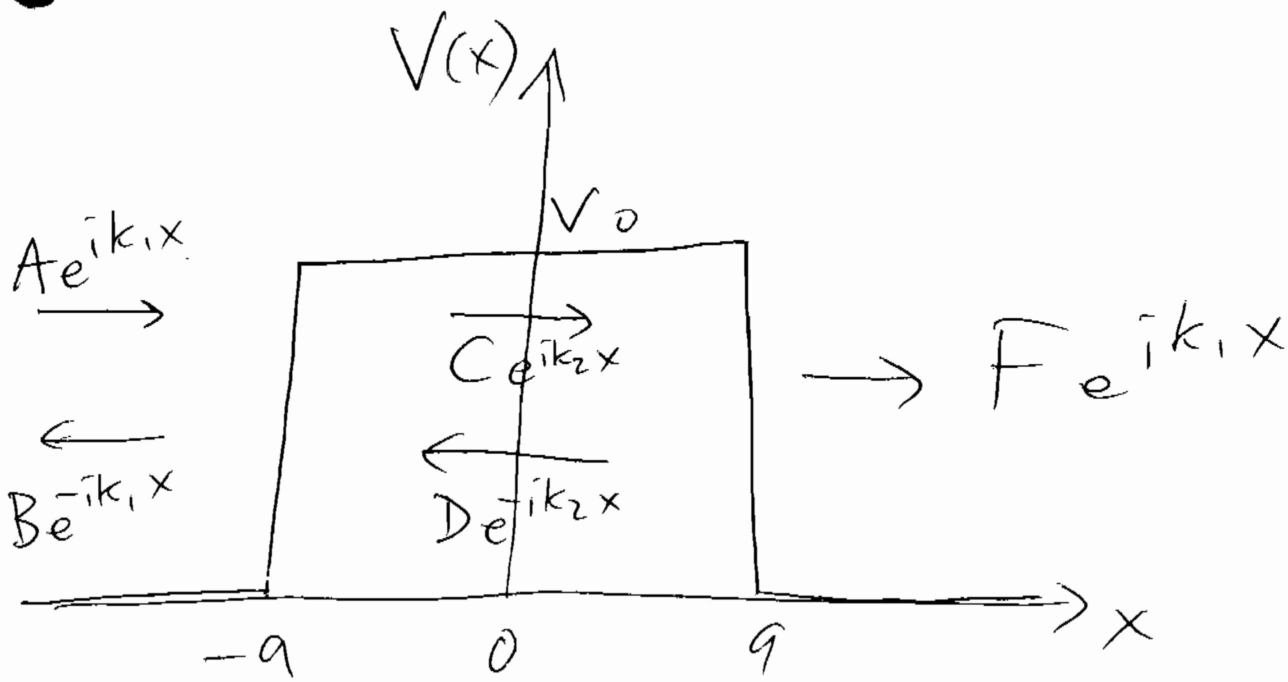
Physics 371 Lecture 10

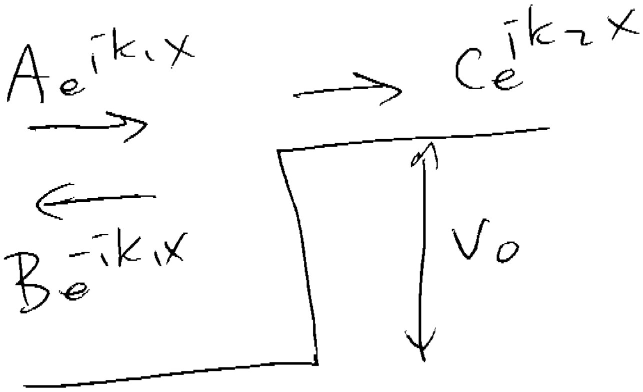
More on piecewise-constant potentials in 1D

i) Scattering from a rectangular barrier revisited ($E > V_0$)



