Physics 371 Lecture 34.

Schrödinger's equation for a charged particle in a (classical) E & M field

Maxwell's equations: (cgs units)

$$\nabla \cdot \vec{B} = 0 \qquad (1)$$

$$\nabla \cdot \vec{E} = 4\pi \rho_e \qquad (2)$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J}_e + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \qquad (3)$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \qquad (4)$$

Vector and scalar potentials:

$(1) \Rightarrow \quad \vec{B} = \nabla \times \vec{A}, \quad \vec{A}(\vec{r},t) =$ vector potential

$\vec{E} = -\nabla V - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \quad V =$ scalar potential

Force on a charged particle

- (classical):

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right).$$

Gauge invariance:

$$\vec{A}' = \vec{A} + \nabla f(\vec{r},t)$$

$$V' = V - \frac{1}{c}\frac{\partial f}{\partial t}$$

- $\vec{B}' = \vec{B} \qquad \vec{E}' = \vec{E}$

Similar symmetry in QM

$$\psi'(\vec{r},t) = e^{i\theta(\vec{r},t)}\,\psi(\vec{r},t)$$

$$\rho(\vec{r},t) = |\psi|^2 \quad \text{unchanged}$$

- But $\vec{p}\,\psi' = \underset{e^{i\theta}}{\wedge}\left(\vec{p} + \hbar\nabla\theta\right)\psi$

Let

$$\vec{p}\,' = \frac{\hbar}{i}\nabla - \hbar\nabla\theta$$

then $\vec{p}\,'\psi' = e^{i\theta}\,\vec{p}\,\psi$

$$\vec{J} = \mathrm{Re}\left\{\psi^* \frac{\vec{p}}{m}\psi\right\} = \mathrm{Re}\left\{(\psi')^* \frac{\vec{p}\,'}{m}\psi'\right\}$$

<u>Schrödinger equation</u>

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + qV(\vec{r},t)\,\psi$$

$$\psi = e^{-i\theta}\,\psi'$$

$$i\hbar\frac{\partial\psi'}{\partial t} + \hbar\frac{\partial\theta}{\partial t}\psi' = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \hbar\nabla\theta\right)^2\psi' + qV(\vec{r},t)\,\psi'$$

$$i\hbar\frac{\partial\psi'}{\partial t} = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \hbar\nabla\theta\right)^2\psi' + q\left(V - \frac{\hbar}{q}\frac{\partial\theta}{\partial t}\right)\psi'$$

Looks like a gauge transformation with

$$\frac{\hbar c}{q}\theta(\vec{r},t) = f(\vec{r},t)$$

$$\theta(\vec{r},t) = \frac{q}{\hbar c} f(\vec{r},t)$$

If we introduce the "kinetic momentum" operator

$\vec{P}_{kin} = \frac{\hbar}{i}\nabla - \frac{q}{c}\vec{A}$, then under a gauge transformation

$$\vec{P}'_{kin} = \vec{P}_{kin} - \frac{q}{c}\nabla f = \vec{P}_{kin} - \hbar\nabla\theta \quad \checkmark$$

The gauge-invariant form of Schrödinger's equation is

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m}\left(\frac{\hbar\nabla}{i} - \frac{q}{c}\vec{A}(\vec{r},t)\right)^2 \psi + qV(\vec{r},t)\,\psi$$

c.f. Classical Hamiltonian $\left(m\vec{v} = \vec{p} - \frac{q}{c}\vec{A}\right)$

