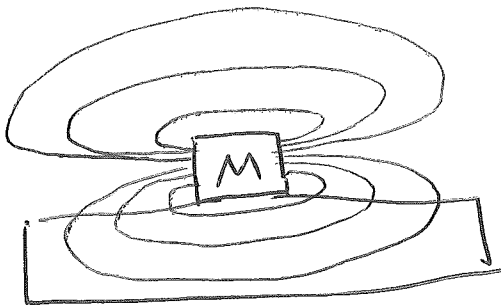


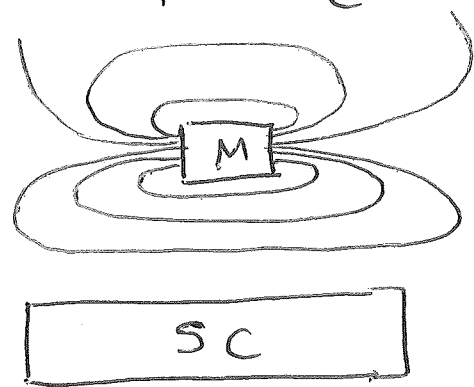
Meissner effect

$T > T_c$



magnetic field penetrates normal metal

$T < T_c$



magnetic field expelled from superconductor

Q: Can we understand the Meissner effect in terms of perfect conductivity?

Classically, 
$$m \frac{d\vec{v}}{dt} = q \vec{E}$$

(charged particle of mass  $m$  in an electric field)

$$\vec{J}_e = n_s q \langle \vec{v} \rangle$$

2

$$n_s = \frac{\# \text{ carriers}}{\text{volume}}$$

$q = \text{charge}$

$$n_s q \frac{d\langle \vec{v} \rangle}{dt} = \frac{n_s q^2}{m} \vec{E}$$

$$\frac{d\vec{J}_e}{dt} = \frac{n_s q^2}{m} \vec{E}$$

$$\frac{d}{dt} \nabla \times \vec{J}_e = \frac{n_s q^2}{m} \nabla \times \vec{E}$$

$$\text{But } \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad (\text{Faraday's law})$$

$$\Rightarrow \frac{d}{dt} \left[ \nabla \times \vec{J}_e + \frac{n_s q^2}{m c} \vec{B} \right] = 0$$

$$\nabla \times \vec{J}_e + \frac{n_s q^2}{m c} \vec{B} = \text{const.}$$

Currents flow only on the surface

of a perfect conductor, so in order 3  
to force  $\vec{B} = 0$  in the interior, we  
must have the integration constant  
in the above equation = 0.

$$\nabla \times \vec{J}_e + \frac{n_s q^2}{m c} \vec{B} = 0$$

London  
equation

The London equation describes the  
Meissner effect. Perfect  
conductivity does not imply  
the Meissner effect, since a  
perfect conductor maintains a  
constant field in its interior,  
but this constant field  
need not be zero!

The Meissner effect is a quantum effect, which occurs in superconductors, but would not occur in a perfect classical conductor.

In order to see why the constant in the London equation is zero, we need to see how the Schrödinger equation is modified in the presence of an (electro)magnetic field:

Maxwell's equations: (cgs units)

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{E} = 4\pi \rho_e$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J}_e + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

# Vector and scalar potentials

5

$$\vec{B} = \nabla \times \vec{A}, \quad \vec{A}(\vec{r}, t) = \text{vector potential}$$

$$\vec{E} = -\nabla V - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}, \quad V = \text{scalar potential}$$

## Force on a charged particle

(classical):

$$\vec{F} = q \left( \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right)$$

Gauge invariance:

$$\vec{A}' = \vec{A} + \nabla f(\vec{r}, t)$$

$$V' = V - \frac{1}{c} \frac{\partial f}{\partial t}$$

$$\vec{B}' = \vec{B}, \quad \vec{E}' = \vec{E}$$

Similar symmetry in QM

$$\psi'(\vec{r}, t) = e^{i\theta(\vec{r}, t)} \psi(\vec{r}, t)$$

$$\rho(\vec{r}, t) = |\Psi|^2 \quad \text{unchanged}$$

(6)

$$\text{But } \vec{p} \Psi' = e^{i\theta} (\vec{p} + \hbar \nabla \theta) \Psi.$$

(Recall  $\vec{p} = \frac{\hbar}{i} \nabla =$  momentum operator.)

