Physics 371 Lecture 24

Angular Momentum

To treat problems with rotational symmetry, it is useful to introduce the angular momentum operator $\vec{L} = \vec{r} \times \vec{p}$.

$$\vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$L_x = y p_z - z p_y$$

$$L_y = z p_x - x p_z$$

$$L_z = x p_y - y p_x$$

$\vec{L}$ is a Hermitian operator,

- $\vec{L}^{+} = \vec{L}$. For example,

$$L_z^{+} = p_y^{+} x^{+} - p_x^{+} y^{+} = p_y x - p_x y$$
$$= x p_y - y p_x = L_z$$

Here, we have used

- $[r_i, p_j] = i\hbar \delta_{ij} \qquad i, j = 1, 2, 3$

$\vec{r} = (r_1, r_2, r_3) = (x, y, z)$

$\vec{p} = (p_1, p_2, p_3) = (p_x, p_y, p_z)$

Example $\quad [x, p_y] = 0$

- $[x, p_y] \psi(x, y, z) = \frac{\hbar}{i}\left(x \frac{\partial \psi}{\partial y} - \frac{\partial}{\partial y}(x \psi)\right)$

$$= 0$$

An essential property of the angular momentum operator is that its components do not commute with one another:

$$[L_x, L_y] = [yP_z - zP_y, zP_x - xP_z]$$

$$= [yP_z, zP_x] + [zP_y, xP_z]$$

$$= y[P_z, z]P_x + x[z, P_z]P_y$$

$$= i\hbar(xP_y - yP_x) = i\hbar L_z$$

In general,

$$[L_i, L_j] = i\hbar \varepsilon_{ijk} L_k ,$$

where ε_{ijk} is the antisymmetric

- unit tensor:

$$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$$

$$\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj}.$$

In other words,

- $[L_y, L_z] = i\hbar L_x$

$$[L_z, L_x] = i\hbar L_y.$$

Another way to express these commutation relations is

$$\vec{L} \times \vec{L} = i\hbar \vec{L}$$

- (note that the cross-product would be zero for ordinary vectors).

The commutation relations imply uncertainty relations, e.g.

$$\Delta L_x \, \Delta L_y \geq \frac{\hbar}{2} \left| \langle L_z \rangle \right| .$$

These uncertainty relations imply that it is impossible to know simultaneously two different components of the $\vec{L}$ (unless $\langle \vec{L} \rangle = 0$, the trivial case).

However, each component of $\vec{L}$ commutes with $\vec{L}^2$:

$$[\vec{L}^2, L_i] = 0 .$$

$$\vec{L}^2 = L_x^2 + L_y^2 + L_z^2$$

$$[\vec{L}^2, L_z] = [L_x^2, L_z] + [L_y^2, L_z]$$

$$= L_x [L_x, L_z] + [L_x, L_z] L_x + L_y [L_y, L_z] + [L_y, L_z] L_y$$

$$= -i\hbar (L_x L_y + L_y L_x) + i\hbar (L_y L_x + L_x L_y) = 0$$

and similarly for L_x and L_y.

Thus we can characterize angular momentum eigenfunctions in terms of the eigenvalues of $\vec{L}^2$ and any one component, say L_z:

$$\vec{L}^2 \Psi_{\lambda m} = \lambda \Psi_{\lambda m}$$

$$L_z \Psi_{\lambda m} = m\hbar \Psi_{\lambda m}$$

The operator method for angular momentum

Define two new operators

$$L_+ = L_x + i L_y$$

$$L_- = L_x - i L_y = L_+^\dagger$$

Clearly $[\vec{L}^2, L_\pm] = 0.$

$$[L_z, L_+] = [L_z, L_x + i L_y]$$
$$= i\hbar L_y + i(-i\hbar L_x)$$
$$= \hbar(L_x + i L_y) = \hbar L_+$$

$$[L_z, L_-] = i\hbar L_y - \hbar L_x = -\hbar L_-$$

$$[L_+, L_-] = [L_x + iL_y, L_x - iL_y]$$

$$= -i[L_x, L_y] + i[L_y, L_x]$$

$$= 2\hbar L_z$$

Furthermore,

$$L_+ L_- = (L_x + iL_y)(L_x - iL_y)$$

$$= L_x^2 + L_y^2 - i[L_x, L_y]$$

$$= L_x^2 + L_y^2 + \hbar L_z$$

$$L_- L_+ = L_x^2 + L_y^2 + i[L_x, L_y]$$

$$= L_x^2 + L_y^2 - \hbar L_z$$

$$\Rightarrow \vec{L}^2 = \frac{1}{2}(L_+ L_- + L_- L_+) + L_z^2$$

From the commutator, we have

$$L_z L_+ = L_+(L_z + \hbar)$$

$$L_z(L_+\psi_{\lambda m}) = L_+(L_z+\hbar)\psi_{\lambda m}$$
$$= (m+1)\hbar\,(L_+\psi_{\lambda m})$$

$\Rightarrow$ $L_+\psi_{\lambda m}$ is an eigenstate of L_z with eigenvalue $(m+1)\hbar$. Similarly,

$$L_z L_-\psi_{\lambda m} = (m-1)\hbar\, L_-\psi_{\lambda m}.$$

L_+ and L_- are known as <u>raising</u> and <u>lowering operators</u>.

$$\vec{L}^2 L_\pm \psi_{\lambda m} = L_\pm \vec{L}^2\psi_{\lambda m} = \lambda L_\pm \psi_{\lambda m}$$

Thus $L_\pm \psi_{\lambda m}$ is an eigenstate of $\vec{L}^2$ with the same eigenvalue λ. $L_\pm$ increase/decrease L_z, but leave $\vec{L}^2$ unchanged.

