Phenomenological theory of superconductivity

We have seen that a superconductor is characterized by a macroscopic condensate wavefunction for Cooper pairs

$$\psi_s(\vec{r}) = \sqrt{n_s(\vec{r})}\, e^{i\theta(\vec{r})}.$$

Below T_c, $|\psi_s(\vec{r})| > 0$; above T_c, $\psi_s(\vec{r}) = 0$. $\psi_s(\vec{r})$ can thus be considered the <u>order parameter</u> of the superconducting state. As $T \nearrow T_c$, $n_s \to 0$, so $\psi_s(\vec{r})$ can be considered a small

parameter for $T_c - T \ll T_c$.
In the absence of an external magnetic field, we can thus write the free energy of a superconductor as a Taylor series:

$$F_s = F_N + \int d^3r \left\{ \frac{\hbar^2}{2m^*} |\nabla \psi|^2 + a|\psi|^2 + \frac{b}{2}|\psi|^4 \right\},$$

where higher-order terms have been omitted, and ~~terms~~ odd powers of ψ are excluded by symmetry. Here it is also assumed that $\psi(\bar{r})$ is a slowly varying function of $\bar{r}$. b is a positive constant, and

a changes sign at $T = T_c$:

$$a(T) = \alpha (T - T_c).$$

In a homogeneous superconductor with no external field ψ is independent of $\vec{r}$:

$$F_s = F_N + aV|\psi|^2 + \frac{1}{2} bV|\psi|^4$$

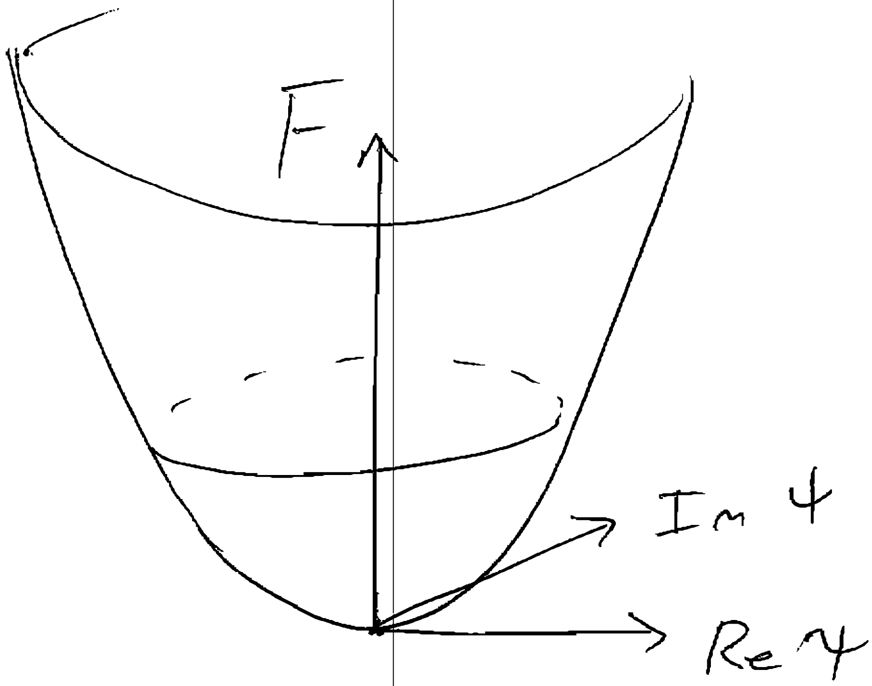
equilibrium:

$$0 = \frac{\partial F_s}{\partial |\psi|^2} = aV + bV|\psi|^2$$

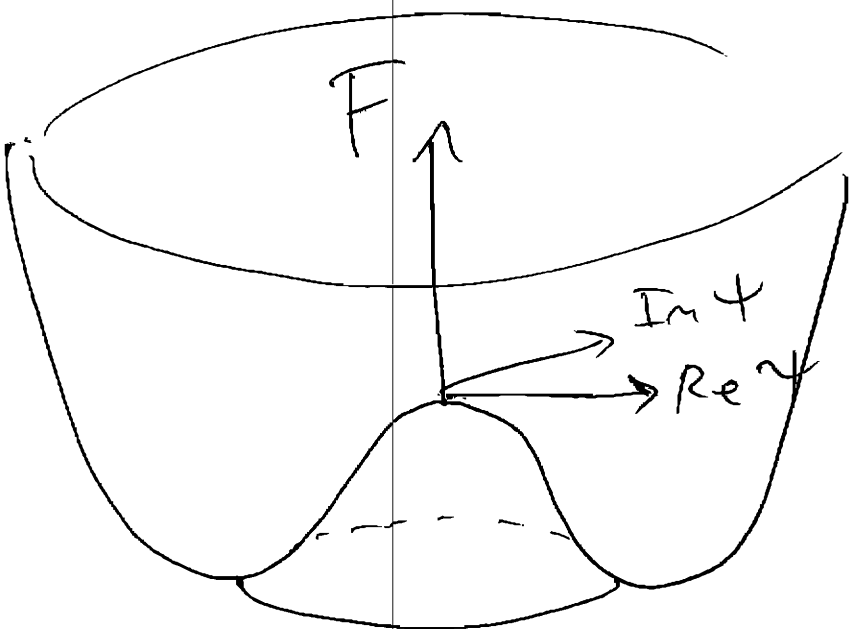
$$|\psi|^2 = -\frac{a}{b} = \frac{\alpha}{b}(T_c - T).$$

Thus $n_s \to 0$ linearly as $T \nearrow T_c$.

Substituting this value of $|\psi|^2$



$T > T_c$



$T < T_c$

"Mexican hat"

back into F_s gives

$$F_s - F_N = -V\frac{\alpha^2}{2b}(T_c - T)^2$$

$$= -\frac{V H_c^2}{8\pi}$$

$$\Rightarrow \quad H_c = \sqrt{\frac{4\pi\alpha^2}{b}}\,(T_c - T).$$

The entropy is given by

$$S = -\left.\frac{\partial F}{\partial T}\right|_{V,N}$$

$$S_s - S_N = V\frac{\alpha^2}{b}(T - T_c)$$

$$C_V = T\left.\frac{\partial S}{\partial T}\right|_{V,N}$$

$$\left.(C_s - C_N)\right|_{T_c} = V\frac{\alpha^2}{b}T_c \Rightarrow$$ discontinuity of specific heat

When a magnetic field is present, the field energy $\vec{B}^2/8\pi$ must be added to F.

Also $\nabla \to \nabla - \frac{iq}{\hbar c}\vec{A}$.

Thus

$$\boxed{F_S = F_N\Big|_{B=0} + \int d^3r \left\{ \frac{\vec{B}^2}{8\pi} + \frac{\hbar^2}{2m^*}\left|\left(\nabla - \frac{iq}{\hbar c}\vec{A}\right)\psi\right|^2 + a|\psi|^2 + \frac{b}{2}|\psi|^4 \right\}}$$

<u>Ginzburg-Landau free energy</u>

The differential equations governing the distribution of $\psi(\vec{r})$ and $\vec{A}(\vec{r})$ are now found by minimizing F_s with respect to the three independent functions ψ, ψ^*, and $\vec{A}$.

The complex quantity ψ is a set of two real quantities, so that ψ and ψ^* must be regarded as independent functions in the variation. Varying the integral with respect to ψ^* and integrating by parts, one finds

$$\delta F = \int d^3r \left\{ -\frac{\hbar^2}{2m^*}\left(\nabla - \frac{iq}{\hbar c}\vec{A}\right)^2 \psi + a\psi + b|\psi|^2\psi \right\} \delta\psi^*$$

$$+ \frac{\hbar^2}{2m} \oint \left(\nabla\psi - \frac{iq}{\hbar c}\vec{A}\psi\right)\cdot d\vec{S}\ \delta\psi^*.$$

Putting $\delta F = 0$ for arbitrary $\delta\psi^*$, we get

$$\frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - \frac{q}{c}\vec{A}\right)^2\psi + a\psi + b|\psi|^2\psi = 0.$$

Varying F_s with respect to ψ gives the complex conjugate equation, and therefore nothing new.