$$\text{Define } \vec{p}' = \frac{\hbar}{i} \nabla - \hbar \nabla \theta$$

$$\text{Then } \vec{p}' \Psi' = e^{i\theta} \vec{p} \Psi$$

$$\vec{J} = \text{Re} \left\{ \Psi^* \frac{\vec{p}}{m} \Psi \right\} = \text{Re} \left\{ (\Psi')^* \frac{\vec{p}'}{m} \Psi' \right\}$$

Physical observables (e.g.  $E$ ,  $\vec{J}$ ) are unchanged under the transformation

$$\Psi \rightarrow e^{i\theta(\vec{r}, t)} \Psi$$

$$\vec{p} \rightarrow \vec{p} - \hbar \nabla \theta$$

# Schrödinger equation

7

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + gV(\vec{r}, t) \psi$$

$$\psi = e^{-i\theta} \psi'$$

$$i\hbar \frac{\partial \psi'}{\partial t} + \hbar \frac{\partial \theta}{\partial t} \psi' = \frac{1}{2m} \left( \frac{\hbar}{i} \nabla - \hbar \nabla \theta \right)^2 \psi' + gV(\vec{r}, t) \psi'$$

$$i\hbar \frac{\partial \psi'}{\partial t} = \frac{1}{2m} \left( \frac{\hbar}{i} \nabla - \hbar \nabla \theta \right)^2 \psi' + g \left( V - \frac{\hbar}{g} \frac{\partial \theta}{\partial t} \right) \psi'$$

Looks like a gauge transformation with

$$f(\vec{r}, t) = \frac{\hbar c}{g} \theta(\vec{r}, t)$$

If we introduce the "kinetic momentum"

$$\vec{p}_{\text{kin}} = \frac{\hbar}{i} \nabla - \frac{g}{c} \vec{A},$$

then under a gauge transformation 8

$$\vec{p}'_{kin} = \vec{p}_{kin} - \frac{q}{c} \nabla f = \vec{p}_{kin} - \hbar \nabla \theta \quad \checkmark$$

The gauge-invariant form of Schrödinger's equation is

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m} \left( \frac{\hbar}{i} \nabla - \frac{q}{c} \vec{A}(\vec{r}, t) \right)^2 \psi + q V(\vec{r}, t) \psi$$

cf. Classical Hamiltonian:

$$H = \frac{1}{2m} \left( \vec{p} - \frac{q}{c} \vec{A} \right)^2 + q V,$$

$$m\vec{v} = \vec{p} - \frac{q}{c} \vec{A}$$

$\vec{p}$  = "canonical momentum"



Quantum mechanically, the electric current is

9

$$\vec{J}_e = q \operatorname{Re} \left\{ \psi^* \vec{\nabla} \psi \right\}$$

$$\equiv \frac{q}{m} \operatorname{Re} \left\{ \psi^* \left( \vec{p} - \frac{q}{c} \vec{A} \right) \psi \right\}$$

$$\vec{J}_e = \frac{q}{m} \operatorname{Re} \left\{ \psi^* \left( \frac{\hbar}{i} \vec{\nabla} - \frac{q}{c} \vec{A} \right) \psi \right\}$$

In a superconductor, the charge carriers condense into a single, macroscopic wave function

$$\psi_s(\vec{r}) = \sqrt{n_s(\vec{r})} e^{i\theta(\vec{r})}$$

$$n_s(\vec{r}) = |\psi_s(\vec{r})|^2$$

$$\Rightarrow \vec{J}_e = \frac{n_s q}{m} \left( \hbar \vec{\nabla} \theta - \frac{q}{c} \vec{A} \right)$$

$$\nabla \times \vec{J}_e = -\frac{n_s e^2}{m c} \nabla \times \vec{A}$$

10

$$\nabla \times \vec{J}_e + \frac{n_s e^2}{m c} \vec{B} = 0 \quad !$$

London equation follows trivially from QM def. of current, provided all carriers are in the same wavefunction  $\psi_s(\vec{r})$ .

Q: If electrons are fermions, and must obey the Pauli exclusion principle, how can they all have the same wave function in a superconductor?!

A: They don't! The charge

Carriers in a SC are

"Cooper pairs" of electrons,

with charge  $q = -2e$

and mass  $m = 2m_e$ . Such

pairs of electrons are bosons,

and can condense into a

single "ground state" wave

function.

