Phys. 371 Midterm 2 solutions

1) Energy eigenfunctions / eigenvalues:

$$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \qquad E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2},$$

$$n = 1, 2, 3, \ldots, \infty$$

$$\Psi(x, t=0) = \sqrt{\frac{1}{3}}\, \psi_1(x) + \sqrt{\frac{2}{3}}\, \psi_2(x)$$

a)

$$\therefore \boxed{\Psi(x,t) = \sqrt{\frac{1}{3}}\, \psi_1(x)\, e^{-\frac{iE_1 t}{\hbar}} + \sqrt{\frac{2}{3}}\, \psi_2(x)\, e^{-\frac{iE_2 t}{\hbar}}}$$

$$\boxed{\rho(x,t) = |\Psi(x,t)|^2 = \frac{1}{3}|\psi_1(x)|^2 + \frac{2}{3}|\psi_2(x)|^2 + \frac{2\sqrt{2}}{3}\psi_1(x)\psi_2(x)\cos\left(\frac{E_2 - E_1}{\hbar}\, t\right)}$$

b) $E = E_1$ or $E = E_2$

$P(E_1) = 1/3 \qquad \psi \to \psi_1(x)$ after measurement

$P(E_2) = 2/3 \qquad \psi \to \psi_2(x)$ " "

Results do not depend on t.

2) a) $$\Delta p_x \, \Delta V(x) \geq \frac{1}{2} \left| \langle [p_x, V(x)] \rangle \right|$$

$$[p_x, V(x)] = \frac{\hbar}{i} \frac{\partial V}{\partial x}$$

$$\boxed{\Delta p_x \, \Delta V(x) \geq \frac{\hbar}{2} \left| \left\langle \frac{\partial V}{\partial x} \right\rangle \right|}$$

b) $$\frac{d}{dt} \langle V(x) \rangle = \frac{1}{i\hbar} \langle [V(x), H] \rangle$$

$$[V(x), H] = \frac{1}{2m} [V(x), p_x^2]$$

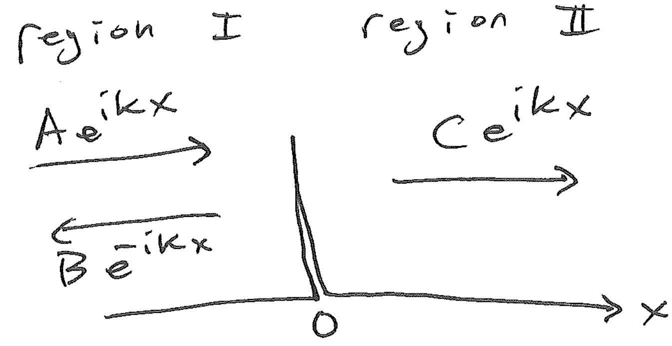
$$= \frac{p_x}{2m} [V(x), p_x] + \frac{1}{2m} [V(x), p_x] p_x$$

$$[V(x), p_x] = i\hbar \frac{\partial V}{\partial x}$$ so

$$[V(x), H] = \frac{i\hbar}{2m} \left(p_x \frac{\partial V}{\partial x} + \frac{\partial V}{\partial x} p_x \right)$$

$$\boxed{\frac{d}{dt} \langle V(x) \rangle = \frac{1}{2m} \left\langle p_x \frac{\partial V}{\partial x} + \frac{\partial V}{\partial x} p_x \right\rangle}$$

3)



$$j_{in} = \frac{\hbar k}{m}|A|^2 \qquad j_{tr} = \frac{\hbar k}{m}|C|^2$$

Trans. probability

$$T = \frac{|j_{tr}|}{|j_{in}|} = \frac{|C|^2}{|A|^2}$$

$$\psi_I(x) = Ae^{ikx} + Be^{-ikx}$$

$$\psi_{II}(x) = Ce^{ikx}$$

$$H\psi_I = E\psi_I \implies E = \frac{\hbar^2 k^2}{2m}$$

i) $\psi_I(0) = \psi_{II}(0)$

$$A + B = C$$

ii) $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \Lambda\delta(x)\psi(x) = E\psi(x)$

$$-\frac{\hbar^2}{2m}\left[\psi_{II}'(0) - \psi_I'(0)\right] + \Lambda\psi_{II}(0) = 0$$

$$ikC - ik(A-B) = \frac{2m\Lambda}{\hbar^2}C$$

$$C - A + B = \frac{2m\Lambda}{i\hbar^2 k} C$$

$$A - B = \left(1 + \frac{2im\Lambda}{\hbar^2 k}\right) C$$

$$\boxed{\text{Let } \frac{\hbar^2}{m\Lambda} = l}$$

ii) $A - B = \left(1 + \frac{2i}{kl}\right) C$

Adding eqs. (i) and (ii) gives

$$2A = \left(2 + \frac{2i}{kl}\right) C$$

$$\frac{C}{A} = \frac{1}{1 + \frac{i}{kl}} = \frac{kl}{kl + i}$$

$$\boxed{T = \frac{|C|^2}{|A|^2} = \frac{(kl)^2}{1 + (kl)^2}} = \boxed{\frac{E}{\frac{m\Lambda^2}{2\hbar^2} + E}}$$