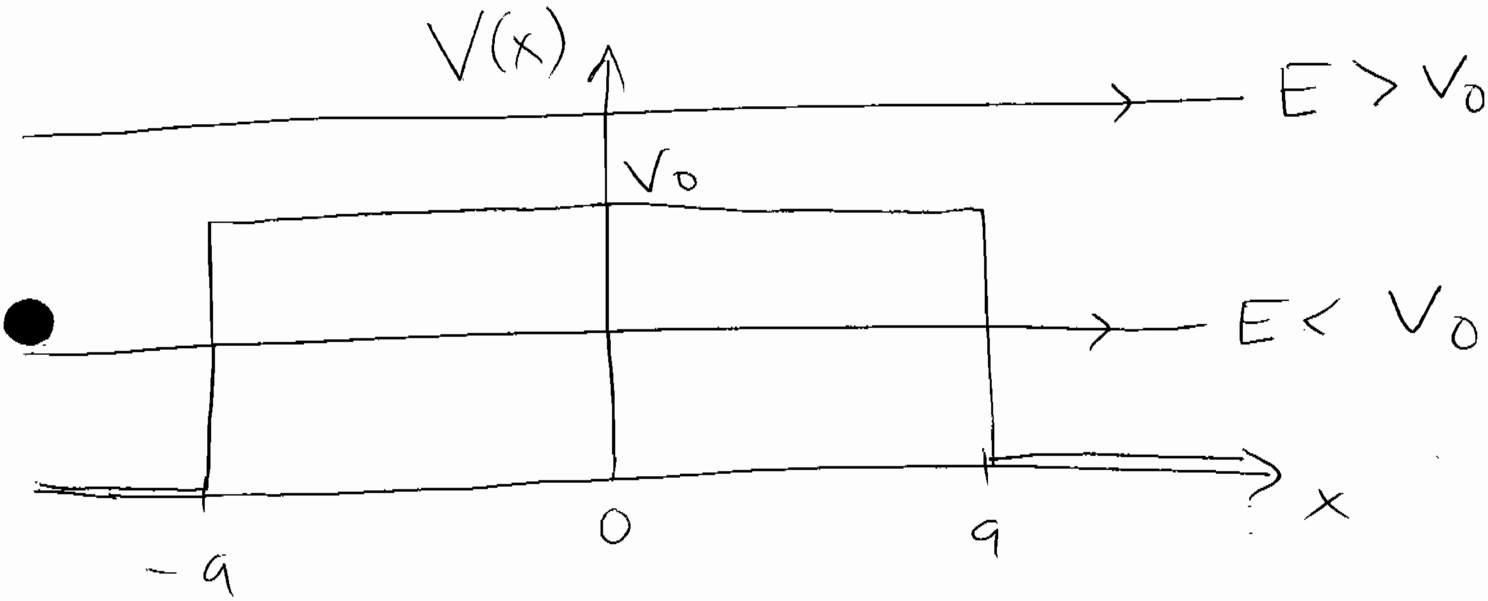
Physics 371 Lecture 9

Tunneling

1) Square barrier



i) $E > V_0$

$$\psi_I = A e^{ik_1 x} + B e^{-ik_1 x}, \quad x < -a$$

$$\psi_{II} = C e^{ik_2 x} + D e^{-ik_2 x}, \quad -a < x < a$$

$$\psi_{III} = F e^{ik_1 x}, \quad x > a$$

$\hbar k_1 = \sqrt{2mE}$

$\hbar k_2 = \sqrt{2m(E - V_0)}$

Boundary conditions:

