Extra Credit Problem solutions

$$\psi = \alpha \psi_n + \beta \psi_{n+1}$$

a) $$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \psi | a + a^\dagger | \psi \rangle$$

$$\langle \psi | a | \psi \rangle = \langle \psi | a^\dagger | \psi \rangle^* = \langle \alpha \psi_n + \beta \psi_{n+1} | a | \alpha \psi_n + \beta \psi_{n+1} \rangle$$

$$= \alpha^* \beta \langle n | a | n+1 \rangle = \alpha^* \beta \sqrt{n+1}$$

$$\boxed{\langle x \rangle = \sqrt{\frac{\hbar (n+1)}{2m\omega}} \left(\alpha^* \beta + \alpha \beta^* \right)}$$

$$\langle p_x \rangle = i \sqrt{\frac{m\hbar\omega}{2}} \langle \psi | a^\dagger - a | \psi \rangle$$

$$\boxed{\langle p_x \rangle = i \sqrt{\frac{m\hbar\omega(n+1)}{2}} \left(-\alpha^* \beta + \alpha \beta^* \right)}$$

b) $$\langle x^2 \rangle = \frac{\hbar}{2m\omega} \langle \psi | (a + a^\dagger)^2 | \psi \rangle = \frac{\hbar}{2m\omega} \langle \psi | 2a^\dagger a + 1 | \psi \rangle$$

$$= \frac{\hbar}{2m\omega} \left[|\alpha|^2 (2n+1) + |\beta|^2 (2n+3) \right]$$

$$\langle p_x^2 \rangle = -\frac{m\hbar\omega}{2} \langle \psi | (a^\dagger - a)^2 | \psi \rangle = \frac{m\hbar\omega}{2} \langle \psi | 2a^\dagger a + 1 | \psi \rangle$$

$$\langle p_x^2 \rangle = \frac{m\hbar\omega}{2}\left[|\alpha|^2(2n+1) + |\beta|^2(2n+3) \right]$$

$|\alpha|^2 + |\beta|^2 = 1$ from $\langle\psi|\psi\rangle = 1$,

so $|\alpha|^2(2n+1) + |\beta|^2(2n+3) = 2n+1 + 2|\beta|^2$

let $\alpha = |\alpha| e^{i\gamma}$ $\quad \beta = |\beta| e^{i\delta}$

$$(\alpha^*\beta + \alpha\beta^*) = 2|\alpha||\beta|\cos(\gamma - \delta)$$

$$(\alpha^*\beta - \beta^*\alpha) = -2i|\alpha||\beta|\sin(\gamma-\delta)$$

$$(\Delta x)^2 = \frac{\hbar}{2m\omega}\left(2n+1+2|\beta|^2 - (n+1)(\alpha^*\beta + \alpha\beta^*)^2\right)$$

$$\boxed{(\Delta x)^2 = \frac{\hbar}{2m\omega}\left(2n+1+2|\beta|^2 - 4(n+1)|\alpha|^2|\beta|^2\cos^2(\gamma-\delta)\right)}$$

$$(\Delta p_x)^2 = \frac{m\hbar\omega}{2}\left(2n+1+2|\beta|^2 + (n+1)(\alpha^*\beta - \alpha\beta^*)^2\right)$$

$$\boxed{(\Delta p_x)^2 = \frac{m\hbar\omega}{2}\left(2n+1+2|\beta|^2 - 4(n+1)|\alpha|^2|\beta|^2\sin^2(\gamma-\delta)\right)}$$

maximum of $\Delta x \Delta p_x$ product occurs when $\cos^2(\gamma-\delta) = \sin^2(\gamma-\delta) = 1/2$.

minimum of $\Delta x \Delta p_x$ product occurs when $\cos^2(\gamma-\delta) = 1$, $\sin^2(\gamma-\delta) = 0$, or vice versa.

$$(\Delta x)^2 (\Delta p_x)^2 \geq \left(\frac{\hbar}{2}\right)^2 \Big((2n+1+2|\beta|^2) \times (2n+1+2|\beta|^2 - 4(n+1)|\alpha|^2|\beta|^2) \Big)$$

$$\geq \left(\frac{\hbar}{2}\right)^2 (2n+1)^2 \quad (\beta \to 0 \text{ limit})$$

$$\boxed{\Delta x \Delta p_x \geq \frac{\hbar}{2}(2n+1) \geq \frac{\hbar}{2}}$$

c) $\alpha = \beta = 1/\sqrt{2}$

$$\psi = \frac{1}{\sqrt{2}} e^{-\frac{iE_n t}{\hbar}} \left(\psi_n + \psi_{n+1} e^{-i\omega t} \right)$$

$$\langle \psi | a | \psi \rangle = \frac{1}{2} \sqrt{n+1}\, e^{-i\omega t}$$

$$\langle x(t)\rangle = \sqrt{\frac{\hbar(n+1)}{2m\omega}}\cos(\omega t)$$

$$\langle p_x(t)\rangle = i\sqrt{\frac{m\hbar\omega(n+1)}{2}}\left(-\frac{1}{2}e^{-i\omega t} + \frac{1}{2}e^{i\omega t}\right)$$

$$= -\sqrt{\frac{m\hbar\omega(n+1)}{2}}\sin(\omega t)$$

<u>Ehrenfest:</u>

$$m\frac{d\langle x(t)\rangle}{dt} = \langle p_x(t)\rangle$$

$$m\frac{d}{dt}\langle x(t)\rangle = -\sqrt{\frac{\hbar m\omega(n+1)}{2}}\sin(\omega t) \checkmark$$