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + qV$$

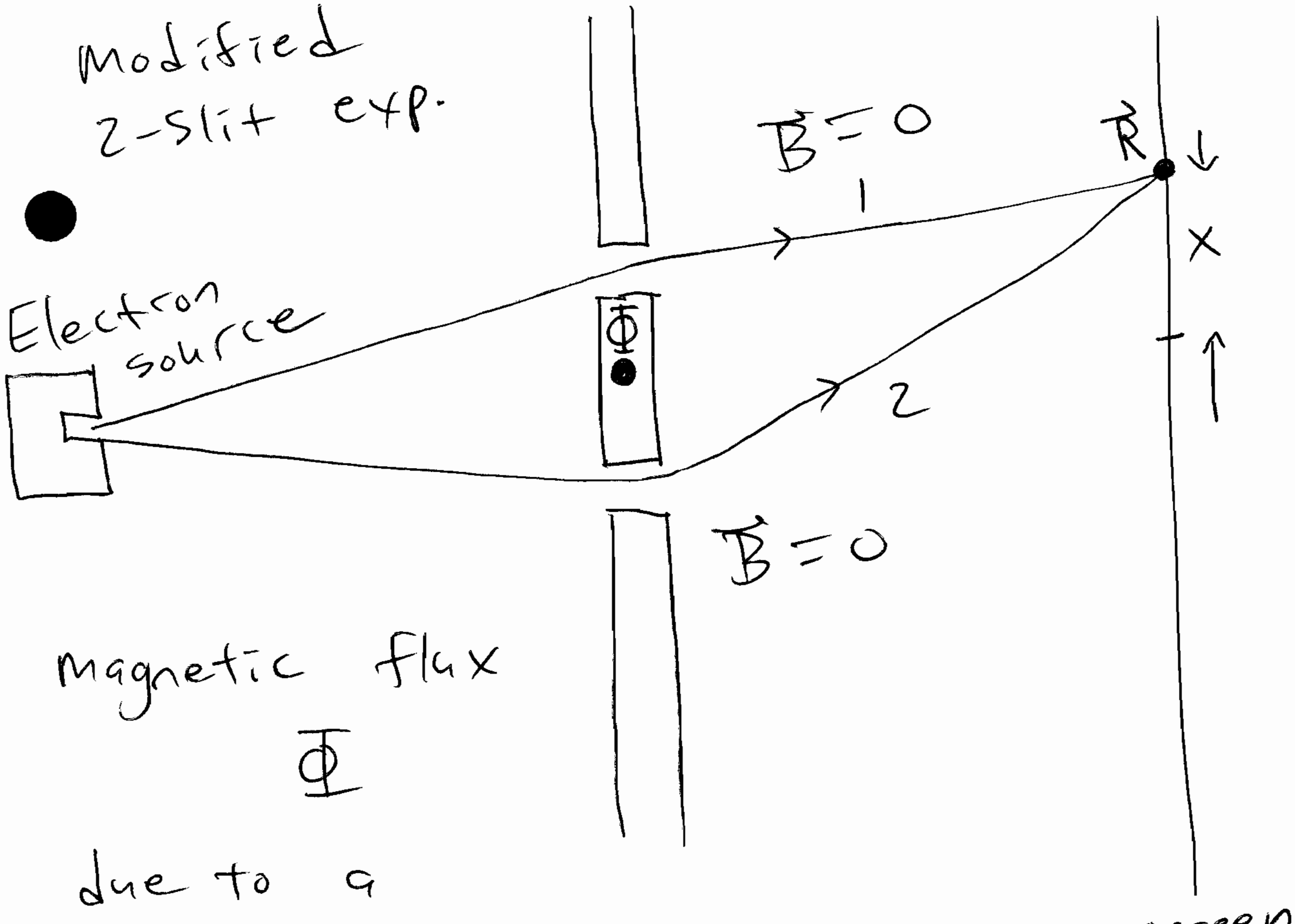
For a constant magnetic field $\vec{B}$

$$\vec{A} = -\frac{1}{2}\vec{r}\times\vec{B},$$

we can rederive the Zeemann effect (see Lecture 31).

More interesting still is the <u>Aharonov-Bohm effect</u>, which describes the QM effect of the vector potential, even when $\vec{B}=0$.

Modified 2-slit exp.



magnetic flux Φ due to a solenoid within central barrier.

$\vec{B}=0$ everywhere the electrons can propagate.

<u>Time-indep. Sch. eq.</u>

$$E\,\Psi = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{q}{c}\vec{A}\right)^2\Psi \qquad (V=0)$$

Outside barrier, $\vec{B}=0$, so $\vec{A}=\nabla f(\vec{r},t) = \frac{\hbar c}{q}\nabla\theta$

$$E\,\Psi = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \hbar\nabla\theta\right)^2\Psi$$

Let $\psi = e^{i\theta(r)}\psi'$.

ψ' satisfies free Sch. eq.

$$E\psi' = -\frac{\hbar^2\nabla^2}{2M}\psi'$$

Interference pattern

$$\rho(\vec{R}) = |\psi_1(\vec{R}) + \psi_2(\vec{R})|^2$$

$$= |\psi_1'(\vec{R})e^{i\theta_1} + \psi_2'(\vec{R})e^{i\theta_2}|^2$$

$$\psi_1'(\vec{R}) = Ce^{ikL_1}, \qquad \psi_2'(\vec{R}) = De^{ikL_2}$$

(suppose $C, D > 0$).

$$\boxed{\rho(\vec{R}) = C^2 + D^2 + 2CD\cos(k\Delta L + \theta_2 - \theta_1)}$$

$\Delta L = L_2 - L_1$

$$\theta_1 = \int_{L_1} \frac{g}{\hbar c}\vec{A}\cdot d\vec{l} \qquad \theta_2 = \int_{L_2} \frac{g}{\hbar c}\vec{A}\cdot d\vec{l}$$

$$\theta_2 - \theta_1 = \oint \frac{q}{\hbar c} \vec{A} \cdot d\vec{\ell}$$

$$= \frac{q}{\hbar c} \Phi = 2\pi \frac{\Phi}{\Phi_0}$$

$\Phi_0 = \frac{hc}{q} =$ "flux quantum"

Interference pattern is shifted by magnetic flux, even though $\vec{B} = 0$ everywhere the particle can propagate! Shift is periodic in Φ with period Φ_0.