},t)$, one has

$$E = \frac{\vec{p}^{\,2}}{2m} + V(\vec{r},t).$$

We postulate that the quantum mechanical wave function $\psi(\vec{r},t)$ of a particle obeys the

Schrödinger equation (1)

$$\boxed{i\hbar \frac{\partial \psi(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \psi(\vec{r},t) + V(\vec{r},t)\psi(\vec{r},t).}$$

The energy carried by an E&M wave is proportional to the time-average of the square of the field, and hence to $|\vec{A}|^2$.

Since $E = nh\nu$, this implies that the # of photons is proportional to $|\vec{A}|^2$. Indeed, the probability to find a photon at a given place and time is proportional to

$$P(\vec{r},t) \propto |\vec{A}(\vec{r},t)|^2.$$

By analogy, we assert, following Max Born, that the probability to observe a material particle is

$$P(\bar{r},t) \propto |\Psi(\bar{r},t)|^2.$$

The total probability should be unity:

$$\boxed{1 = \int d^3r\, |\Psi(\bar{r},t)|^2.}$$

If $\Psi(\bar{r},t)$ satisfies this equation, it is said to be <u>normalized</u>.

Since the Schrödinger equation is linear in ψ, we have the important

$\boxed{\begin{array}{l}\underline{\text{Superposition principle:}}\\ \text{If } \psi_1 \text{ satisfies Eq. (1)} \\ \text{and } \psi_2 \text{ satisfies Eq. (1),} \\ \text{then so does} \\ \quad \phi(\vec{r},t) = a\,\psi_1(\vec{r},t) + b\,\psi_2(\vec{r},t) \end{array}}$

Normalization requires

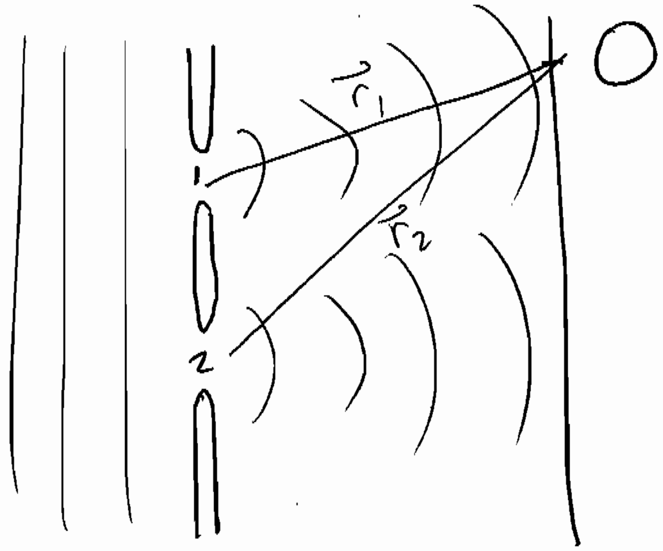
$$1 = \int d^3r \left|\phi(\vec{r},t)\right|^2$$

$$= |a|^2 \int d^3r \left|\psi_1(\vec{r},t)\right|^2 + |b|^2 \int d^3r \left|\psi_2\right|^2$$

$$+ a^* b \int d^3r\, \psi_1^* \psi_2 + a b^* \int d^3r\, \psi_1 \psi_2^* .$$

- If $\int d^3r \, \psi_1^* \psi_2 = 0$, then ψ_1 & ψ_2 are said to be "orthogonal." In that case, we must have $|a|^2 + |b|^2 = 1$.

- <u>Example</u> 2-slit experiment



$$P_1 = |\psi_1|^2$$

$$P_2 = |\psi_2|^2$$

$$P_{12} = |\psi_1 + \psi_2|^2$$

- $$P_{12} = (\psi_1^* + \psi_2^*)(\psi_1 + \psi_2)$$
$$= \psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_1^*\psi_2 + \psi_2^*\psi_1$$

$$P_{12} = P_1 + P_2 + \Delta P$$

$$\Delta P = \psi_1^* \psi_2 + \psi_1 \psi_2^*$$

suppose $\psi_1 = A e^{i\vec{k}\cdot\vec{r}_1} = \sqrt{P_1}\, e^{i\vec{k}\cdot\vec{r}_1}$

$\psi_2 = B e^{i\vec{k}\cdot\vec{r}_2} = \sqrt{P_2}\, e^{i\vec{k}\cdot\vec{r}_2}$

Then $\Delta P = \sqrt{P_1 P_2}\left(e^{-i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)} + e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)} \right)$

$= \sqrt{P_1 P_2}\; 2\cos\left[\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)\right]$