Varying F_s with respect to $\vec{A}$ gives (see advanced texts on electromagnetism):

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} ,$$

where

$$\vec{J} = -\frac{i e \hbar}{2m^*} \left(\psi^* \nabla \psi - \psi \nabla \psi^* \right)$$

$$- \frac{e^2}{m^* c} |\psi|^2 \vec{A}$$

$$= \vec{J}_s$$

($\vec{J}_N = 0$ in equilibrium!). Thus we indentify the parameter m^* as the mass of a Cooper pair $= 2m_e$.

<u>The boundary conditions</u> on ψ and $\vec{A}$ are

$$\vec{n} \cdot \left(\frac{\hbar}{i} \nabla\psi - \frac{q}{c} \vec{A}\psi \right) = 0$$

and $\vec{B}$ = continuous.

The Ginzburg-Landau theory has two characteristic length scales:

i) <u>Penetration depth</u> In a weak field, one can neglect the dependence of $|\psi|^2$ on $\vec{B}$, and assume $|\psi|^2 = \alpha(T_c - T)/b$ throughout the sample. Then

$$\vec{J} = \left(\frac{q\hbar}{m^*} \nabla\theta - \frac{q^2}{m^* c} \vec{A} \right) |\psi|^2$$

and $\nabla \times \nabla \times \vec{B} = \frac{4\pi}{c} \nabla \times \vec{J}$

$$\Rightarrow \quad \nabla^2 \vec{B} = \frac{1}{\lambda^2} \vec{B}$$

$$\boxed{\lambda^2 = \frac{m^* c^2}{4\pi n_s q^2}}$$

same as London penetration depth

$$\boxed{\lambda = \sqrt{\frac{m^* c^2 b}{4\pi q^2 \alpha (T_c - T)}}}$$

ii) <u>Correlation length</u>

In the absence of a magnetic field, let's rescale ψ by its value for a homogeneous superconductor

$$\Psi = \frac{\psi}{\psi_0}, \quad \psi_0 = \sqrt{\frac{\alpha (T_c - T)}{b}}$$

$$\Rightarrow -\xi^2 \nabla^2 \psi - \psi + |\psi|^2 \psi = 0$$

$$\boxed{\xi(T) = \frac{\hbar}{\sqrt{2m^* \alpha (T_c - T)}}}$$

The Ginzburg-Landau parameter is the temperature-independent ratio:

$$K = \frac{\lambda(T)}{\xi(T)} = \frac{m^* c}{\hbar |q|} \sqrt{\frac{b}{2\pi}}$$

ξ characterizes the length-scale over which ψ varies in inhomogeneous problems.

Type-II Superconductors

If $K > \frac{1}{\sqrt{2}}$, it is possible for superconductivity to persist for $H > H_c$ (thermodynamic critical field). Superconductivity is then destroyed at a higher field H_{c_2} determined as follows.

Near H_{c_2} $|\psi|$ is small, so we can neglect the cubic term in the non-linear Schrödinger equation for ψ:

$$\frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - \frac{q}{c}\vec{A}\right)^2\psi = -a\psi.$$

we define H_{c_2} as the largest

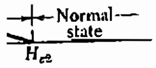
field for which this equation has a nonzero solution. This equation is just the Schrödinger equation for a particle of charge q in a uniform magnetic field. The lowest eigenvalue is

$$\frac{\hbar \Omega_c}{2}, \text{ where } \Omega_c = \frac{|q| H_{c2}}{m^* c}.$$