### Penetration depth

At the surface of a SC, the

currents which screen out

magnetic fields from the

interior flow, and the magnetic

field can penetrate a short distance.

Combining the London

12

equation  $\nabla \times \vec{J}_e = -\frac{n_s q^2}{mc} \vec{B}$

and Ampere's law (for static fields)

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J}_e \quad \text{gives}$$

$$\nabla \times (\nabla \times \vec{B}) = -\nabla^2 \vec{B} = \frac{4\pi}{c} \nabla \times \vec{J}_e$$

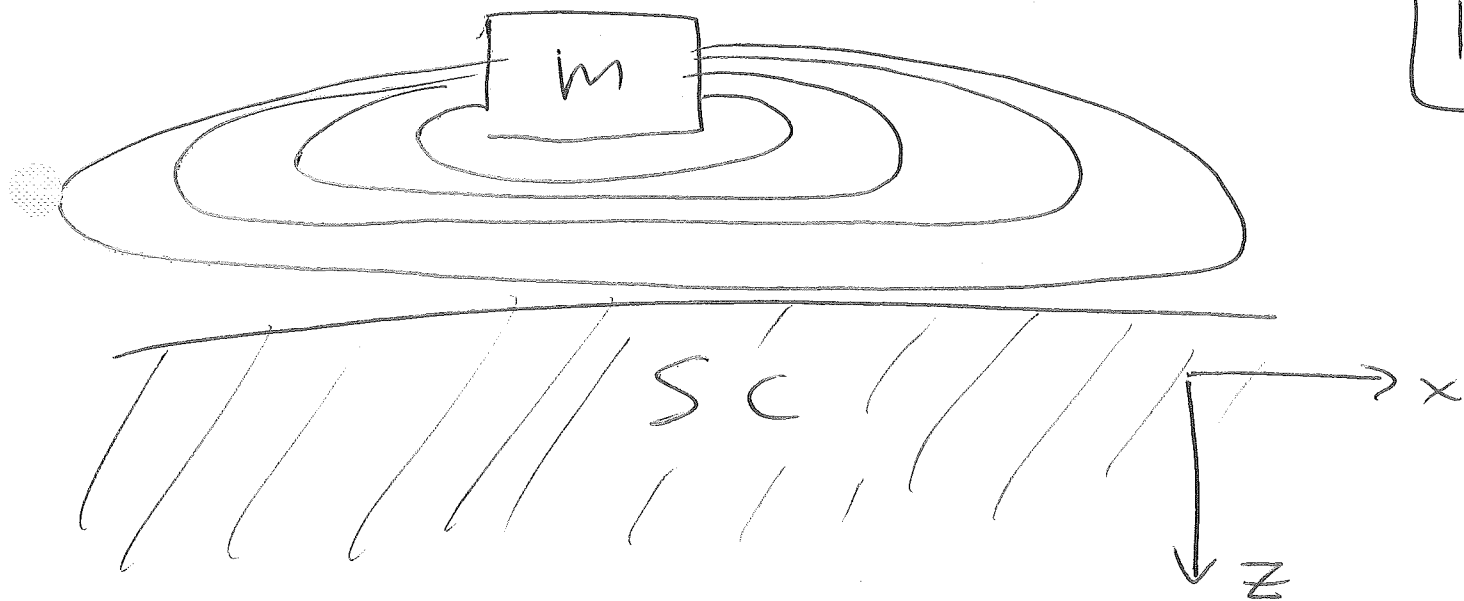
$$\Rightarrow \nabla^2 \vec{B} = \frac{4\pi n_s q^2}{mc^2} \vec{B}$$

Define the London penetration

depth

$$\lambda_L = \sqrt{\frac{mc^2}{4\pi n_s q^2}}$$

$$\nabla^2 \vec{B} = \lambda_L^{-2} \vec{B}$$



Neglecting edge effects,

$$\vec{B}(\vec{r}) = \hat{x} B_x(z)$$

$$\frac{d^2 B_x}{dz^2} = \frac{1}{\lambda_L^2} B_x(z)$$

$$\text{Sol'n: } B_x(z) = B_x(0) e^{-z/\lambda_L}$$

⇒ Field penetrates exponentially,  
with a decay length  $\lambda_L$ .