Wave mechanics

Planck: \[ E = h \nu = \frac{h}{\lambda} \omega \]
\[ (\omega = 2\pi \nu) \]
"angular frequency"

de Broglie: \[ p = \frac{\hbar}{\lambda} = \frac{h}{\lambda} k \]
\[ (k = \frac{2\pi}{\lambda}) \]
"wave number"

In terms of their complementary wave/particle properties, the photon and material particles, such as the electron, are
pluging this into the wave equation

\[ A(x, t) = \exp \left( ik \cdot r - \omega t \right) \]

is the wave solution to the free space wave equation.

Here, the vector potential is

\[ (\partial_2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) A(\mathbf{r}, t) = 0 \]
equation gives

$$(- \frac{k^2}{c^2} + \frac{\omega^2}{c^2})\hat{E} = 0$$

or $\omega = c |k|$. For a free particle of mass $m$, one has (neglecting relativistic effects)

$$E = \frac{p^2}{2m}$$

$$\hbar \omega = \frac{\hbar^2 k^2}{2m}.$$ 

A plane wave

$$\psi(\vec{r}, t) = A e^{(\vec{k} \cdot \vec{r} - \omega t)}$$

for such a particle must
satisfy a different type of wave equation. Notice that

\[ D^2 \psi = \hbar^2 k^2 \psi \]

\[ \frac{\partial \psi}{\partial t} = -i \hbar \omega \psi \]

\[ \bar{p} \psi = \hbar k \psi = \frac{\hbar \omega}{i} \psi \]

\[ E \psi = \hbar \omega \psi = i \hbar \frac{\partial \psi}{\partial t} \]

\[ (E - \frac{\bar{p}^2}{2m}) \psi = 0 \]

\[ \left[ i \hbar \frac{\partial}{\partial t} - \left( \frac{\hbar \omega}{i} \right)^2 \right] \psi = 0 \]

\[ i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} D^2 \psi \]
In general, for a particle in an external potential \( V(\vec{r},t) \), one has

\[
E = \frac{\vec{p}^2}{2m} + V(\vec{r},t).
\]

We postulate that the quantum mechanical wave function \( \psi(\vec{r},t) \) of a particle obeys the Schrödinger equation (1)

\[
i\hbar \frac{\partial \psi(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r},t) + V(\vec{r},t) \psi(\vec{r},t).
\]
The energy carried by an E&M wave is proportional to the time-average of the square of the field and hence to $|\mathbf{A}|^2$.

Since $E = nh\nu$, this implies that the number of photons is proportional to $|\mathbf{A}|^2$. Indeed, the probability to find a photon at a given place and time is proportional to $P(\mathbf{r},\mathbf{t}) \propto \overline{|\mathbf{A}(\mathbf{r},\mathbf{t})|^2}$. 


By analogy, we assert, following Max Born, that the probability to observe a particle is

\[ P(\vec{r},t) \propto |\Psi(\vec{r},t)|^2. \]

The total probability should be unity:

\[ 1 = \int d^3r |\Psi(\vec{r},t)|^2. \]

If \( \Psi(\vec{r},t) \) satisfies this equation, it is said to be normalized.
Since the Schrödinger equation is linear in $\Psi$, we have the important

Superposition principle:

If $\Psi_1$ satisfies Eq. (1) and $\Psi_2$ satisfies Eq. (1), then so does

$$\Phi(\vec{r},t) = a \Psi_1(\vec{r},t) + b \Psi_2(\vec{r},t)$$

Normalization requires

$$1 = \int d^3r \left| \Phi(\vec{r},t) \right|^2$$

$$= |a|^2 \int d^3r |\Psi_1(\vec{r},t)|^2 + |b|^2 \int d^3r |\Psi_2(\vec{r},t)|^2$$

$$+ a^* b \int d^3r \Psi_1^*(\vec{r},t) \Psi_2(\vec{r},t) + a b^* \int d^3r \Psi_1(\vec{r},t) \Psi_2^*(\vec{r},t).$$
If \( \int d^3r \, \psi_1^* \psi_2 = 0 \), then \( \psi_1 \) and \( \psi_2 \) are said to be "orthogonal." In that case, we must have \( |a_1|^2 + |b_1|^2 = 1 \).

**Example**  
2-slit experiment

\[
P_1 = |a_1|^2
\]
\[
P_2 = |a_2|^2
\]
\[
P_{12} = |a_1 + a_2|^2
\]
\[
P_{12} = (\psi_1^* + \psi_2^*) (\psi_1 + \psi_2)
\]
\[
= \psi_1^* \psi_1 + \psi_2^* \psi_2 + \psi_1^* \psi_2 + \psi_2^* \psi_1
\]
\[ P_{12} = P_1 + P_2 + \Delta P \]

\[ \Delta P = t_1^* t_2 + t_1 t_2^* \]

Suppose

\[ \chi_1 = A e^{i k \cdot \vec{r}_1} = \sqrt{P_1} e^{i k \cdot \vec{r}_1} \]

\[ \chi_2 = B e^{i k \cdot \vec{r}_2} = \sqrt{P_2} e^{i k \cdot \vec{r}_2} \]

Then

\[ \Delta P = \sqrt{P_1 P_2} \left( e^{-i k \cdot (\vec{r}_1 - \vec{r}_2)} + e^{i k \cdot (\vec{r}_1 - \vec{r}_2)} \right) \]

\[ = \sqrt{P_1 P_2} \ 2 \cos[k \cdot (\vec{r}_1 - \vec{r}_2)] \]