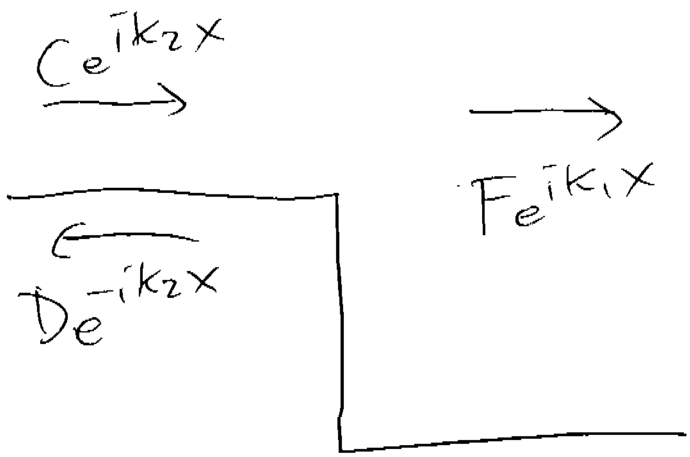
- Solve using Huygen's principle in 1D

- For a single step up, we would have



$$C = \frac{2k_1}{k_1+k_2} A$$

- For a single step down, we would have



$$D = \frac{k_2-k_1}{k_1+k_2} C$$

$$F = \frac{2k_2}{k_1+k_2} C$$

- Let $t_1 = \dfrac{2k_1}{k_1+k_2}$ $\quad r_1 = \dfrac{k_2-k_1}{k_1+k_2}$

$$t_2 = \frac{2k_2}{k_1+k_2} \qquad r_2 = \frac{k_2-k_1}{k_1+k_2}$$

- $$\frac{F}{A} = t_1 e^{i2k_2a} t_2 + t_1 e^{i2k_2a} r_2 e^{i2k_2a} r_1 e^{i2k_2a} t_2 + t_1 e^{i2k_2a} r_2 e^{i2k_2a} r_1 e^{i2k_2a} r_2 e^{i2k_2a} \times r_1 e^{i2k_2a} t_2 + \cdots$$

- $$= t_1 t_2 e^{i2k_2a} \left(1 + r_1 r_2 e^{i4k_2a} + (r_1 r_2)^2 e^{i8k_2a} + \cdots \right)$$

$$= \frac{t_1 t_2 e^{i2k_2a}}{1 - r_1 r_2 e^{i4k_2a}}$$

- Now $t_1 t_2 = \frac{4k_1k_2}{(k_1+k_2)^2} = T_1$

(trans. prob. for a single step).

and $r_1 r_2 = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 = R_1$

(refl. prob. for a single step).

Note that $T_1 + R_1 = 1$.

$$t_{12} = \frac{F}{A} = \frac{T_1 e^{i 2k_2 a}}{1 - R_1 e^{i 4k_2 a}}$$

$$T_{12} = |t_{12}|^2 = \frac{T_1^{\,2}}{\left(1 - R_1 e^{i4k_2 a}\right)\left(1 - R_1 e^{-i4k_2 a}\right)}$$

$$= \frac{T_1^{\,2}}{1 + R_1^2 - 2R_1 \cos 4k_2 a}$$

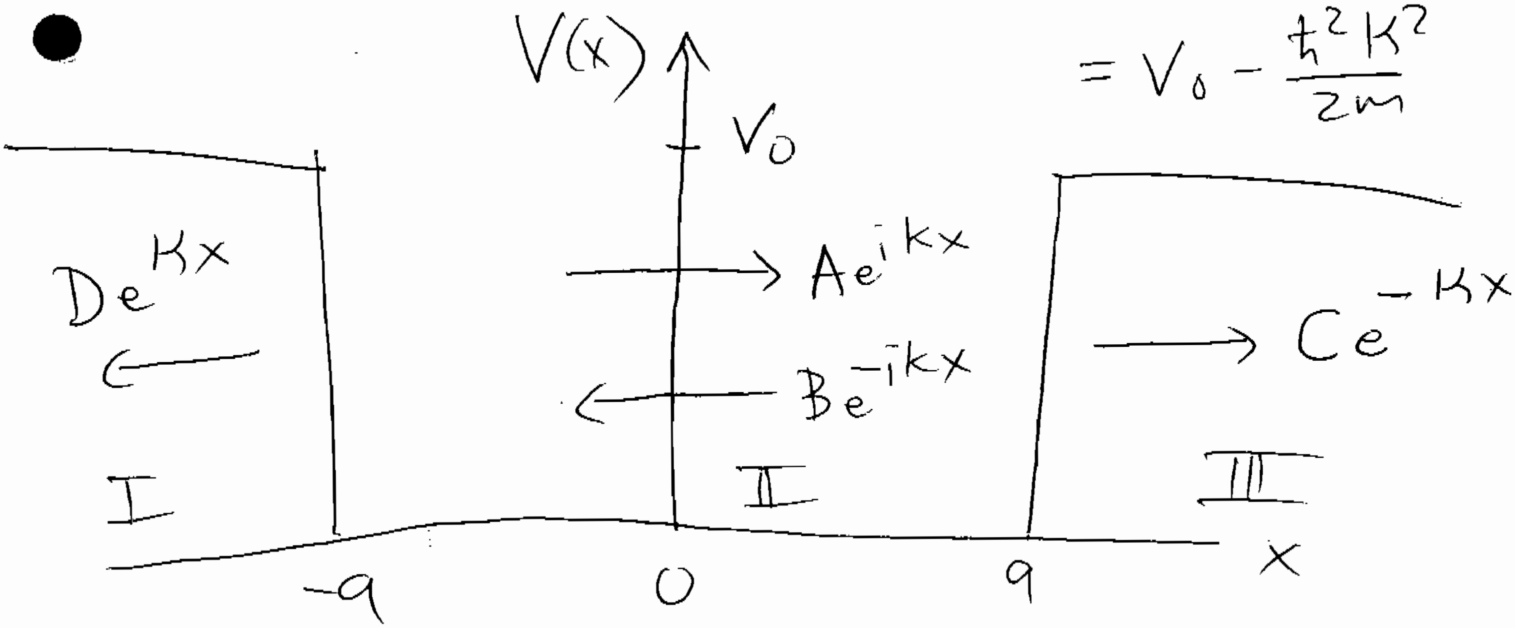
$$= \frac{1}{1 + \frac{4R_1}{(1-R_1)^2} \sin^2(2k_2 a)}$$

