Physics 371 Midterm 1

Solutions to practice problems

1) a) $1 = \int_0^L dx\, |\psi(x)|^2 = A^2 \int_0^L dx\, x^2 (x-L)^2$

$$1 = A^2 \int_0^L dx\, (x^4 - 2Lx^3 + L^2 x^2)$$

$$= A^2 \left(\frac{L^5}{5} - \frac{2L^5}{4} + \frac{L^5}{3} \right) = \frac{A^2 L^5}{30}$$

$$A = \sqrt{30}\, L^{-5/2}$$

b) $\langle x \rangle = L/2$ since $|\psi(x)|^2$ is symmetric about $x = L/2$.

c) $\langle E \rangle = \int_0^L dx\, \psi^*(x) \left(-\frac{\hbar^2}{2m}\right) \frac{d^2}{dx^2} \psi(x)$

$$= A^2 \int_0^L dx\, x(L-x) \frac{\hbar^2}{m}$$

$$= A^2 \frac{\hbar^2}{m} \left(\frac{L^3}{2} - \frac{L^3}{3} \right) = \frac{A^2 \hbar^2 L^3}{6m}$$

$$= \frac{5\hbar^2}{mL^2} > \frac{\pi^2 \hbar^2}{2mL^2} = E_1$$

(but not by much!)

2) $$\langle E \rangle = \frac{\langle p_x^2 \rangle}{2m} + \frac{\alpha}{16}\langle x^4 \rangle$$

a)

$$\geq \frac{\langle p_x^2 \rangle}{2m} + \frac{\alpha}{16}\langle x^2 \rangle^2$$

By definition,

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$(\Delta p_x)^2 = \langle p_x^2 \rangle - \langle p_x \rangle^2$$

$$\Rightarrow \langle x^2 \rangle = (\Delta x)^2 + \langle x \rangle^2 \geq (\Delta x)^2$$

$$\langle p_x^2 \rangle = (\Delta p_x)^2 + \langle p_x \rangle^2 \geq (\Delta p_x)^2$$

$$\Rightarrow \langle E \rangle \geq \frac{(\Delta p_x)^2}{2m} + \frac{\alpha}{16}(\Delta x)^4$$ ✓

b) Energy would be lowered by decreasing Δp_x and/or Δx, but $\Delta x \Delta p_x \geq \hbar/2$. Best case scenario:

$$\Delta x \Delta p_x = \hbar/2$$

$$\Delta p_x = \frac{\hbar}{2\Delta x}$$

$$\langle E \rangle \geq \frac{\hbar^2}{8m(\Delta x)^2} + \frac{\alpha}{16}(\Delta x)^4$$

$$0 = \left.\frac{\partial \langle E \rangle}{\partial \Delta x}\right|_{E_{min}} = -\frac{\hbar^2}{4m(\Delta x)^3} + \frac{\alpha}{4}(\Delta x)^3$$

$$\Rightarrow \quad \Delta x^6 = \frac{\hbar^2}{m\alpha} \quad ; \quad \Delta x^2 = \left(\frac{\hbar^2}{m\alpha}\right)^{1/3}$$

$$\langle E \rangle \geq \frac{\hbar^2}{8m}\left(\frac{m\alpha}{\hbar^2}\right)^{1/3} + \frac{\alpha}{16}\left(\frac{\hbar^2}{m\alpha}\right)^{2/3}$$

$$E_{min} \sim \frac{1}{8}\left(\frac{\hbar^4\alpha}{m^2}\right)^{1/3} + \frac{1}{16}\left(\frac{\hbar^4\alpha^2}{m^2}\right)^{1/3}$$

$$\boxed{E_{min} \sim \frac{3}{16}\left(\frac{\hbar^4\alpha}{m^2}\right)^{1/3}}$$