Normalization

$$L_\pm \Psi_{\lambda m} = \alpha_\pm \Psi_{\lambda, m\pm 1}$$

Find $\alpha_\pm$.

$$L_+ \Psi_{\lambda m} = \alpha_+ \Psi_{\lambda m+1}$$

- $$\langle L_+ \Psi_{\lambda m} | L_+ \Psi_{\lambda m} \rangle = |\alpha_+|^2 \underbrace{\langle \Psi_{\lambda m+1} | \Psi_{\lambda m+1} \rangle}_{=1}$$

simplify notation

$$\langle \Psi_{\lambda m} | L_- L_+ \Psi_{\lambda m} \rangle \equiv \langle \lambda m | L_- L_+ | \lambda m \rangle$$

$$= |\alpha_+|^2$$

$$|\alpha_+|^2 = \langle \lambda m | \vec{L}^2 - L_z^2 - \hbar L_z | \lambda m \rangle$$

- $$= \lambda - \hbar^2 m(m+1)$$

$$\Rightarrow \quad L_+ \psi_{\lambda m} = \sqrt{\lambda - \hbar^2 m(m+1)}\; \psi_{\lambda\, m+1}$$

also $$L_- \psi_{\lambda m} = \sqrt{\lambda - \hbar^2 m(m-1)}\; \psi_{\lambda\, m-1}$$

Now $\vec{L}^2 = L_x^2 + L_y^2 + L_z^2$

$$\lambda = \langle \lambda m | \vec{L}^2 | \lambda m \rangle = \langle \lambda m | L_x^2 + L_y^2 | \lambda m \rangle + m^2 \hbar^2$$

$$\Rightarrow \quad \lambda \geq m^2 \hbar^2$$

since $\langle J_i^2 \rangle \geq 0$ for any hermitian operator J_i.

$$|m| \leq \sqrt{\lambda / \hbar^2}$$

$$-\sqrt{\lambda/\hbar^2} \leq m \leq \sqrt{\lambda/\hbar^2}$$

But for any particular value of m, we can always generate $m \pm 1$ by using the raising or lowering operator. We cannot keep doing so ad infinitum. We must have

$$L_+ \psi_{\lambda m_{max}} = 0$$

$$\Rightarrow \quad \lambda = \hbar^2 m_{max} (m_{max} + 1)$$

and $\quad L_- \psi_{\lambda m_{min}} = 0$

$$\Rightarrow \quad \lambda = \hbar^2 m_{min} (m_{min} - 1).$$

Furthermore,

$$m_{max} - m_{min} = n \geq 0,$$

where $n \in \mathbb{Z}$ is the number

of times we must apply L_+ to $\psi_{\lambda m_{min}}$ to generate $\psi_{\lambda m_{max}}$. The only possible solution is

$$m_{max} - m_{min} = n \equiv 2\ell$$

$$\lambda = \hbar^2 \ell(\ell+1)$$

$$-\ell \le m \le \ell \qquad (2\ell+1 \text{ values})$$

Rewrite $\psi_{\lambda m} \equiv \psi_{\ell m}$

$$\boxed{\begin{aligned} \vec{L}^2 \psi_{\ell m} &= \hbar^2 \ell(\ell+1)\, \psi_{\ell m} \\ L_z \psi_{\ell m} &= m\hbar\, \psi_{\ell m} \\ L_\pm \psi_{\ell m} &= \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)}\; \psi_{\ell m\pm 1} \end{aligned}}$$

As we shall see, for orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$, the quantum number

$$l = \text{integer.}$$

However, the angular momentum algebra only requires

$$n = 2l = \text{integer.}$$

Half-odd integers arise from "spin," the intrinsic angular momentum of certain elementary particles, which has no classical analogue — or from a combination of spin and orbital angular momentum.