This is the same answer we got before by another method.

ii) Bound states in a rectangular potential well

$(E < V_0)$

$$E = \frac{\hbar^2 k^2}{2m}$$
$$= V_0 - \frac{\hbar^2 K^2}{2m}$$



Standard method:

$\psi_I(-a) = \psi_{II}(-a)$

$\psi_I'(-a) = \psi_{II}'(-a)$

$\psi_{II}(a) = \psi_{III}(a)$

$\psi_{II}'(a) = \psi_{III}'(a)$

One of the coefficients A, B, C, D can be fixed by normalization. The remaining four variables (including E) can be determined by solving this system of four equations.

(See Goswami, 4.2)

Another (easier) way:

$$r = \frac{B}{A} = \frac{k - iK}{k + iK} \equiv e^{i\theta}$$

A bound state, or standing wave, is a stationary solution:

$$e^{ik2a} r e^{ik2a} r = 1$$

$$r^2 e^{i4ka} = 1$$

$$r e^{i2ka} = \pm 1 \qquad \text{let } L = 2a$$

$$e^{i(kL+\theta)} = \pm 1$$

$$e^{i(kL+\theta)} = e^{i\pi n} = \begin{cases} +1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$$

$$\boxed{\begin{gathered} kL + \theta(k) = n\pi, \\ \theta = -2\tan^{-1}\left(\frac{K}{k}\right) \end{gathered}}$$

$\Rightarrow$ determines allowed values of k_n and hence E_n.

Special case: $V_0 \gg E \Rightarrow K \gg k$

$\tan^{-1}(\infty) = \frac{\pi}{2}$ $\qquad k_n L - \pi = n\pi$

$$k_n = \frac{(n+1)\pi}{L}, \quad n = 0, 1, 2, \ldots$$

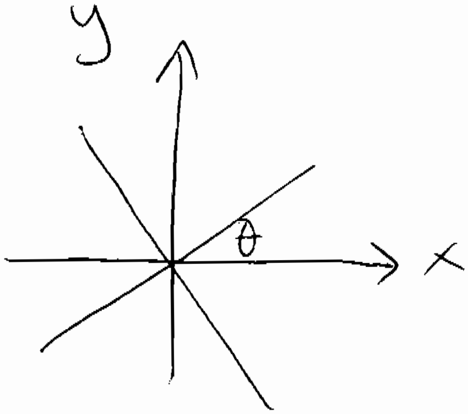
$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 \pi^2}{2mL^2}(n+1)^2$$

$\Rightarrow$ particle in a box.

One can also write

$$\tan^{-1}\left(\frac{K}{k}\right) = k_n a - \frac{n\pi}{2}$$

$$\frac{K}{k} = \tan\left(k_n a - \frac{n\pi}{2}\right)$$



$$\frac{K}{k} = \tan ka, \quad n \text{ even}$$

or

$$\frac{K}{k} = -\cot ka, \quad n \text{ odd}$$

$$Lk - 2\tan^{-1}\left(\frac{K}{k}\right) = n\pi$$

- If $L \to 0$ or $V_0 \to 0$, then only one solution exists : $n = 0$.