Thus

$$-a = |a| = \frac{|q| \hbar H_{c2}}{2m^* c}$$

$$\Rightarrow H_{c2} = \frac{2m^* c}{|q| \hbar} \alpha (T_c - T)$$



:tor exhibiting a
ıavior is called a
nductor and the
d on the vertical
ıg magnetization
ı field H_{c1} lower
:en H_{c1} and H_{c2},
men is a normal
: area under the
, units in all parts

ler the condi-
;. 4a. This ap-
ıder placed in
ls exhibit this
soft supercon-
uctors to have
ıgnets.
of Fig. 4b and
s in Fig. 5a) or
in the normal
short. We shall
tion" of super-

roperties up to
and the upper
:t is said to be
· (Fig. 5b) than
ynamics of the
·tor is threaded
0 kG (41 teslas)
›oint of helium,

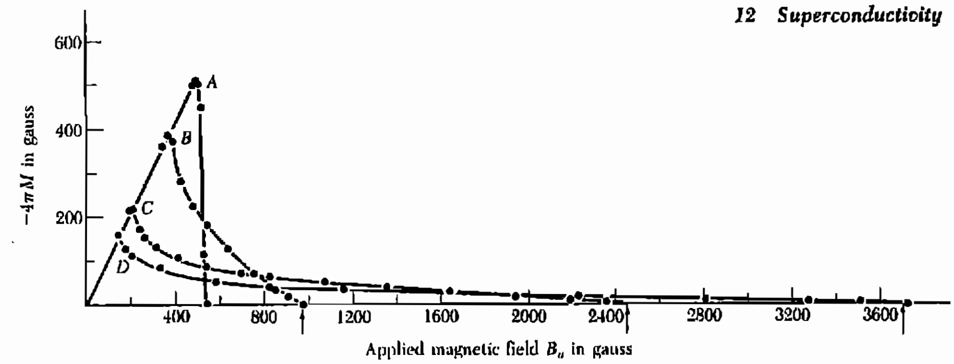


Figure 5a Superconducting magnetization curves of annealed polycrystalline lead and lead-indium alloys at 4.2 K. (*A*) lead; (*B*) lead–2.08 wt. percent indium; (*C*) lead–8.23 wt. percent indium; (*D*) lead–20.4 wt. percent indium. (After Livingston.)

