Lecture 26 Physics 371

The angular momentum matrices

$$\vec{L}^2 Y_{\ell m} = \hbar^2 \ell(\ell+1) Y_{\ell m}$$

$$L_z Y_{\ell m} = \hbar m Y_{\ell m}$$

$$\Rightarrow \langle \ell' m' | \vec{L}^2 | \ell m \rangle = \hbar^2 \ell(\ell+1) \delta_{\ell\ell'} \delta_{mm'}$$

$$\langle \ell' m' | L_z | \ell m \rangle = \hbar m \delta_{\ell\ell'} \delta_{mm'}$$

If we order the basis functions as follows, $Y_{00}, Y_{11}, Y_{10}, Y_{1-1}, Y_{22}, Y_{21}, \cdots$ then we can write the matrix representation of $\vec{L}^2$ as:

$$
(\vec{L}^2) = \hbar^2 \begin{pmatrix}
0 & & & & & & & & & \\
& 2 & & & & & & & & \\
& & 2 & & & & & & & \\
& & & 2 & & & & & & \\
& & & & 6 & & & & & \\
& & & & & 6 & & & & \\
& & & & & & 6 & & & \\
& & & & & & & 6 & & \\
& & & & & & & & 6 & \\
& & & & & & & & & 12 \\
& & & & & & & & & & \ddots
\end{pmatrix},
$$

where all the off-diagonal elements are zero. The boxes correspond to the $2l+1$ functions associated with the same eigenvalue of $\vec{L}^2$. Similarly,

$$(L_z) = \hbar \begin{pmatrix} 0 &&&&&&&&&& \\ & 1 &&&&&&&&& \\ && 0 &&&&&&&& \\ &&& -1 &&&&&&& \\ &&&& 2 &&&&&& \\ &&&&& 1 &&&&& \\ &&&&&& 0 &&&& \\ &&&&&&& -1 &&& \\ &&&&&&&& -2 && \\ &&&&&&&&& 3 & \\ &&&&&&&&&& \ddots \end{pmatrix}$$

Although these matrices are infinite dimensional, each set of $2l+1$ functions associated with a given value of l forms a <u>subspace</u> of dimension $2l+1$.

- Furthermore, the $2l+1$ spherical harmonics Y_{lm}, $m=-l,\dots,l$

form a complete, orthonormal basis in that subspace.

The operators $L_\pm$, L_x, and L_y can also be represented by matrices:

$$L_\pm Y_{lm} = \hbar\sqrt{l(l+1) - m(m\pm 1)}\; Y_{l,m\pm 1}$$

$$\langle l'm' | L_+ | lm\rangle = \hbar\sqrt{l(l+1) - m(m+1)}\; \delta_{ll'}\,\delta_{m',m+1}$$

$$\langle l'm' | L_- | lm\rangle = \hbar\sqrt{l(l+1) - m(m-1)}\; \delta_{ll'}\,\delta_{m',m-1}$$

In the subspace with $l=1$, $L_\pm$ are 3×3 matrices

$$L_+ = \hbar\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad L_- = \hbar\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

In this matrix representation, the eigenfunctions $Y_{\ell m}$ become column vectors with $2\ell+1$ components

$$Y_{11} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad Y_{10} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad Y_{1,-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$L_+ Y_{11} = 0 \qquad L_+ Y_{10} = \hbar\sqrt{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \hbar\sqrt{2}\, Y_{11}$$

$$L_+ Y_{1,-1} = \hbar\sqrt{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \hbar\sqrt{2}\, Y_{10}, \text{ etc.}$$

In the subspace with $\ell = 1$, the matrices for L_x and L_y are

$$(L_x) = \frac{1}{2}(L_+) + \frac{1}{2}(L_-) = \hbar \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$(L_y) = \frac{1}{2i}(L_+) - \frac{1}{2i}(L_-) = \hbar \begin{pmatrix} 0 & \frac{-i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

Spin-1/2

It is also possible to construct matrix representations for half-odd integer values of the angular momentum quantum #, although there are no spherical harmonics in that case.

The case where the total angular momentum quantum # is 1/2 is of particular interest, since it represents the intrinsic angular momentum, or spin, of many elementary particles, such as electrons, quarks, etc. (also protons and neutrons). For spin, the angular momentum quantum # is denoted by s:

$$\vec{S}^2 \psi_{sm} = \hbar^2 s(s+1)\psi_{sm}$$

$$S_z \psi_{sm} = \hbar m \psi_{sm},$$

where ψ_{sm} is a spin wavefunction to be determined.

$\underline{S = 1/2}$ $\qquad s(s+1) = \frac{1}{2}\left(\frac{3}{2}\right) = \frac{3}{4}$

$m = \pm 1/2$

Let $\psi_{\frac{1}{2},\frac{1}{2}} = \psi_{\uparrow}$, $\psi_{\frac{1}{2},-\frac{1}{2}} = \psi_{\downarrow}$.

$$S_z \psi_{\uparrow} = \frac{\hbar}{2}\psi_{\uparrow}$$

$$S_z \psi_{\downarrow} = -\frac{\hbar}{2}\psi_{\downarrow}$$

$$S_+ \psi_{\downarrow} = \hbar\sqrt{s(s+1) - m(m+1)}\;\psi_{\uparrow}$$

$$= \hbar\sqrt{\frac{3}{4} - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)}\;\psi_{\uparrow}$$

$$= \hbar\;\psi_{\uparrow}$$

Similarly,

$$S_- \psi_{\uparrow} = \hbar\psi_{\downarrow}$$

Matrix representations

$$(\vec{S}^2) = \hbar^2 \begin{pmatrix} 3/4 & 0 \\ 0 & 3/4 \end{pmatrix}$$

$$(S_z) = \hbar \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

$$(S_+) = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$(S_-) = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\psi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \psi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$(S_x) = \frac{1}{2}(S_+) + \frac{1}{2}(S_-)$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(S_y) = \frac{1}{2i}(S_+) - \frac{1}{2i}(S_-)$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

<u>The Pauli matrices</u> are defined by $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Properties: $\sigma_i^2 = \mathbb{1}$

$[\sigma_x, \sigma_y] = 2i\,\sigma_z$ (and cyclic permutations)

$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}\,\mathbb{1}$

(anticommutation)

Eigenvectors of σ_x

$$\sigma_x \psi = \lambda \psi$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$$0 = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1$$

$\Rightarrow \lambda = \pm 1$ (eigenvalues)

$$\begin{pmatrix} \mp 1 & 1 \\ 1 & \mp 1 \end{pmatrix} \begin{pmatrix} a_\pm \\ b_\pm \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mp a_\pm + b_\pm = 0 \qquad b_\pm = \pm a_\pm$$

$$\psi_{S_x = \pm \hbar/2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(\psi_\uparrow \pm \psi_\downarrow \right)$$