Physics 371 Lecture 16

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1) Measurements of compatible observables: Commuting operators

Recall $[\hat{Q},\hat{P}] = \hat{Q}\hat{P} - \hat{P}\hat{Q}$.

If $[\hat{Q},\hat{P}] = 0$, then

- $\langle \hat{Q}\hat{P} \rangle = \langle \hat{P}\hat{Q} \rangle$ i.e., the result does not depend on the order of measurement

Theorem If $[\hat{P},\hat{Q}] = 0$ and either $\hat{P}$ or $\hat{Q}$ has non-degenerate eigenvalues, its eigenfunctions are also eigenfunctions of the other operator.

Proof: Given $[\hat{P}, \hat{Q}] = 0$ and

- $\hat{P}\psi_i = p_i \psi_i$, where all p_i are distinct.

Then, $\hat{Q}\hat{P}\psi_i = \hat{Q}\, p_i \psi_i = p_i(\hat{Q}\psi_i)$.

But $\hat{Q}\hat{P}\psi_i = \hat{P}\hat{Q}\psi_i = \hat{P}(\hat{Q}\psi_i)$

$$\Rightarrow \quad \hat{P}(\hat{Q}\psi_i) = p_i(\hat{Q}\psi_i).$$

- $\hat{Q}\psi_i$ is an eigenvector of $\hat{P}$ with eigenvalue p_i. This leads to a contradiction unless

$\hat{Q}\psi_i = q_i \psi_i$, where q_i is a complex number. Therefore ψ_i is also an eigenfunction of $\hat{Q}$.

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Let us write

$\psi = \sum_i c_i \psi_i$. If the variable p is measured, the result p_i will be obtained with probability $|c_i|^2$, assuming ψ is normalized. After the measurement, the wavefunction will be ψ_i. Subsequent measurements of q or p will yield q_i and p_i, respectively, and ψ will not be altered by further measurements of these variables. q and p are said to be <u>compatible</u>.

Q: What happens if $[\hat{Q},\hat{P}] \neq 0$? (4

2) Commutators and uncertainty relations

If $[\hat{Q},\hat{P}] \neq 0$, it is clear that $\hat{Q}$ and $\hat{P}$ do not have the same eigenfunctions.

- measurement of q forces Ψ into an eigenfunction of $\hat{Q}$. A subsequent measurement of $\hat{P}$ forces Ψ into an eigenfunction of $\hat{P}$, and destroys the information about the variable q gleaned from the previous measurement.
- This is the root of the

Uncertainty principle.

Generalized uncertainty principle

$$\Delta Q \, \Delta P \geq \frac{1}{2} \left| \langle [\hat{Q}, \hat{P}] \rangle \right| ,$$

where $(\Delta Q)^2 = \langle \hat{Q}^2 \rangle - \langle \hat{Q} \rangle^2$

$(\Delta P)^2 = \langle \hat{P}^2 \rangle - \langle \hat{P} \rangle^2 .$

Proof: Let $\Delta \hat{Q} = \hat{Q} - \langle \psi | \hat{Q} | \psi \rangle$ for a given wavefunction ψ. Similarly, define $\Delta \hat{P} = \hat{P} - \langle \psi | \hat{P} | \psi \rangle$.

$$\langle (\Delta \hat{Q})^2 \rangle = \langle \hat{Q}^2 \rangle + \langle \hat{Q} \rangle^2 - 2 \langle \hat{Q} \rangle^2$$

$$= \langle \hat{Q}^2 \rangle - \langle \hat{Q} \rangle^2 = (\Delta Q)^2$$

Also $\langle (\Delta\hat{P})^2 \rangle = (\Delta P)^2$.

Now $(\Delta Q)^2 (\Delta P)^2 = \langle \psi | (\Delta\hat{Q})^2 | \psi \rangle$
$\times \langle \psi | (\Delta\hat{P})^2 | \psi \rangle$

$= \langle \Delta\hat{Q}\psi | \Delta\hat{Q}\psi \rangle \langle \Delta\hat{P}\psi | \Delta\hat{P}\psi \rangle$,

since $\Delta\hat{Q}$ and $\Delta\hat{P}$ are hermitian.

