Phys 371 Midterm 3
Solutions

1) a) $E \longleftrightarrow \hat{H} = \frac{P_x^2}{2m} + V(x)$

$$\Delta x \Delta E \geq \frac{1}{2}\left|\langle [x, H] \rangle\right|$$

$$[x, H] = \frac{1}{2m}[x, P_x^2] = \frac{1}{2m}\left[P_x [x, P_x] + [x, P_x] P_x\right]$$

$$= i\hbar \frac{P_x}{m}$$

$$\Delta x \Delta E \geq \frac{\hbar}{2m}\left|\langle P_x \rangle\right| \qquad \text{Q.E.D.}$$

b) $$\frac{d\langle x \rangle}{dt} = \frac{1}{i\hbar}\langle [x, H] \rangle + \cancel{\left\langle \frac{\partial x}{\partial t} \right\rangle} \rightarrow 0$$

$$= \frac{1}{m}\langle P_x \rangle$$

$$\Delta x \Delta E \geq \frac{\hbar}{2}\left|\frac{d\langle x \rangle}{dt}\right|$$

$$\frac{\Delta x}{\left|\frac{d\langle x \rangle}{dt}\right|} \Delta E \equiv \Delta t \Delta E \geq \frac{\hbar}{2} \qquad \text{Q.E.D.}$$

2) $x = \sqrt{\frac{\hbar}{2m\omega}}\,(a^+ + a)$

$$p_x = i\sqrt{\frac{m\hbar\omega}{2}}\,(a^+ - a)$$

a) $$\langle n|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\langle n|a^+|n\rangle + \langle n|a|n\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\,\underbrace{\cancel{\langle n|n+1\rangle}}_{0} + \sqrt{n}\,\underbrace{\cancel{\langle n|n-1\rangle}}_{0}\right)$$

$$= 0$$

$$\langle n|p_x|n\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(\underbrace{\cancel{\langle n|a^+|n\rangle}}_{0} - \underbrace{\cancel{\langle n|a|n\rangle}}_{0}\right) = 0$$

b) $$\langle n|x^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|(a^+)^2 + a^2 + a^+a + aa^+|n\rangle$$

$$= \frac{\hbar}{2m\omega}\langle n|2a^+a + 1|n\rangle = \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right)$$

b) $$\langle n|p_x^2|n\rangle = -\frac{m\hbar\omega}{2}\left(\langle n|(a^+)^2 + a^2 - a^+a - aa^+|n\rangle\right)$$

$$= \frac{m\hbar\omega}{2}\langle n|2a^+a + 1|n\rangle = m\hbar\omega\left(n + \frac{1}{2}\right)$$

c) $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)}$

$\Delta p_x = \sqrt{\langle p_x^2 \rangle - \langle p_x \rangle^2} = \sqrt{m\hbar\omega\left(n+\frac{1}{2}\right)}$

$\Delta x \, \Delta p_x = \hbar\left(n+\frac{1}{2}\right) \geq \frac{\hbar}{2}$

equality holds for $n=0$ (ground state).

→ min. uncertainty in g.s.

---

3) a) $L_x = y p_z - z p_y$

$L_y = z p_x - x p_z$

$L_z = x p_y - y p_x$

$[L_x, L_y] = [y p_z - z p_y, \; z p_x - x p_z]$

$= [y p_z, z p_x] + [z p_y, x p_z]$

$= y [p_z, z] p_x + x [z, p_z] p_y$

$= -i\hbar \, y p_x + i\hbar \, x p_y = i\hbar L_z$ Q.E.D.