"Vector model" for eigenstates of $\vec{L}^2$, L_z

$$\vec{L}^2 \psi_{\ell m} = \hbar^2 \ell(\ell+1)\psi_{\ell m}$$

$$L_z \psi_{\ell m} = \hbar m \psi_{\ell m}$$

$$L_z^2 \psi_{\ell m} = (\hbar m)^2 \psi_{\ell m}$$

$$(L_x^2 + L_y^2)\psi_{\ell m} = (\vec{L}^2 - L_z^2)\psi_{\ell m}$$
$$= \hbar^2(\ell(\ell+1) - m^2)\psi_{\ell m}$$

$\Rightarrow$ The wavefunction $\psi_{\ell m}$ has definite values of $\vec{L}^2$, L_z, and $L_x^2 + L_y^2$. L_x and L_y individually are uncertain, because

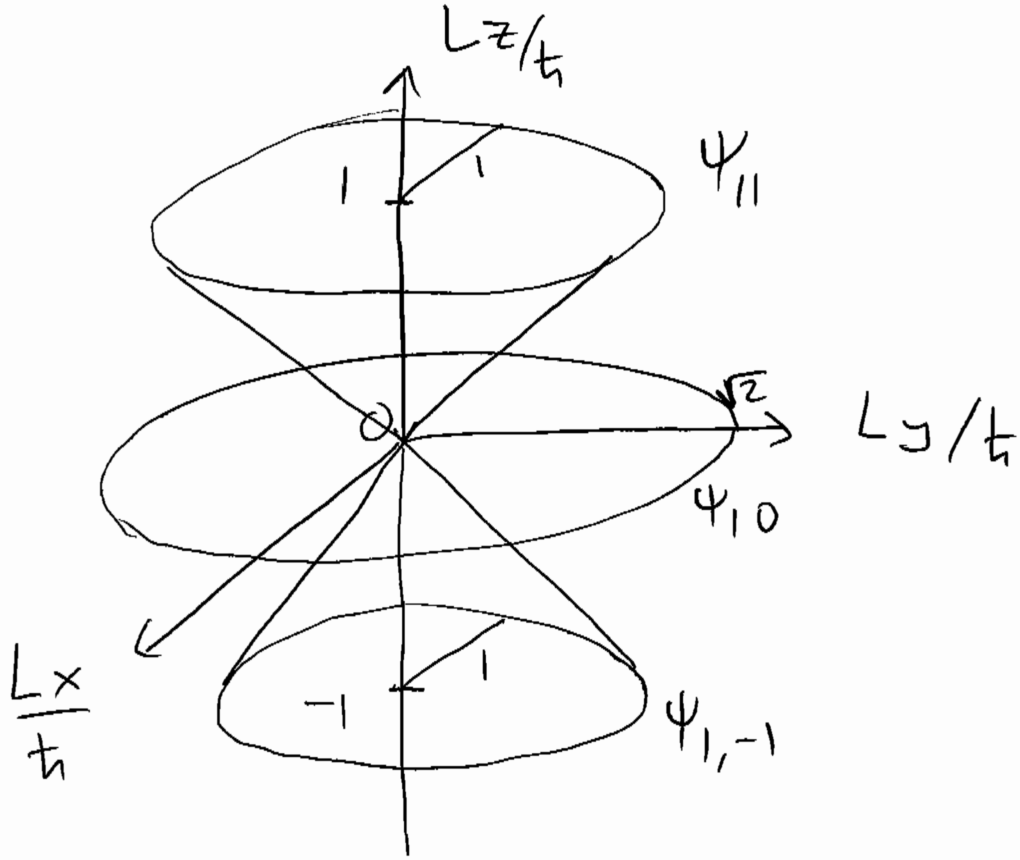
$$\Delta L_x \Delta L_y \geq \frac{\hbar}{2}|\langle L_z \rangle|.$$

Example 1 $\quad \ell = 1, \quad m = -1, 0, 1$

$$\ell(\ell+1) = 2$$

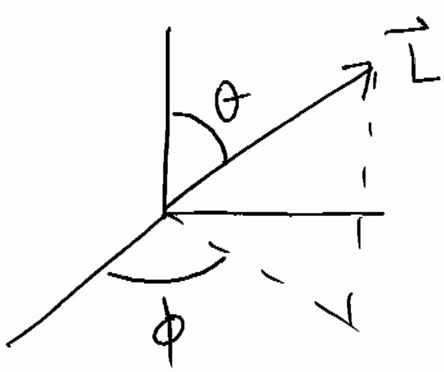
$$(L_x^2+L_y^2)\Psi_{1,\pm1} = \hbar^2(2-1)\Psi_{1,\pm1}$$
$$= \hbar^2\,\Psi_{1,\pm1}$$

$$(L_x^2+L_y^2)\Psi_{10} = 2\hbar^2\,\Psi_{10}$$



$m=0$: $\vec{L}$ lies in x-y plane

$m=\pm1$: $\vec{L}$ lies on upper (lower) cone



θ known

φ uncertain

Example 2 $\quad l = 0 \;,\; m = 0$

$$L_x = L_y = L_z = 0$$

Example 3 $\quad l = \frac{1}{2} \equiv s \quad$ (spin)

$$m = -\frac{1}{2}, \frac{1}{2} \;, \quad 2s+1 = 2$$

$$\vec{S}^2 = \hbar^2 s(s+1) = \frac{3}{4}\hbar^2$$

$$\left(S_x^2 + S_y^2\right)\psi_{\frac{1}{2},\pm\frac{1}{2}} = \left(\vec{S}^2 - S_z^2\right)\psi_{\frac{1}{2},\pm\frac{1}{2}}$$

$$= \hbar^2 \frac{1}{2}\, \psi_{\frac{1}{2},\pm\frac{1}{2}}$$