Physics 371
Lecture 6

Separation of space and time in the Schrödinger equation

Often, $V = V(x)$, independent of time. Then one can write

$$\Psi(x,t) = \psi(x)\,\chi(t)$$

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x)\Psi$$

$$i\hbar\, \psi(x) \frac{\partial \chi(t)}{\partial t} = -\frac{\hbar^2}{2m}\chi(t)\frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\psi\chi$$

$$\frac{i\hbar}{\chi(t)}\frac{\partial \chi}{\partial t} = -\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{\partial^2 \psi}{\partial x^2} + V(x)$$

R.H.S. depends only on x, L.H.S. depends only on t $\Rightarrow$ both sides must be constants:

$$\frac{i\hbar}{\chi}\frac{\partial \chi}{\partial t} = -\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{\partial^2 \psi}{\partial x^2} + V(x) = E$$

$$\frac{\partial \chi}{\partial t} = \frac{-iE}{\hbar}\chi(t)$$

$$\chi(t) = \chi(0)\, e^{-i\frac{Et}{\hbar}}$$

$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi(x) = E\psi(x)}$$

time-independent Schrödinger equation (in 1D)

Each solution $\psi_n(x)$ of this equation will correspond to a particular value of the energy, E_n:

$$\Psi_n(x,t) = \psi_n(x)\, e^{-i\frac{E_n t}{\hbar}}$$

These are called stationary states or <u>energy eigenstates</u>

It is clear from the correspondence principle that E_n is the energy of the system:

$$\psi_n^*(x)\, E_n \psi_n(x) = \psi_n^*(x)\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi_n(x)$$

$$E_n \int_{-\infty}^{\infty} dx\, |\psi_n|^2 = \int_{-\infty}^{\infty} dx\, \psi_n^*\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi_n(x)$$

$$E_n = \langle T + V \rangle_n .$$

Q: Can E_n be complex ?

$$\rho(x,t) = \Psi^*(x,t)\, \Psi(x,t)$$

$$= \psi^*(x)\, \psi(x)\, e^{-\frac{i(E_n - E_n^*)}{\hbar} t}$$

Assuming conservation of particles,

$$0 = \frac{d}{dt}\int_{-\infty}^{\infty} \rho(x,t)\, dx = \int_{-\infty}^{\infty} \frac{\partial \rho(x,t)}{\partial t}\, dx$$

$$= -\frac{i}{\hbar}(E_n - E_n^*) \int_{-\infty}^{\infty} |\psi(x)|^2 dx \;\; e^{-\frac{i(E_n - E_n^*)}{\hbar} t}$$

$\Rightarrow$ $E_n = E_n^*$ for a closed system. Not only $\langle E \rangle$, but also the eigenenergies must be real.

Q: What is the uncertainty

- ΔE in an eigenstate E_n?

$$\Delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2}$$

$$= \sqrt{\langle \hat{H}^2 \rangle_n - E_n^2}$$

- $$\hat{H} = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$$

$$\hat{H} \psi_n(x) = E_n \psi_n(x)$$

$$\hat{H}^2 \psi_n(x) = E_n^2 \psi_n(x)$$

$$\int_{-\infty}^{\infty} dx \, \psi_n^*(x) \, \hat{H}^2 \psi(x) = E_n^2 \int_{-\infty}^{\infty} dx \, |\psi_n(x)|^2$$

