Physics 460 Solutions

Problem set # 1

a) The total # of states with energy $\leq \mathcal{E}$ is given by the area of a disk in phase space

$$N(\mathcal{E}) = 2\int_{|k| \leq \sqrt{\frac{2m\mathcal{E}}{\hbar^2}}} \left(\frac{L}{2\pi}\right)^2 d^2k \quad ,$$

where the prefactor is for spin.

$$N(\mathcal{E}) = \frac{A}{2\pi^2}\, \pi\, \frac{2m\mathcal{E}}{\hbar^2} = \frac{mA\mathcal{E}}{\pi\hbar^2}$$

$$D(\mathcal{E}) = \frac{dN}{d\mathcal{E}} = \frac{mA}{\pi\hbar^2}$$

At zero temperature,
The total # of particles
is

$$N = 2 \int_{|k| \leq k_F} \left(\frac{L}{2\pi}\right)^2 d^2k$$

$$= \frac{A}{2\pi^2} \pi k_F^2 = \frac{A k_F^2}{2\pi}$$

$$= \frac{mA}{\hbar^2 \pi} \varepsilon_F \qquad \left(\varepsilon_F = \frac{\pi \hbar^2}{m} \frac{N}{A} \right)$$

Thus $D(\varepsilon) = \dfrac{N}{\varepsilon_F}$

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b) $N = \int_0^\infty d\varepsilon \, D(\varepsilon) f(\varepsilon)$

$$\frac{N}{A} = \frac{m}{\pi \hbar^2} \int_0^\infty \frac{d\varepsilon}{e^{\beta(\varepsilon - \mu)} + 1}$$

$$\varepsilon_F = \frac{\pi \hbar^2}{m} \frac{N}{A} = \int_0^\infty \frac{d\varepsilon}{e^{\beta(\varepsilon - \mu)} + 1}$$

Let $x = \beta(\varepsilon - \mu)$

$$\varepsilon_F = \frac{1}{\beta}\int_{-\beta\mu}^{\infty} \frac{dx}{e^x + 1}$$

$$= \frac{1}{\beta}\int_{-\beta\mu}^{\infty} \frac{dx\ e^{-x}}{1+e^{-x}}$$

$$= -\frac{1}{\beta}\ln\left(1+e^{-x}\right)\Big|_{-\beta\mu}^{\infty}$$

$$= \frac{1}{\beta}\ln\left(1+e^{\beta\mu}\right)$$

$$\beta\varepsilon_F = \ln\left(1+e^{\beta\mu}\right)$$

$$e^{\beta\varepsilon_F} = 1 + e^{\beta\mu}, \quad \beta\mu = \ln\left(e^{\beta\varepsilon_F} - 1\right)$$

$$\boxed{\mu = \beta^{-1}\ln\left(e^{\beta\varepsilon_F} - 1\right)}$$

Note that $\lim_{T \to 0} \mu(T) = \varepsilon_F$, as it should.

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c) $$E = \int_0^\infty d\varepsilon \frac{D(\varepsilon)\,\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$$

$$= \frac{N}{\varepsilon_F} \int_0^\infty d\varepsilon \frac{\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$$

$$C_V = \frac{\partial E}{\partial T} = \frac{\partial E}{\partial \beta}\frac{\partial \beta}{\partial T} = -\frac{1}{k_B T^2}\frac{\partial E}{\partial \beta}$$

$$= \frac{N}{\varepsilon_F}\frac{1}{k_B T^2}\int_0^\infty d\varepsilon \frac{\varepsilon(\varepsilon-\mu)\,e^{\beta(\varepsilon-\mu)}}{\left(e^{\beta(\varepsilon-\mu)}+1\right)^2}$$

$$= \frac{N}{4\varepsilon_F}\frac{1}{k_B T^2}\int_0^\infty d\varepsilon \frac{\varepsilon(\varepsilon-\mu)}{\cosh^2\left[\beta\frac{(\varepsilon-\mu)}{2}\right]}$$

Let $\beta\frac{(\varepsilon-\mu)}{2} = x$

- $$C_V = \frac{N}{4\varepsilon_F} \frac{1}{k_B T^2} \left(\frac{2}{\beta}\right)^3 \int_{-\frac{\beta\mu}{2}}^{\infty} dx \, \frac{x^2 + \frac{\beta\mu}{2} x}{\cosh^2 x}$$

$$C_V \underset{T \ll T_F}{\simeq} \frac{2N}{\varepsilon_F} k_B^2 T \underbrace{\int_{-\infty}^{\infty} dx \, \frac{x^2 + \cancelto{0}{\frac{\beta\mu}{2} x}}{\cosh^2 x}}$$

$$= 2 N k_B \frac{k_B T}{\varepsilon_F} \times \frac{\pi^2}{6}$$

- $$\boxed{C_V = \frac{\pi^2}{3} N k_B \frac{k_B T}{\varepsilon_F}}$$

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