$\boxed{\text{An attractive potential always forms at least one bound state in 1D.}}$

- $\theta \to 0$ as $E \to V_0$, so the total # of bound states is, in general, determined by

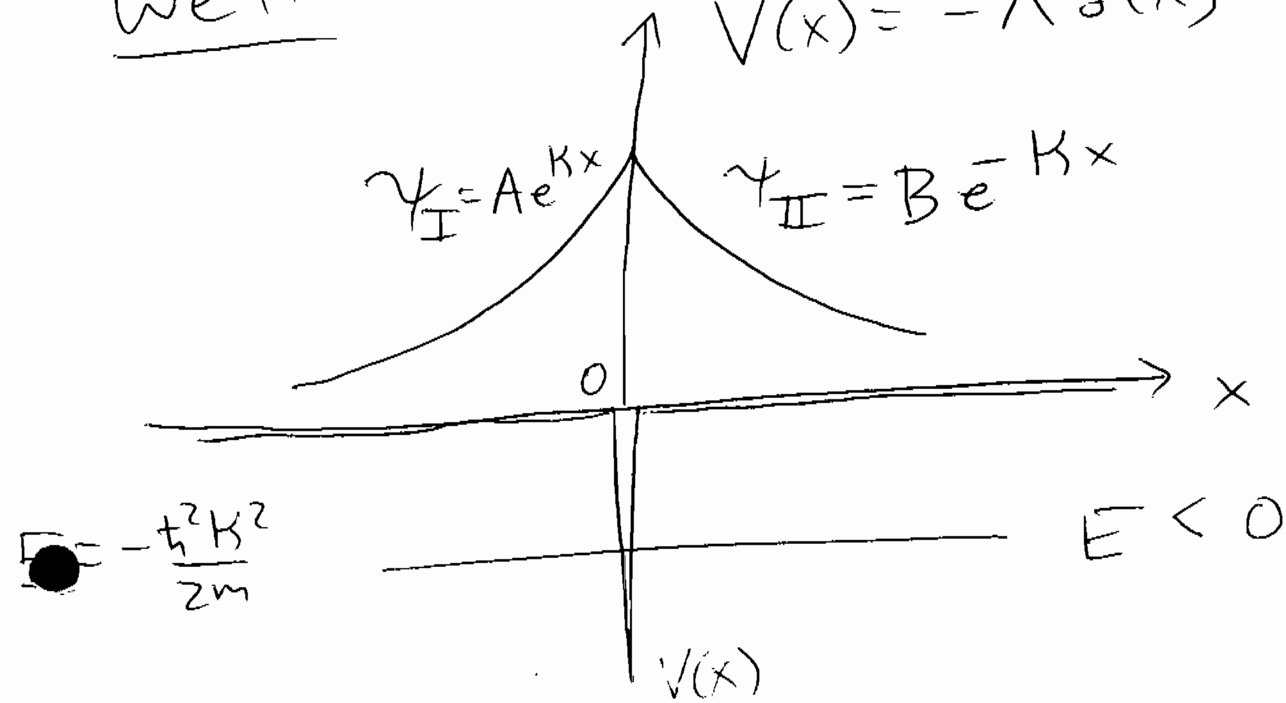
$$V_0 > E_n = \frac{\hbar^2 k_n^2}{2m} \simeq \frac{\hbar^2 \pi^2 n^2}{2m L^2}$$

$$n_{max} \simeq \text{Int}\sqrt{\frac{2mL^2 V_0}{\pi^2 \hbar^2}}$$

Physics 371

1) The delta-function potential well

$V(x) = -\Lambda\,\delta(x)$



$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \Lambda\,\delta(x)\,\psi(x) = E\,\psi(x)$$

For $x > 0$, one has

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\left(Be^{-Kx}\right) = E\,Be^{-Kx}$$

$$-\frac{\hbar^2 K^2}{2m}Be^{-Kx} = E\,Be^{-Kx}$$

$$\Rightarrow \quad E = -\frac{\hbar^2 K^2}{2m}$$

similarly for $x < 0$. Exponentially growing solutions are ruled out because they can not be normalized. Boundary conditions at $x = 0$:

i) $\psi_I(0) = \psi_{II}(0)$

$$A = B$$

ii) $$\int_{-\epsilon}^{\epsilon} \left(\frac{d^2\psi}{dx^2} + \frac{2m\Lambda}{\hbar^2} \delta(x)\psi(x) \right) dx = -\frac{2m}{\hbar^2}\int_{-\epsilon}^{\epsilon} E\psi(x)\, dx$$

$$\psi'(\epsilon) - \psi'(-\epsilon) + \frac{2m\Lambda}{\hbar^2}\psi(0) \simeq \cancel{-2\epsilon E\psi(0)\left(\frac{2m}{\hbar^2}\right)} \to 0$$

$$\psi'(0^+) - \psi'(0^-) = -\frac{2m\Lambda}{\hbar^2}\psi(0)$$

$$\psi_{II}'(0) - \psi_I'(0) = -\frac{2m\Lambda}{\hbar^2}\psi_I(0)$$

$$-KA - KA = -\frac{2m\Lambda}{\hbar^2} A$$

$$K = \frac{m\Lambda}{\hbar^2} \Rightarrow E = -\frac{m\Lambda^2}{2\hbar^2}$$

There is exactly one bound state.

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