Phys 371 Midterm 3
Solutions

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a^\dagger + a), \qquad p_x = i\sqrt{\frac{m\hbar\omega}{2}}\,(a^\dagger - a)$$

a)
$$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\langle \psi | a^\dagger + a | \psi \rangle$$
$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\,\langle \psi_0 - \psi_1 | a^\dagger + a | \psi_0 - \psi_1 \rangle$$
$$= -\frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(\underbrace{\langle \psi_0 | a | \psi_1 \rangle}_{1} + \underbrace{\langle \psi_1 | a^\dagger | \psi_0 \rangle}_{1}\right)$$
$$= -\sqrt{\frac{\hbar}{2m\omega}}$$

b)
$$\langle p_x \rangle = i\sqrt{\frac{m\hbar\omega}{2}}\,\langle \psi | a^\dagger - a | \psi \rangle$$
$$= \frac{i}{2}\sqrt{\frac{m\hbar\omega}{2}}\,\langle \psi_0 - \psi_1 | a^\dagger - a | \psi_0 - \psi_1 \rangle$$
$$= \frac{i}{2}\sqrt{\frac{m\hbar\omega}{2}}\left(\underbrace{\langle \psi_0 | a | \psi_1 \rangle}_{1} - \underbrace{\langle \psi_1 | a^\dagger | \psi_0 \rangle}_{1}\right)$$
$$= 0$$

c)
$$\psi(t) = \frac{1}{\sqrt{2}}\left(\psi_0 - \psi_1 e^{-i\omega t}\right) e^{-i\frac{\omega}{2}t}$$

from $\psi_n(t) = \psi_n(0)\, e^{-\frac{iE_n t}{\hbar}}, \quad E_n = \hbar\omega(n + \frac{1}{2})$

$$\langle x(t)\rangle = -\frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(\langle\psi_0|a|\psi_1\rangle e^{-i\omega t} + \langle\psi_1|a^\dagger|\psi_0\rangle e^{i\omega t}\right)$$

$$= -\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)$$

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2) a) $\Delta A\,\Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

so $\Delta L_z\,\Delta L_x \geq \frac{1}{2}|\langle[L_z,L_x]\rangle|$

$$[L_z, L_x] = [xp_y - yp_x,\ yp_z - zp_y]$$
$$= [xp_y, yp_z] + [yp_x, zp_y]$$
$$= x[p_y, y]p_z + z[y, p_y]p_x$$
$$= -i\hbar\, xp_z + i\hbar\, zp_x$$
$$= i\hbar(zp_x - xp_z) = i\hbar L_y$$

$\therefore$ $\Delta L_z\,\Delta L_x \geq \frac{1}{2}|\langle L_y\rangle|$

b) $[L_x, \vec{L}^2] = [L_x, L_x^2 + L_y^2 + L_z^2]$

$$= [L_x, L_y^2] + [L_x, L_z^2]$$
$$= L_y[L_x,L_y] + [L_x,L_y]L_y$$
$$+ L_z[L_x,L_z] + [L_x,L_z]L_z$$

$[L_x, L_z] = -i\hbar L_y$ from part (a)

$[L_x, L_y] = i\hbar L_z$ by symmetry

$$\Rightarrow [L_x, \vec{L}^2] = i\hbar(L_y L_z + L_z L_y)$$
$$-i\hbar(L_z L_y + L_y L_z) = 0 \quad \checkmark$$

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3) a) $S_x \psi_\lambda = \lambda \psi_\lambda$, let $\psi_\lambda = \begin{pmatrix} a \\ b \end{pmatrix}$

$$(S_x - \lambda \mathbb{1})\psi_\lambda = 0$$

$$0 = |S_x - \lambda \mathbb{1}| = \begin{vmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{vmatrix} = \lambda^2 - \left(\frac{\hbar}{2}\right)^2$$

$$\Rightarrow \boxed{\lambda = \pm\frac{\hbar}{2}}$$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\begin{pmatrix} a \\ b \end{pmatrix}$$

$$b = \pm a$$

$$\boxed{\psi_{\pm\frac{\hbar}{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}}$$

b) Possible outcomes are

$$S_x = \pm \frac{\hbar}{2}.$$

$$C_+ = \langle + | \chi \rangle, \quad C_- = \langle - | \chi \rangle$$

$$C_\pm = \langle \pm | \chi \rangle = \frac{1}{\sqrt{2}} (1 \ \ \pm 1) \begin{pmatrix} a \\ b \end{pmatrix} = \frac{a \pm b}{\sqrt{2}}$$

$$P\left(S_x = \pm \frac{\hbar}{2}\right) = |C_\pm|^2 = \frac{|a \pm b|^2}{2}$$

$$P_+ + P_- = \frac{|a+b|^2}{2} + \frac{|a-b|^2}{2}$$

$$= \frac{(a^* + b^*)(a+b) + (a^* - b^*)(a-b)}{2}$$

$$= \frac{|a|^2 + |b|^2 + \cancel{a^* b} + \cancel{a b^*} + |a|^2 + |b|^2 \cancel{- a^* b} \cancel{- a b^*}}{2}$$

$$= |a|^2 + |b|^2 = \underline{1}$$

(assuming χ is normalized).