The simple harmonic oscillator (see Goswami, Ch. 7)

\[ H = \frac{p^2}{2m} + \frac{1}{2} kx^2 \]

Classically, oscillating solutions

\[ x(t) = A \cos(\omega t + \delta), \quad \omega = \sqrt{\frac{k}{m}} \]

\[ \Rightarrow H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \]

Any potential can be approximate as a harmonic oscillator about its minimum.
\[ V(x) = V(x_0) + \frac{V'(x_0)}{2!} (x-x_0)^2 + \frac{V''(x_0)}{3!} (x-x_0)^3 + \ldots \]

\[ k = \frac{d^2V}{dx^2} \bigg|_{x_0} \]

Many systems described by (coupled) harmonic oscillators, photons, phonons, vibrations of molecules, nuclei, etc.
Schrödinger equation

\[-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \frac{m\omega^2 x^2}{2} \psi = E \psi\]

Dimensionless variables

Let \( \epsilon = \frac{2E}{\hbar \omega} \), \( \xi = \sqrt{\frac{m\omega}{\hbar}} x \)

\[\Rightarrow \frac{d^2 \psi}{d\xi^2} + (\epsilon - \xi^2) \psi = 0\]

Asymptotic behavior

For large \( |\xi| \) (\( \xi^2 \gg 1 \)),

\[\frac{d^2 \psi}{d\xi^2} - \xi^2 \psi \approx 0.\]
Solutions:  

\[ \psi \sim A e^{s^{3/2}} + B e^{-s^{3/2}} \]

Insert into Sch. equation:

\[ \frac{d^2 \psi}{ds^2} = s^{3} \psi + \psi \sim s^{3}\psi \]

Thus  \( \psi(s) \sim e^{-s^{3/2}} \)  

\[ \lim_{|s| \to \infty} \]

Factoring out the asymptotic behavior, let  \( \psi(s) = e^{-s^{3/2}} h(s) \).

\[ \Rightarrow \left| \frac{d^2 h}{ds^2} - 2s \frac{dh}{ds} + (\epsilon - 1) h(s) = 0 \right| \]

Hermite's equation
Power series solution

\[ h(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots \]

\[ h'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \]

\[ h''(x) = 2a_2 + 2 \cdot 3 a_3 x + 3 \cdot 4 a_4 x^2 + \cdots \]

Coefficient of each power of \(x\) must vanish.

1) \[ 2a_2 + (e-1) a_0 = 0 \]

2) \[ 2 \cdot 3 a_3 + (e-1-2) a_1 = 0 \]

3) \[ 3 \cdot 4 a_4 + (e-1-4) a_2 = 0 \]

\vdots
In general,

\[(k+1)(k+2) q_{k+2} + (e - 1 - 2k) q_k = 0\]

\[
q_{k+2} = -\frac{e - 1 - 2k}{(k+1)(k+2)} q_k
\]

Recursion relation

The recursion relation relates all even coefficients to \(q_0\) and all odd coefficients to \(q_1\). The even and odd coefficients are independent of one another.
This reflects the fact that the potential is symmetric in $x_0$

$V(-x) = V(x)$. Thus, the energy eigenstates must be either even or odd functions of $x$ (eigenstates of parity).

For the even eigenstates, we can choose $q_0 = 1$ and $q_1 = 0$. For the odd eigenstates, we can choose
\( q_0 = 0 \) and \( q_1 = 1 \).

Given \( E \), and the parity of the state (even or odd), the remaining terms in the power series follow from the recursion relation.

For large \( k \), one has

\[
\frac{a_{k+2}}{a_k} \sim \frac{2}{k}.
\]

Notice that

\[
e^\xi^2 = 1 + \xi^2 + \frac{\xi^4}{2!} + \frac{\xi^6}{3!} + \ldots
\]
That is, the coefficient of $\xi^k$ in the expansion of $e^{\xi^2}$ is $b_k = \frac{1}{(k/2)!}$.

$$\frac{b_{k+2}}{b_k} = \frac{(k/2)!}{(k/2 + 1)!} = \frac{1}{k + 1} \sim \frac{2}{k}.$$ 

Thus $h(\xi) \sim e^{\xi^2}$ if the power series continues for $k \to \infty$.

But then

$$\Psi(\xi) = e^{-\xi^2/2} h(\xi) \sim e^{\xi^2/2} \quad \xi \to \infty.$$ 

This just resurrects the discarded, non-normalizable term.
According to the postulates of QM, \( \psi \) must be normalizable. The only way for this to happen is if the polynomial \( h(\xi) \) terminates at some finite order \( n \). This will happen if

\[ E = 1 - 2n = 0 \]

\[ E = \frac{\hbar \omega}{2} c = \frac{\hbar \omega}{2} (n + \frac{1}{2}) \]

\( n = 0, 1, 2, \ldots, \infty \)

allowed energy levels