$(\Delta Q)^2 (\Delta P)^2 \geq |\langle \Delta\hat{Q}\psi | \Delta\hat{P}\psi \rangle|^2$

(schwartz inequality)

$\geq \left[\mathrm{Im} \left(\langle \Delta\hat{Q}\psi | \Delta\hat{P}\psi \rangle \right) \right]^2$

$= \left| \dfrac{\langle \Delta\hat{Q}\psi | \Delta\hat{P}\psi \rangle - \langle \Delta\hat{P}\psi | \Delta\hat{Q}\psi \rangle}{2i} \right|^2$

$= \left| \langle [\Delta\hat{Q}, \Delta\hat{P}] \rangle / 2 \right|^2$

$= \left| \langle [\hat{Q}, \hat{P}] \rangle / 2 \right|^2$

$$\Rightarrow \Delta Q \, \Delta P \geq \frac{1}{2} \left| \langle [\hat{Q}, \hat{P}] \rangle \right|.$$

Example $\hat{x} = x \quad \hat{p}_x = \frac{\hbar}{i} \frac{\partial}{\partial x}$

$$[\hat{x}, \hat{p}_x] \psi = \frac{\hbar}{i} \left(x \frac{\partial}{\partial x} \psi - \frac{\partial}{\partial x} (x \psi) \right)$$

$$= \frac{\hbar}{i} \left[x \psi'(x) - \psi(x) - x \psi'(x) \right]$$

$$= i\hbar \, \psi(x)$$

$$\Rightarrow [\hat{x}, \hat{p}_x] = i\hbar$$

$$\Delta x \, \Delta p_x \geq \frac{1}{2} \left| \langle [\hat{x}, \hat{p}_x] \rangle \right| = \frac{\hbar}{2}$$

The familiar form of the uncertainty principle.

Aside: The Schwartz inequality for two vectors states

$$|\vec{a}|\,|\vec{b}| \geq |\vec{a}\cdot\vec{b}|.$$

In terms of two complex functions $f(x)$ and $g(x)$, one has analogously

$$\int |f(x)|^2\,dx \int |g(x)|^2\,dx \geq \left|\int f^*(x)\,g(x)\,dx\right|^2.$$

3) Time evolution of expectation values

$$\frac{d}{dt}\langle\hat{Q}\rangle = \frac{d}{dt}\langle\Psi|\hat{Q}\Psi\rangle$$

$$= \left\langle\frac{\partial\Psi}{\partial t}\Big|\hat{Q}\Psi\right\rangle + \left\langle\Psi\Big|\frac{\partial\hat{Q}}{\partial t}\Psi\right\rangle + \left\langle\Psi\Big|\hat{Q}\frac{\partial\Psi}{\partial t}\right\rangle$$

write it out longhand

But $\frac{\partial \Psi}{\partial t} = \frac{\hat{H}\Psi}{i\hbar}$ and

$$\frac{\partial \Psi^*}{\partial t} = -\frac{\hat{H}\Psi^*}{i\hbar},$$ so

$$\frac{d\langle\hat{Q}\rangle}{dt} = -\frac{1}{i\hbar}\langle \hat{H}\Psi | \hat{Q}\Psi\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle + \frac{1}{i\hbar}\langle \Psi | \hat{Q}\hat{H}\Psi\rangle$$

$$\frac{d}{dt}\langle\hat{Q}\rangle = -\frac{1}{i\hbar}\langle \Psi | (\hat{H}\hat{Q} - \hat{Q}\hat{H})\Psi\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle$$

$$= -\frac{1}{i\hbar}\langle [\hat{H},\hat{Q}]\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle$$

$$= \frac{1}{i\hbar}\langle [\hat{Q},\hat{H}]\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle$$

Thus the expectation value is constant if $\hat{Q}$ has no explicit time dependence and it commutes with the Hamiltonian.