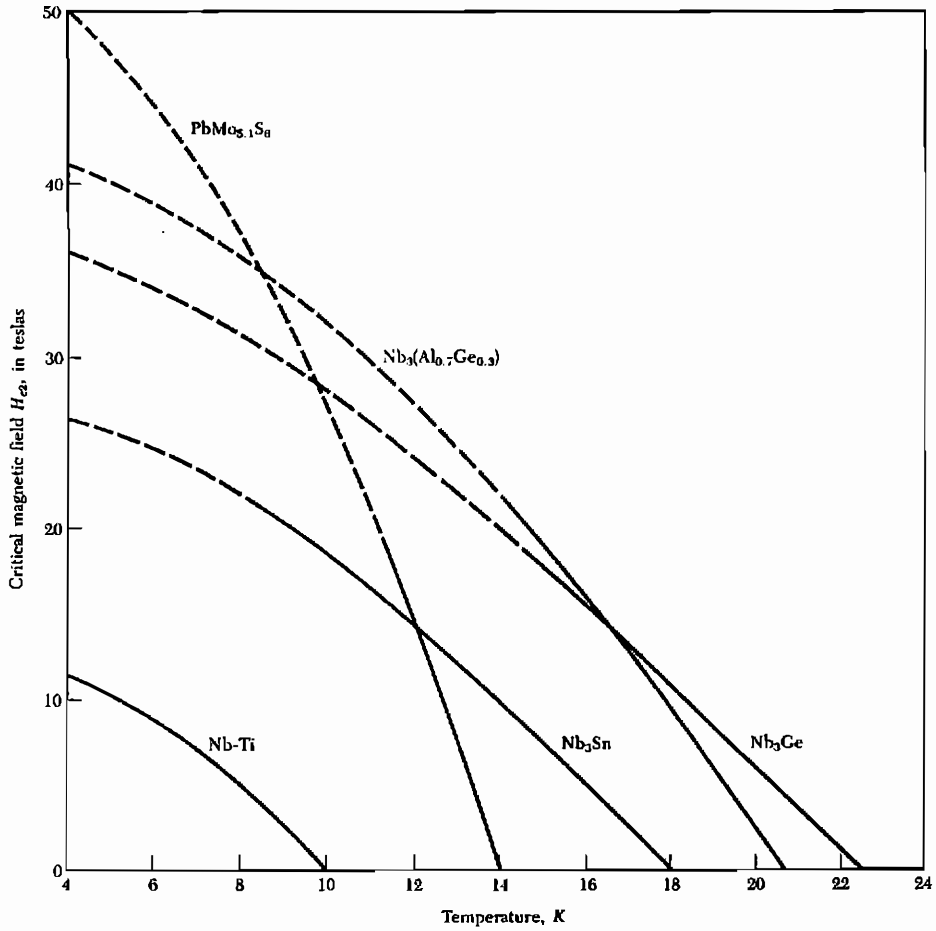


Figure 5b Stronger magnetic fields than any now contemplated in practical superconducting devices are within the capability of certain Type II materials. These materials cannot be exploited, however, until their critical current density can be raised and until they can be fabricated as finely divided conductors. (Magnetic fields of more than about 20 teslas can be generated only in pulses, and so portions of the curves shown as broken lines were measured in that way.)

But $H_c = \sqrt{\frac{4\pi}{b}}\,\alpha\,(T_c - T)$.

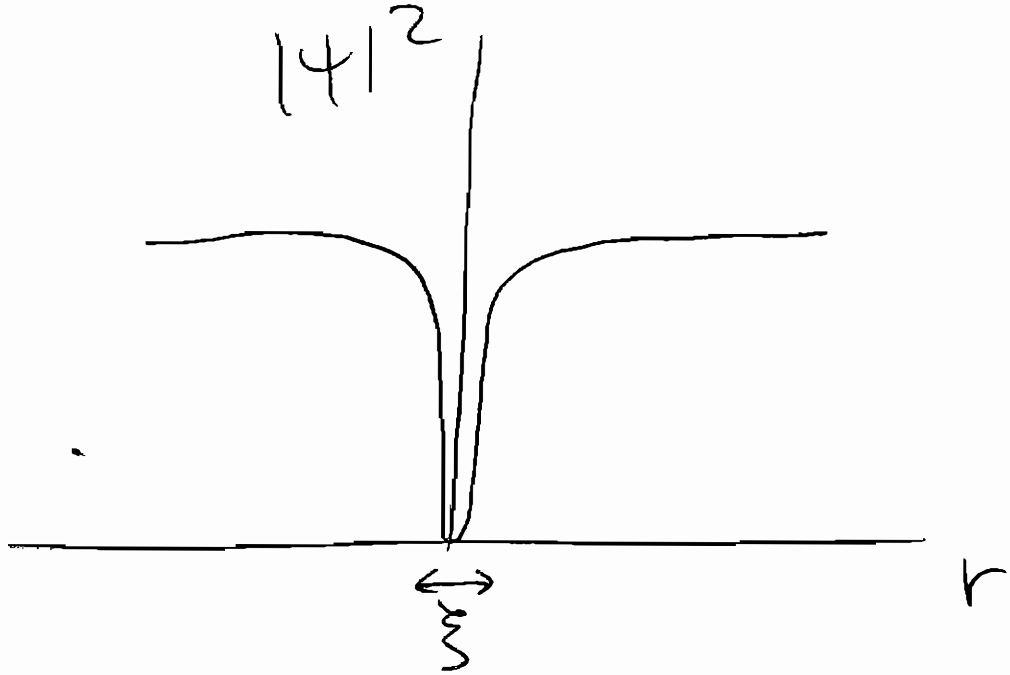
So
$$\boxed{\frac{H_{c_2}}{H_c} = \frac{2m^*c}{|g|\hbar}\sqrt{\frac{b}{4\pi}} = \sqrt{2}\,\kappa}$$

