Physics 371 Lecture 28

Motion in a central potential

1) Two-body problem

$$\hat{H} = \frac{\vec{P}_1^{\,2}}{2m_1} + \frac{\vec{P}_2^{\,2}}{2m_2} + V(|\vec{r}_1 - \vec{r}_2|)$$

- Define $M = m_1 + m_2$ total mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{reduced mass}$$

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M} \qquad \vec{r} = \vec{r}_1 - \vec{r}_2$$

C.M. coord. relative coord.

- $\vec{P} = \vec{P}_1 + \vec{P}_2$ $\qquad \vec{p} = \mu \frac{\vec{P}_1}{m_1} - \mu \frac{\vec{P}_2}{m_2}$

C.M. momentum relative momentum

- $$\hat{H} = \frac{\vec{P}^{\,2}}{2M} + \frac{\vec{p}^{\,2}}{2\mu} + V(|\vec{r}|)$$

$\Rightarrow$ separable

C.M. motion is that of free particle of mass M. Relative motion is that of particle of mass μ moving in central potential $V(r)$:

- 2) $$\hat{H}_{rel} = \frac{\vec{p}^{\,2}}{2\mu} + V(r) = -\frac{\hbar^2 \nabla^2}{2\mu} + V(r)$$

(drop subscript "rel" from now on.)

$\hat{H}$ is invariant under rotations about the point $\vec{r} = 0$. This suggests that we utilize

- spherical polar coordinates.

Classically,

- $$\vec{L}^2 = (\vec{r}\times\vec{p})^2 = \vec{r}^2\,\vec{p}^2 - (\vec{r}\cdot\vec{p})^2$$

so
$$\vec{p}^2 = \frac{1}{r^2}(\vec{r}\cdot\vec{p})^2 + \frac{\vec{L}^2}{r^2}$$
$$= p_r^2 + \frac{\vec{L}^2}{r^2},$$

- where $p_r = \frac{1}{r}\vec{r}\cdot\vec{p}$.

However, quantum mechanically

$$p_r^{\dagger} = \vec{p}\cdot\vec{r}\,\frac{1}{r} \neq p_r .$$

We must find the appropriate Hermitian operator for p_r^2.

- One might try $p_r = \frac{\hbar}{i}\frac{\partial}{\partial r}$, but p_r^2, so defined does not give the correct radial part of the Laplacian in spherical polar coordinates.

It turns out that the correct Hermitian operator is

$$p_r^2 = -\frac{\hbar^2}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}$$

$$\hat{H} = -\frac{\hbar^2 \nabla^2}{2\mu} + V(r)$$

$$= -\frac{\hbar^2}{2\mu r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\vec{L}^2}{2\mu r^2} + V(r)$$

- Since $[\vec{L}, r] = 0$, $[\vec{L}, \hat{H}] = 0$.

$\Rightarrow$ energy eigenstates also ang. mom. eigenstates!

The fact that $[\hat{L}, \hat{H}] = 0$ is intimately connected to the invariance of $\hat{H}$ under rotations (see Goswami 12.2).

3) Time-independent Schrödinger eq.

$$\hat{H}\,\Psi(r,\theta,\phi) = E\,\Psi(r,\theta,\phi)$$

Let $\Psi(r,\theta,\phi) = \psi(r)\,Y(\theta,\phi)$

$$-\frac{\hbar^2}{2\mu r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)Y + \frac{\psi(r)}{2\mu r^2}\,\hat{L}^2\,Y(\theta,\phi)$$
$$+V(r)\,\psi(r)\,Y(\theta,\phi) = E\,\psi\,Y$$

- $$\frac{1}{\Psi(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Psi}{\partial r}\right) + \frac{2\mu r^2}{\hbar^2}\left(E - V(r)\right)$$

$$= \frac{1}{Y(\theta,\phi)}\frac{\vec{L}^2}{\hbar^2}Y(\theta,\phi)$$

$$= \ell(\ell+1)$$

4) Radial equation

- $$0 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) + \frac{2\mu}{\hbar^2}\left(E - V(r) - \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right)\Psi$$

$$V_{eff}(r) = V(r) + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}$$

centrifugal barrier

- Let $u(r) = r\Psi(r)$

- $$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)\frac{u}{r} = \frac{d^2}{dr^2}\frac{u}{r} + \frac{2}{r}\frac{d}{dr}\frac{u}{r}$$
$$= \frac{1}{r}\frac{d^2u}{dr^2}$$

$$\Rightarrow 0 = \frac{d^2u}{dr^2} + \frac{2\mu}{\hbar^2}\left(E - V(r) - \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right)u$$

- For bound states, the boundary conditions are

$u(r) \to 0$ as $r \to 0$

$u(r) \to 0$ as $r \to \infty$

- Effective 1D problem:

$$\boxed{\frac{d^2u}{dr^2} + \frac{2\mu}{\hbar^2}\left(E - V_{eff}(r)\right)u = 0}$$