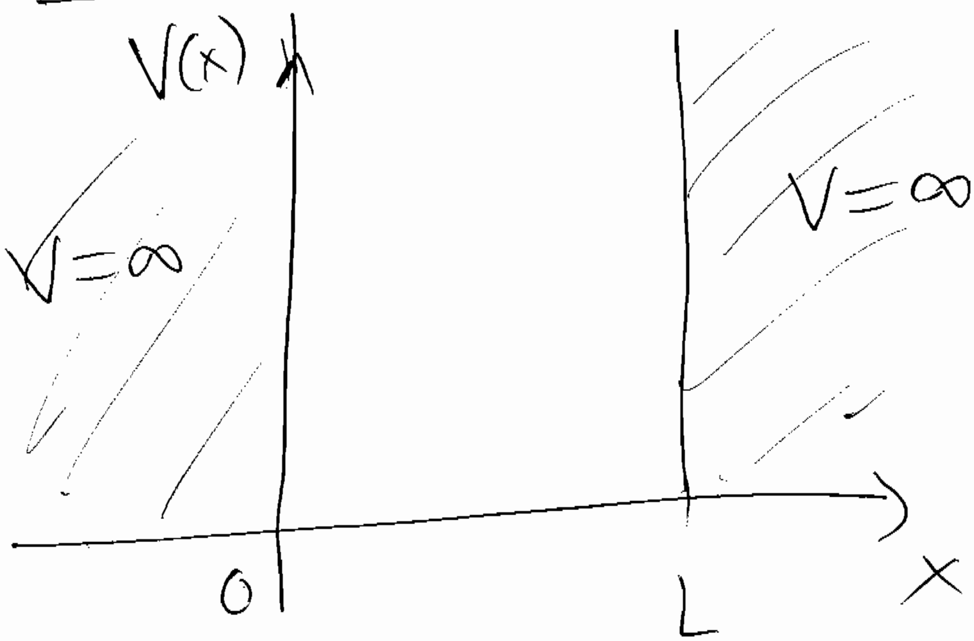
- $$\langle \hat{H}^2 \rangle = E_n^2$$

- $\Delta E = 0$. This is the essential property of an energy eigenstate: it is a state of definite energy. c.f. momentum eigenstates.

gen. sol'n: $\psi(x,t) = \sum_n C_n \psi_n(x) e^{-\frac{iE_n t}{\hbar}}$

- Examples

i) particle in a box



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- $$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} = E\psi(x), \quad 0 < x < L$$

$$\psi(x) = 0, \quad x \leq 0 \text{ or } x \geq L$$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi(x) = -K^2\psi(x)$$

- Define $\frac{2mE}{\hbar^2} = K^2$.

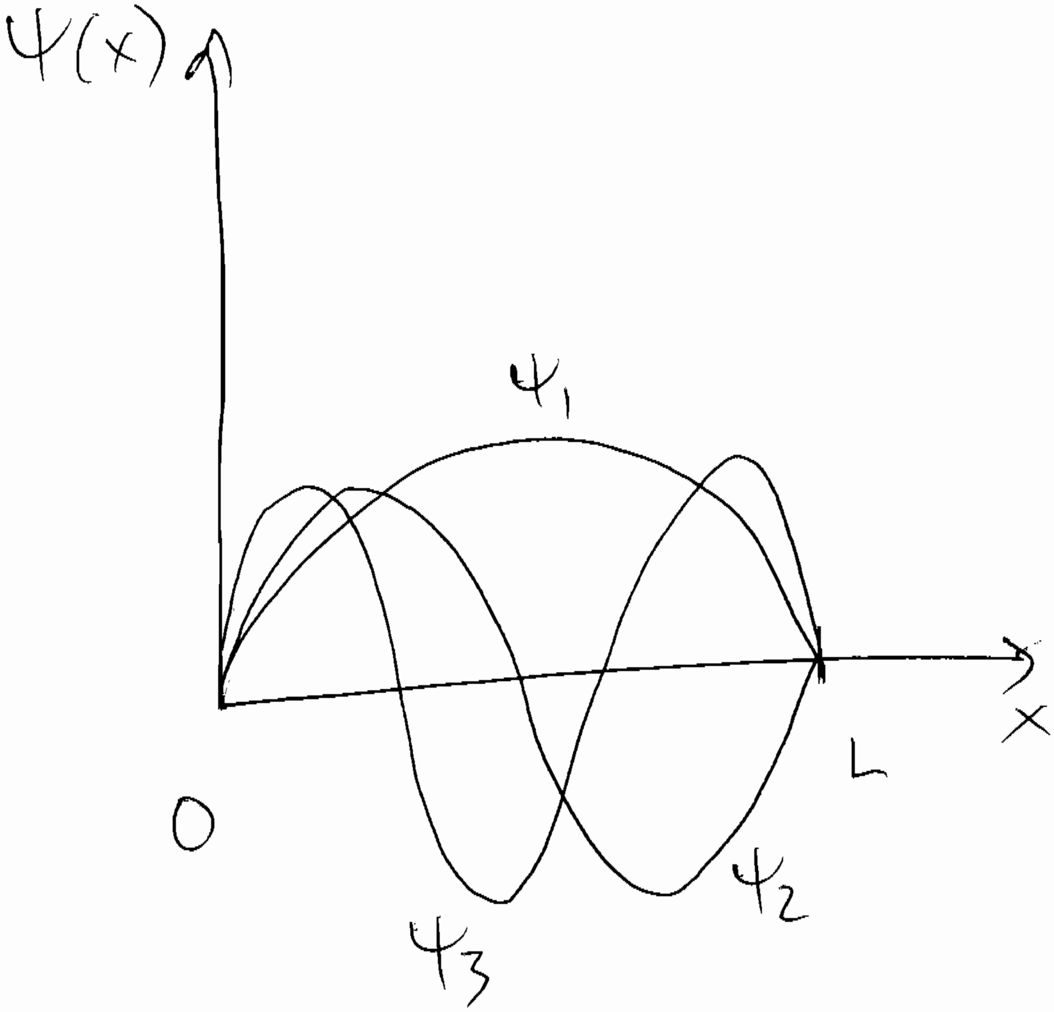
Solutions are:

$$\psi(x) = A\cos Kx + B\sin Kx$$

$$0 = \psi(0) = A$$

- $$0 = \psi(L) = B\sin KL \Rightarrow K_n L = n\pi$$

$$n = 1, 2, 3, \ldots$$


ψ(x)
ψ1
ψ2
ψ3
0
L
x

$$\psi_n(x) = B \sin\left(\frac{n\pi x}{L}\right)$$

$$\int_0^L dx\, |\psi_n(x)|^2 = 1$$

$$B^2 \int_0^L dx\, \sin^2\left(\frac{n\pi x}{L}\right) = 1$$

$$B^2 \frac{L}{2} = 1$$

$$B = \sqrt{\frac{2}{L}}$$

$$\boxed{\begin{gathered} \psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \\ E_n = \frac{\hbar^2 K_n^2}{2m} = \frac{\hbar^2 \pi^2 n^2}{2m L^2} \\ n = 1, 2, 3, \ldots \end{gathered}}$$

$$\Phi(x,t) = a\psi_1(x)e^{-\frac{iE_1 t}{\hbar}} + b\psi_2(x)e^{-\frac{iE_2 t}{\hbar}}$$

Proof:

$$i\hbar\frac{\partial\Phi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Phi}{\partial x^2} + V(x)\Phi(x,t)$$

$$E_1 a\psi_1(x)e^{-\frac{iE_1 t}{\hbar}} + E_2 b\psi_2(x)e^{-\frac{iE_2 t}{\hbar}}$$

$$= ae^{-\frac{iE_1 t}{\hbar}}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi_1(x)$$

$$+ be^{-\frac{iE_2 t}{\hbar}}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi_2(x)$$

Q: What is $\rho(x,t) = |\Phi(x,t)|^2$?

$$\rho(x,t) = \left(a^*\psi_1(x)e^{\frac{iE_1 t}{\hbar}} + b^*\psi_2(x)e^{\frac{iE_2 t}{\hbar}}\right)$$

$$\times\left(a\psi_1(x)e^{-\frac{iE_1 t}{\hbar}} + b\psi_2(x)e^{-\frac{iE_2 t}{\hbar}}\right)$$

$$\rho(x,t) = |a|^2 |\psi_1(x)|^2 + |b|^2 |\psi_2(x)|^2$$
$$+ a^* b\, \psi_1(x)\psi_2(x)\, e^{i\frac{E_1-E_2}{\hbar}t}$$
$$+ a b^* \psi_1(x)\psi_2(x)\, e^{-i\frac{E_1-E_2}{\hbar}t}$$

Let $a = |a| e^{i\alpha}$, $b = |b| e^{i\beta}$

$$\rho(x,t) = |a|^2 \psi_1^2(x) + |b|^2 \psi_2^2(x)$$
$$+ 2|a||b|\psi_1(x)\psi_2(x)$$
$$\times \cos\left(\frac{E_1-E_2}{\hbar}t + \beta - \alpha\right)$$

The probability distribution oscillates at a frequency

$$\Omega_{12} = \frac{E_2 - E_1}{\hbar}$$

$\rightarrow$ gen. sol'n $\psi(x,t)$