b) $[\vec{L}^2, L_z] = [L_x^2 + L_y^2 + L_z^2, L_z]$

$$= [L_x^2, L_z] + [L_y^2, L_z]$$

$$= L_x[L_x, L_z] + [L_x, L_z]L_x$$

$$+ L_y[L_y, L_z] + [L_y, L_z]L_y$$

$$= -i\hbar \cancel{L_x L_y} - i\hbar \cancel{L_y L_x}$$

$$+ i\hbar \cancel{L_y L_x} + i\hbar \cancel{L_x L_y} = 0 \quad \text{Q.E.D.}$$

c) $\vec{L}^2 Y_{\ell m} = \hbar^2 \ell(\ell+1) Y_{\ell m}$

$$L_z Y_{\ell m} = \hbar m Y_{\ell m}$$

$\ell = 0, 1, 2, 3, \ldots, \infty$

For each ℓ, $m = -\ell, -\ell+1, \ldots, \ell$ ($2\ell+1$ values)

$$\cancel{\Delta x \, \Delta p_x = \left(n + \tfrac{1}{2}\right) \sqrt{\frac{\hbar}{m\omega}} \sqrt{m\hbar\omega}}$$

$$\cancel{= \left(n + \tfrac{1}{2}\right) \hbar \geq \frac{\hbar}{2}} \quad \checkmark$$

2

---

4) (a) $S_y = \frac{\hbar}{2} \sigma_y$

$$S_y \chi_\pm = \lambda_\pm \chi_\pm$$

$$\frac{\hbar}{2} \sigma_y \chi_\pm = \lambda_\pm \chi_\pm$$

$$\sigma_y \chi_\pm = \tilde{\lambda}_\pm \chi_\pm \, , \quad \lambda_\pm = \frac{\hbar}{2} \tilde{\lambda}_\pm$$

$$\det \begin{pmatrix} -\tilde{\lambda} & -i \\ i & -\tilde{\lambda} \end{pmatrix} = \tilde{\lambda}^2 + i^2 = \tilde{\lambda}^2 - 1 = 0$$

$$\tilde{\lambda}_\pm = \pm 1 \implies \boxed{\lambda_\pm = \pm \frac{\hbar}{2}}$$

$$\begin{pmatrix} \mp 1 & -i \\ i & \mp 1 \end{pmatrix} \begin{pmatrix} a_\pm \\ b_\pm \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mp a_{\pm} - i b_{\pm} = 0$$

$$i a_{\pm} \mp b_{\pm} = 0 \implies b_{\pm} = \pm i a_{\pm}$$

$$\boxed{\chi_{+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \chi_{-} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}}$$

(b) $$\begin{pmatrix} a \\ b \end{pmatrix} = c_{+}\chi_{+} + c_{-}\chi_{-}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} c_{+} + c_{-} \\ i(c_{+} - c_{-}) \end{pmatrix}$$

$$\sqrt{2}\, a = c_{+} + c_{-}$$

$$-i\sqrt{2}\, b = c_{+} - c_{-}$$

$$2c_{+} = \sqrt{2}(a - ib) \qquad c_{+} = \frac{a - ib}{\sqrt{2}}$$

$$2C_- = \sqrt{2}(a+ib)$$

$$C_- = \frac{a+ib}{\sqrt{2}}$$

$$\boxed{\begin{array}{l} \text{Possible outcomes are } S_y = \pm\frac{\hbar}{2}. \\ \\ P(S_y = \frac{\hbar}{2}) = |C_+|^2 = \frac{1}{2}|a-ib|^2 \\ \\ P(S_y = -\frac{\hbar}{2}) = |C_-|^2 = \frac{1}{2}|a+ib|^2 \end{array}}$$

$$|C_+|^2 + |C_-|^2 = \frac{1}{2}\left[(a-ib)(a^*+ib^*) + (a+ib)(a^*-ib^*)\right]$$

$$= \frac{1}{2}\left[|a|^2+|b|^2 \cancel{+iab^*} \cancel{-ia^*b} + |a|^2 + |b|^2 \cancel{+ia^*b} \cancel{-iab^*}\right]$$

$$= |a|^2 + |b|^2 = 1$$ if χ is normalized.