- $\psi_I(-a) = \psi_{II}(-a)$ $\qquad$ $\psi_I'(-a) = \psi_{II}'(-a)$

  $\psi_{II}(a) = \psi_{III}(a)$ $\qquad$ $\psi_{II}'(a) = \psi_{III}'(a)$

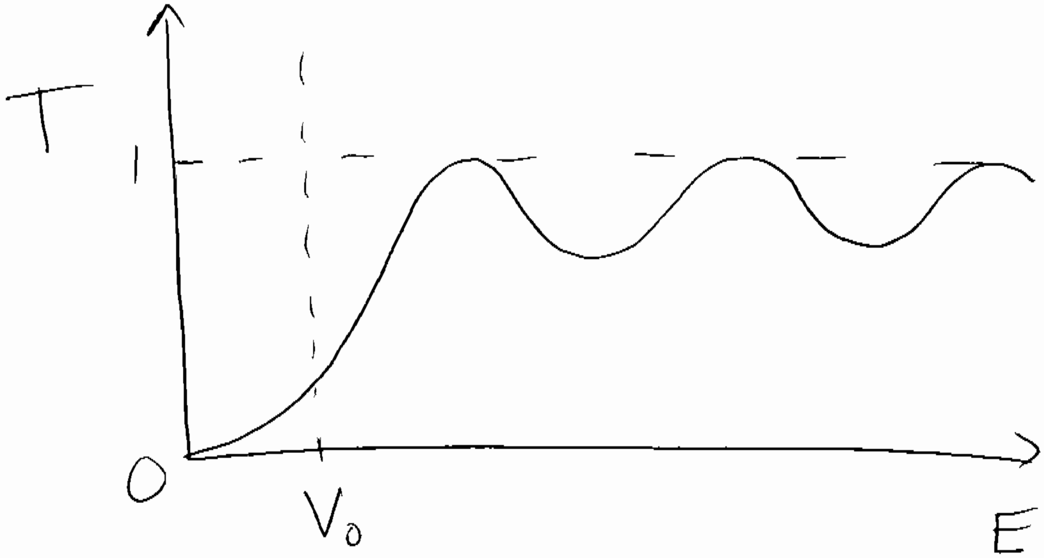
$$T = \frac{|\bar{J}_{tr}|}{|\bar{J}_{in}|} = \frac{|F|^2}{|A|^2}$$

- $$R = \frac{|\bar{J}_{re}|}{|\bar{J}_{in}|} = \frac{|B|^2}{|A|^2}$$

$$\bar{J}_{in} = \frac{\hbar k_1}{m} |A|^2 \Rightarrow A$$

4 eq.s & 4 unknowns $\Rightarrow$

- $$T = \frac{1}{1 + \frac{V_0^2}{4E(E-V_0)} \sin^2(2k_2 a)}$$



$T = 1$ whenever $2k_2a = n\pi$,

i.e., when $2a = \dfrac{n\lambda}{2}$

(constructive interference).

Minima in T correspond to destructive interference between the direct wave and the multiply reflected wave:

e^{i2k_2a}

$e^{i(2k_2a + 4k_2a)}$

$e^{i4k_2a} = -1$ $\qquad 4a = (n+\frac{1}{2})\lambda$

(ii) $E < V_0$

- $$\psi_{II} = C e^{K_2 x} + D e^{-K_2 x}$$

$$\hbar K_2 = \sqrt{2m(V_0 - E)}$$

$$\Rightarrow \quad T = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2(2K_2 a)}$$

- $$\lim_{E \to V_0} T(E) = \frac{1}{1 + \frac{V_0}{4(V_0 - E)} (2K_2 a)^2}$$

$$= \frac{1}{1 + \frac{2mV_0 a^2}{\hbar^2}}$$

- $$= \frac{1}{1 + (K_1 a)^2}$$

- Note that $T \neq 0$ even when $E < V_0$. The particle has a nonzero probability to "tunnel" through the energetically forbidden barrier. How is this possible?

- $$\Delta E \, \Delta t \geq \frac{\hbar}{2}$$

If the particle stays in the barrier for only a very short time, then

$$\Delta E \sim \frac{\hbar}{2\Delta t} > V_0 - E.$$

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Tunneling time

- $$\Delta t \lesssim \frac{\hbar}{2(V_0 - E)}.$$

Any longer dwell time would run into a problem with energy conservation.