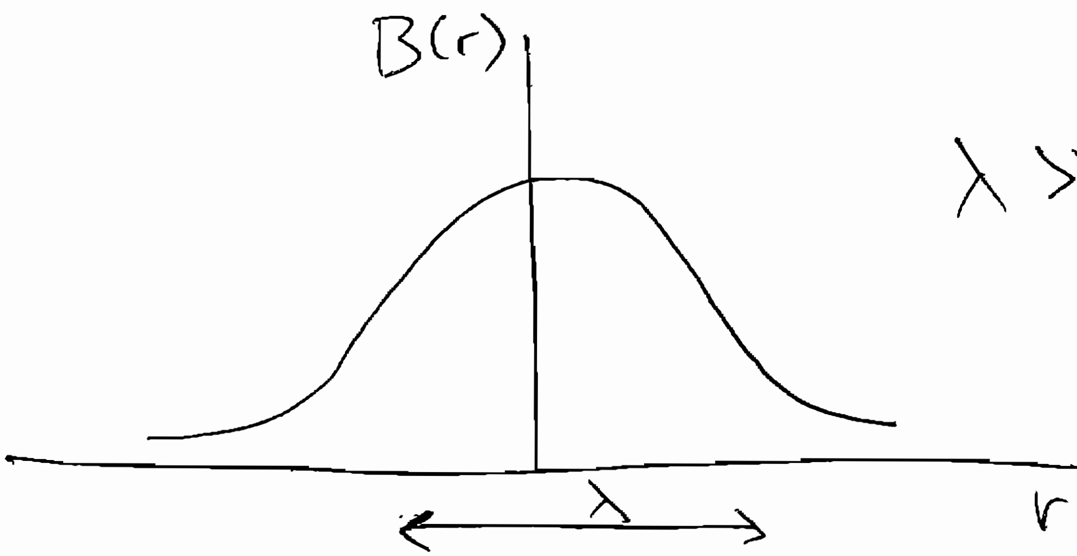
If $\kappa > \frac{1}{\sqrt{2}}$, then $H_{c_2} > H_c$.

In type-II superconductors, the the interface between normal state and superconducting state has a <u>negative</u> surface tension.

The Meissner effect is not complete in type-II superconductors.

Above a <u>lower-critical field</u> H_{c_1}, magnetic field penetrates the superconductor through nodes of the superconducting order parameter. Consider the case $\kappa \gg 1$, which is technologically relevant:





$\lambda \gg \xi$

For $r > \xi$:

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} = \frac{4\pi q \hbar n_s}{m^*}\left(\nabla\theta - \frac{q}{\hbar c}\vec{A}\right)$$

$$\vec{A} + \lambda^2 \nabla \times \vec{B} = \frac{\hbar c}{q}\nabla\theta$$

$$\oint \vec{A}\cdot d\vec{l} + \lambda^2 \oint \nabla\times\vec{B}\cdot d\vec{l} = \frac{\hbar c}{q} 2\pi s$$

For a large enough loop, $\nabla\times\vec{B} = 0$.

$$\oint \vec{A}\cdot d\vec{l} = \Phi = \frac{hc}{2e}s = \phi_0 s.$$

Thus the magnetic flux through such a vortex is quantized. Typically $s = 1$, so $\Phi = \phi_0$.

The lower-critical field is the minimum field to nucleate a vortex: (16)