Example

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

$$= \frac{\hat{P}_x^{\,2}}{2m} + V(x)$$

i) $$\frac{d}{dt}\langle x\rangle = -\frac{1}{i\hbar}\langle [\hat{H}, \hat{x}]\rangle$$

$$= -\frac{1}{2i\hbar m}\langle [\hat{P}_x^{\,2}, \hat{x}]\rangle$$

$$= -\frac{1}{2i\hbar m}\langle \hat{P}_x[\hat{P}_x, \hat{x}] + [\hat{P}_x, \hat{x}]\hat{P}_x\rangle$$

$$= \frac{1}{m}\langle \hat{P}_x\rangle$$

$$m\frac{d}{dt}\langle x\rangle = \langle \hat{P}_x\rangle$$

ii) $$\frac{d}{dt}\langle P_x\rangle = \frac{1}{i\hbar}\langle [\hat{P}_x, \hat{H}]\rangle$$

$$= \frac{1}{i\hbar}\langle [\hat{P}_x, V(x)]\rangle$$

$$[\hat{p}_x, V(x)]\psi(x) = \frac{\hbar}{i}\frac{d}{dx}\big(V(x)\psi(x)\big) - V(x)\frac{\hbar}{i}\frac{d\psi}{dx}$$

$$= \frac{\hbar}{i}V'(x)\,\psi(x)$$

$$\Rightarrow \quad [\hat{p}_x, \hat{V}(x)] = \frac{\hbar}{i}\frac{dV}{dx}$$

$$\frac{d}{dt}\langle p_x\rangle = -\left\langle \frac{dV}{dx}\right\rangle .$$

Thus the expectation values obey the classical equations of motion! This result is known as <u>Ehrenfest's theorem</u>, and holds quite generally.

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Parity

Define the <u>parity operator</u> $\hat{P}$:

$$\hat{P} f(x) = f(-x)$$

$$\hat{P}^2 f(x) = \hat{P} f(-x) = f(x)$$

$$\Rightarrow \hat{P}^2 = \mathbb{1} \quad \text{unit operator}$$

<u>Eigenfunctions & eigenvalues</u>:

Let $\hat{P} \psi_p(x) = p \psi_p(x)$

$$\hat{P}^2 \psi_p(x) = p^2 \psi_p(x) = \psi_p(x)$$

$$\Rightarrow p^2 = 1, \quad p = \pm 1$$

i) $p = +1$ $\quad \psi_p(-x) = \psi_p(x)$ (even function)

ii) $p = -1$ $\quad \psi_p(-x) = -\psi_p(x)$ (odd function)

$\Rightarrow$ The eigenfunctions of the parity operator are the even and odd functions.

[The same holds true for wavefunctions in three dimensions:

$$\hat{P}\,\psi(\vec{x}) = \psi(-\vec{x}), \text{ etc.}$$]

Even potentials $\quad V(-x) = V(x)$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x).$$

Consider an energy eigenstate:

$$\hat{H}\,\psi(x) = E\,\psi(x)$$

$$\hat{P}\hat{H}\,\psi(x) = E\,\hat{P}\,\psi(x) = E\,\psi(-x)$$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{d(-x)^2} + V(-x)\right)\psi(-x) = E\,\psi(-x)$$

$$\hat{H}\,\psi(-x) = E\,\psi(-x).$$

- $\Rightarrow \psi(-x)$ is an eigenstate of $\hat{H}$ with the same energy eigenvalue as $\psi(x)$! If the energy E is nondegenerate, then $\psi(-x) = e^{i\theta}\psi(x)$. That is,

- $$\hat{P}\,\psi(x) = e^{i\theta}\psi(x).$$

But $\hat{P}^2\psi(x) = e^{i\theta}\hat{P}\,\psi(x) = e^{i2\theta}\psi(x)$

$$\psi(x) \;\;=\;\; e^{i2\theta}\psi(x)$$

$$\Rightarrow e^{i2\theta} = 1, \quad e^{i\theta} = p = \pm 1$$

- Thus the energy eigenstates in a symmetric potential are also eigenstates of parity.