Physics 460

Problem set # 9 Solutions

Kittel 10.3) $\vec{J} = \vec{J}_N + \vec{J}_s$

$$\vec{J}_N = \sigma_0 \vec{E} \qquad \nabla \times \vec{J}_s = -\frac{c}{4\pi\lambda_L^2}\vec{B}$$

Maxwell's equations

1) $\nabla \cdot \vec{B} = 0$

2) $\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$

3) $\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

Take the curl of equation (2)

$$\nabla \times (\nabla \times \vec{B}) - \frac{1}{c}\frac{\partial}{\partial t}\nabla \times \vec{E} = \frac{4\pi}{c}\nabla \times (\vec{J}_N + \vec{J}_s)$$

$$\nabla \underbrace{\cancel{(\nabla \cdot \vec{B})}}_{0} - \nabla^2 \vec{B} + \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = \frac{4\pi}{c}\left[\sigma_0 (\nabla \times \vec{E}) - \frac{c}{4\pi\lambda_L^2}\vec{B}\right]$$

wave equation:

$$-\nabla^2 \vec{B} + \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} + \frac{4\pi\sigma_0}{c^2}\frac{\partial \vec{B}}{\partial t} + \lambda_L^{-2}\vec{B} = 0$$

Look for a solution of the form $\vec{B}(\vec{x},t) = B_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$$\Rightarrow k^2 - \frac{\omega^2}{c^2} - \frac{4\pi\sigma_0\omega}{c^2} i + \lambda_L^{-2} = 0$$

or $$\boxed{k^2c^2 = \omega^2 + 4\pi\sigma_0\omega i - \frac{c^2}{\lambda_L^2}}$$

Q.E.D.

b) $\sigma_0 = \frac{ne^2\tau}{m}$

$$k^2c^2 = \omega^2 + i\frac{4\pi ne^2}{m}\omega\tau - \frac{c^2}{\lambda_L^2}$$

The plasma frequency $\omega_p^2 = \frac{4\pi ne^2}{m}$.

[See Eq.(10.9) in Kittel.]

$$\Rightarrow k^2c^2 = -\frac{c^2}{\lambda_L^2} + i\omega_p^2\omega\tau + \omega^2$$

For $\omega \to 0$, one has

$k = \pm \frac{i}{\lambda_L}$. This describes the

Meissner effect:

$$B(\vec{x}) = B_0 e^{-|\vec{x}|/\lambda_L}.$$

The condition for the normal electrons to be unimportant in the dispersion relation is

$$\omega_p^2\omega\tau \ll \frac{c^2}{\lambda_L^2}$$

$$\omega\tau \ll c^2/\lambda_L^2\omega_p^2$$

Using $\omega_p^2 = \frac{4\pi n e^2}{m}$ and

$$\lambda_L^2 = \frac{m_s c^2}{4\pi n_s q^2} = \frac{2m c^2}{4\pi n_s (2e)^2} = \frac{mc^2}{8\pi n_s e^2}$$

we find the criterion

$$\boxed{\omega\tau \ll \frac{2n_s}{n}} \sim \mathcal{O}(1)$$

for $T \ll T_c$.

10.4) $B_z - \lambda^2 \nabla^2 B_z = 0$

By symmetry $\quad B_z = B_z(\rho)$

$$\nabla^2 B_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial B_z}{\partial\rho}\right) = \frac{\partial^2 B_z}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial B_z}{\partial\rho}$$

$$2\pi\int_0^\infty d\rho\,\rho\, B_z(\rho) = \Phi_0$$

$$\frac{\partial^2 B_z}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial B_z}{\partial\rho} - \frac{1}{\lambda^2}B_z = 0$$

$$\rho^2\frac{\partial^2 B_z}{\partial\rho^2} + \rho\frac{\partial B_z}{\partial\rho} - \frac{\rho^2}{\lambda^2}B_z = 0$$

i) For $\rho \ll \lambda$, we can neglect the last term. Then

$$\rho\frac{d}{d\rho}B_z'(\rho) + B_z'(\rho) \cong 0$$

$$\frac{dB_z'}{B_z'} = -\frac{d\rho}{\rho}$$

$$\ln B_z' = -\ln\rho + C$$

$$B_z' = \frac{e^C}{\rho} \equiv \frac{D}{\rho}$$

$$dB_z = \frac{D}{\rho}\, d\rho \qquad B_z = D \ln \rho + A$$

$$B_z(\rho) = -D \ln \frac{1}{\rho} + \ln F \qquad F = e^A$$

$$B_z(\rho) = -D \ln \frac{F}{\rho} \qquad (\rho \gg \lambda)$$

(ii) For $\rho \gg \lambda$

$$\lambda^2 B_z''(\rho) + \frac{\lambda^2}{\rho} B_z'(\rho) - B_z = 0$$

$$\cancel{\lambda^2 \nabla^2 B_z = \frac{\lambda}{\rho} \partial}$$

$$\lambda^2 \nabla^2 B_z - B_z = 0$$

let $x = \frac{\rho}{\lambda}$

$$\frac{\lambda^2}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial B_z}{\partial \rho}\right) - B_z = 0$$

$$\frac{1}{x} \frac{\partial}{\partial x}\left(x \frac{\partial B_z}{\partial x}\right) - B_z = 0 , \qquad x \gg 1$$

$$\frac{\partial^2 B_z}{\partial x^2} + \underbrace{\frac{1}{x}\frac{\partial B_z}{\partial x}}_{\text{small}} - B_z = 0 \quad , \quad x \gg 1$$

$$\frac{\partial^2 B_z}{\partial x^2} \simeq B_z$$

$$B_z(x) = A_1 e^{-x} + \underbrace{A_2 e^{+x}}$$

$\hookrightarrow$ doesn't satisfy B.C. at $x \to \infty$

$$B_z(x) = A_1 e^{-x}$$

$$B_z(\rho) = A_1 e^{-\rho/\lambda}$$

$$2\pi \int_0^\infty d\rho\, \rho\, A_1 e^{-\rho/\lambda} \approx \Phi_0$$

$$\int_0^\infty d\rho\, \rho\, e^{-\rho/\lambda} \simeq \frac{\Phi_0}{2\pi A_1}$$

$$\lambda^2 \cong \frac{\Phi_0}{2\pi A_1} \qquad A_1 \simeq \frac{\Phi_0}{2\pi\lambda^2}$$

$$\boxed{B_z(\rho) \simeq \frac{\Phi_0}{2\pi\lambda^2} e^{-\rho/\lambda} \quad , \quad \rho \gg \lambda}$$

To match up to the solution for $\rho \ll \lambda$, we need the two solutions to agree when $\rho \approx \lambda$ and for their 1st derivatives to agree:

$$-D \ln \frac{F}{\lambda} = \frac{\Phi_0}{2\pi\lambda^2}$$

$$\frac{D}{\lambda} = -\frac{1}{\lambda}\frac{\Phi_0}{2\pi\lambda^2} \Rightarrow D = -\frac{\Phi_0}{2\pi\lambda^2}$$

$$\Rightarrow F = \lambda$$

$$\boxed{B_z(\rho) = \frac{\Phi_0}{2\pi\lambda^2} \ln(\lambda/\rho), \quad \rho \ll \lambda}$$