- <u>Large / long barrier</u>

$T \ll 1$: $\sinh^2(2K_2 a) \sim \frac{e^{+4K_2 a}}{4}$

$$T \sim \frac{16E(V_0 - E)}{V_0^2} e^{-4K_2 a}$$

$$\ln T \simeq -4K_2 a$$

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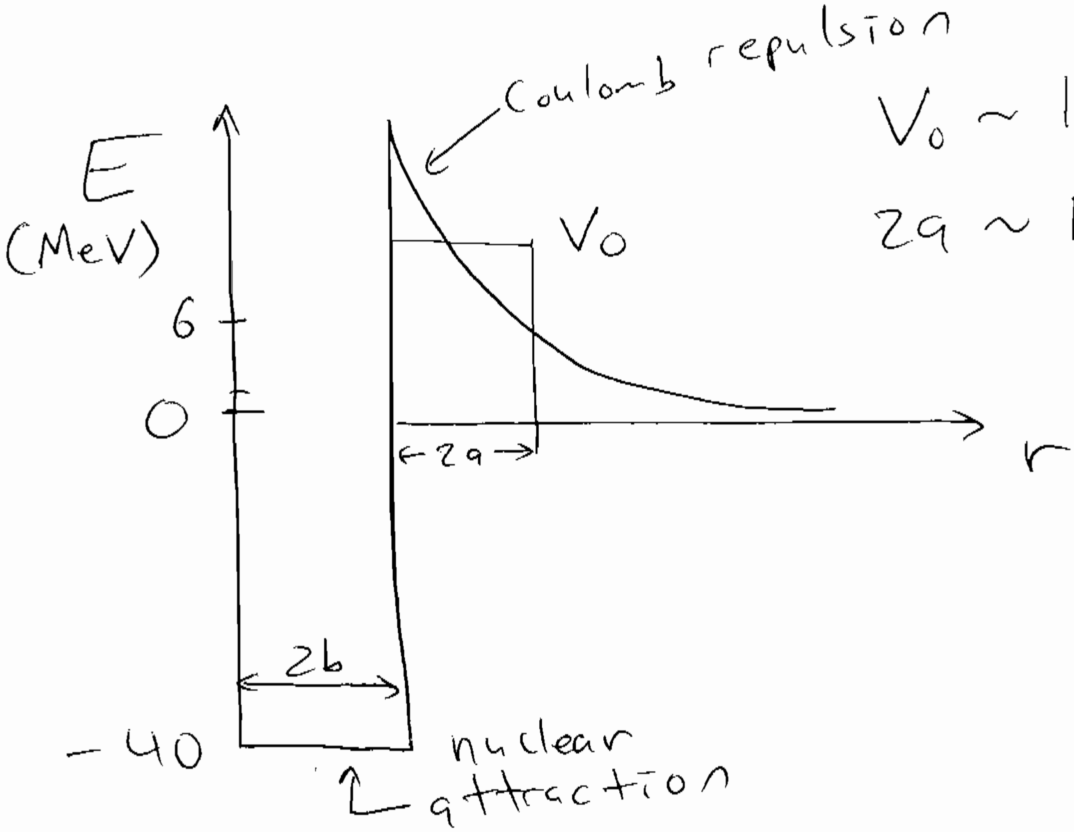
Example Decay of a heavy nucleus via α emission.



$V_0 \sim 16$ MeV

$2a \sim 10^{-14}$ m

$$\ln T \sim -4K_2 a \sim -59$$

$$K_2 = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$$

$$T \sim 10^{-25}$$

$m \sim 4$ amu

Escape rate

$$\Gamma = \omega e^{-4 K_2 a}$$

ω — attempt frequency; $e^{-4K_2 a}$ — tunneling probability

$$\omega = 2\pi \frac{v}{2b} = \frac{\pi}{b}\sqrt{\frac{2E_K}{m}} \sim 10^{21}\, s^{-1}$$

$$\Gamma \sim 10^{-4}\, s^{-1} \quad (\tau \sim 3\text{ hrs.})$$

If we increased barrier width by 20%, then

$$T \sim 10^{-30} \quad \text{and} \quad \Gamma \sim 10^{-9}\, s^{-1}$$

$$(\tau \sim 100\text{ yrs}).$$