$$\Phi_0 \approx \pi \lambda^2 H_{c_1}$$

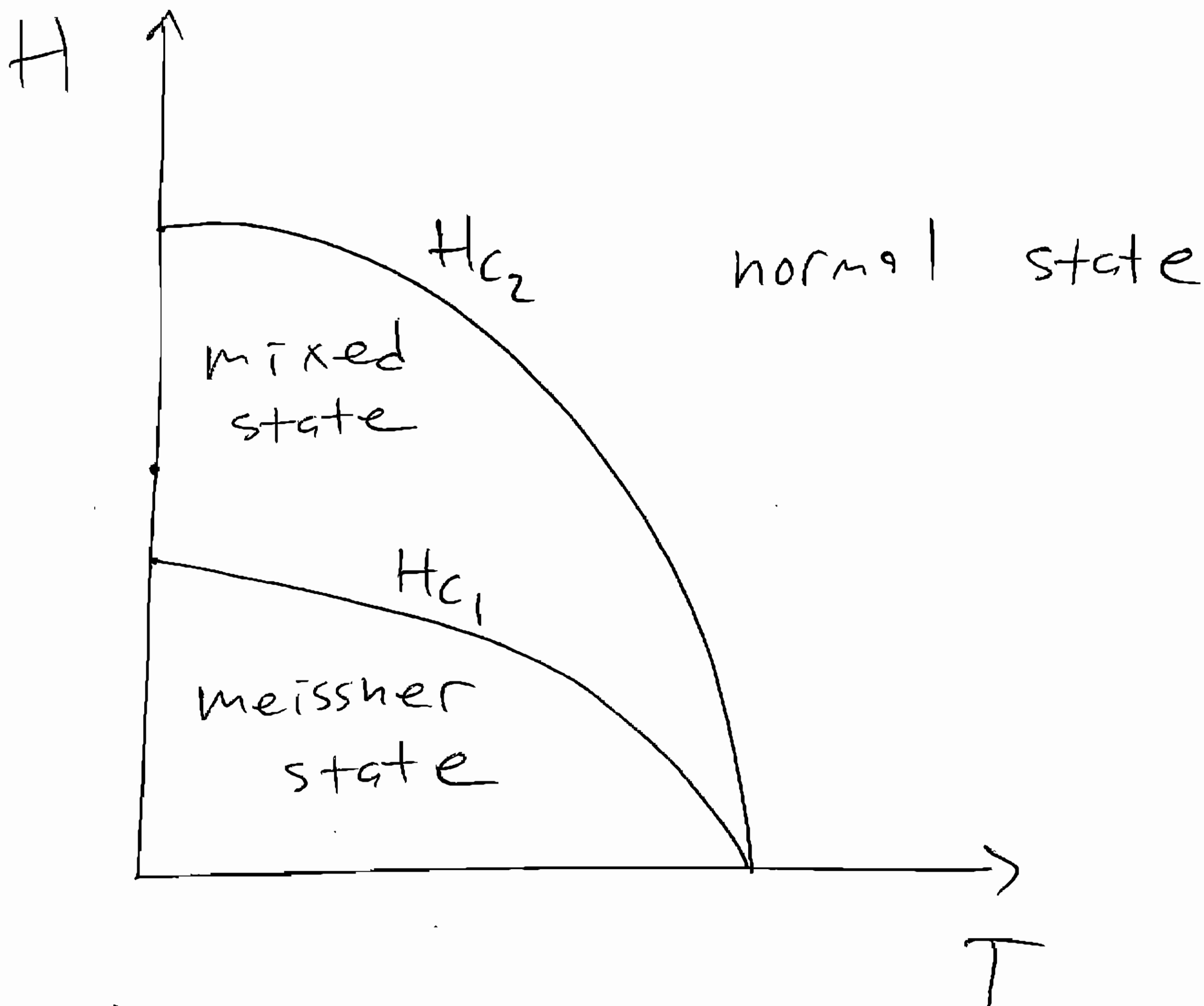
$$H_{c_1} \approx \frac{\Phi_0}{\pi \lambda^2} \sim \frac{H_c}{\kappa}$$

One can also write

$$H_{c_2} = \frac{\Phi_0}{2\pi \xi^2}.$$ For $H_{c_1} < H < H_{c_2}$,

one has an <u>Abrikosov lattice</u> of magnetic flux tubes/vortices (show image). H_{c_2} corresponds to the minimum spacing between vortices $\sim \xi$.

Phase diagram



Pinning of vortices

by defects still gives perfect conductivity in mixed phase.

$\Rightarrow$ superconducting magnets.