Phenomenological theory of superconductivity

We have seen that a superconductor is characterized by a macroscopic condensate wave function for Cooper pairs

\[ \Psi_s(\vec{r}) = \sqrt{n_s(\vec{r})} e^{i\theta(\vec{r})} \]

Below \( T_c \), \( |\Psi_s(\vec{r})| > 0 \); above \( T_c \), \( \Psi_s(\vec{r}) = 0 \). \( \Psi_s(\vec{r}) \) can thus be considered the order parameter of the superconducting state. As \( T \to T_c \), \( n_s \to 0 \), so \( \Psi_s(\vec{r}) \) can be considered a small
parameter for $T_c - T << T_c$. In the absence of an external magnetic field, we can thus write the free energy of a superconductor as a Taylor series:

$$F_s = F_N + \int d^3r \left\{ \frac{\hbar^2}{2m^*} \nabla \Psi^2 + a |\Psi|^2 + \frac{b}{2} |\Psi|^4 \right\},$$

where higher-order terms have been omitted, and terms odd powers of $\Psi$ are excluded by symmetry. Here it is also assumed that $\Psi(r)$ is a slowly varying function of $r$. $b$ is a positive constant, and
a changes sign at $T = T_c$:

$$a(T) = \alpha(T - T_c).$$

In a homogeneous superconductor with no external field $\psi$ is independent of $\bar{r}$. 

$$F_s = F_n + aV 14l^2 + \frac{1}{2} b V (4l^4$$

in equilibrium:

$$0 = \frac{\partial F_s}{\partial l^2} = aV + bV 14l^2$$

$$14l^2 = -\frac{a}{b} = \frac{\alpha(T_c - T)}{b}.$$

Thus $N_s \rightarrow 0$ linearly as $T \rightarrow T_c$. 

Substituting this value of $14l^2$
$T > T_c$

$T < T_c$

"Mexican hat"
back into $F_s$ gives

$$F_s - F_N = -V \frac{\alpha^2}{2b} (T_c - T)^2$$

$$= -V \frac{H_c^2}{8\pi}$$

$$\Rightarrow H_c = \sqrt{\frac{4\pi \alpha^2}{b}} (T_c - T)$$.

The entropy is given by

$$S = -\left. \frac{\partial F}{\partial T} \right|_{V,N}$$

$$S_s - S_N = V \frac{\alpha^2}{b} (T - T_c)$$

$$C_V = T \left. \frac{\partial S}{\partial T} \right|_{V,N}$$

$$\left(\frac{C_s - C_N}{T_c}\right) = V \frac{\alpha^2}{b} T_c \Rightarrow \text{discontinuity of specific heat}$$
When a magnetic field is present, the field energy $\frac{\mathbf{B}^2}{8\pi}$ must be added to $F$.

Also $\nabla \to \nabla - \frac{i\mathbf{B}}{\hbar c} \mathbf{A}$.

Thus

$$F_S = F_N \bigg|_{B=0} + \int d^3r \left\{ \frac{\mathbf{B}^2}{8\pi} + \frac{\hbar^2}{2m^*} \left| \nabla - \frac{i\mathbf{A}}{\hbar c} \right|^2 \right\}$$

$$+ g_1 |\psi|^2 + \frac{b}{2} |\psi|^4 + \frac{3}{2}$$

**Ginzburg-Landau free energy**

The differential equations governing the distribution of $\psi(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$ are now found by minimizing $F_S$ with respect to the three independent functions $\psi$, $\psi^*$, and $\mathbf{A}$. 
The complex quantity $\psi$ is a set of two real quantities, so that $\psi$ and $\psi^*$ must be regarded as independent functions in the variation. Varying the integral with respect to $\psi^*$ and integrating by parts, one finds

$$\delta F = \int d^3r \left\{ \frac{\varepsilon - \frac{\hbar^2}{2m} \left( \nabla - \frac{e}{\hbar c} \vec{A} \right)^2}{\hbar^2} \psi^* \right. \right.$$

$$+ a \psi + b |\psi|^2 \right\} \delta \psi^*$$

$$+ \frac{\hbar^2}{2m} \int \left( \nabla \psi^* - \frac{e}{\hbar c} \vec{A} \psi^* \right) \cdot \vec{\nabla} \delta \psi^*.$$

Putting $\delta F = 0$ for arbitrary $\delta \psi^*$, we get

$$\frac{1}{2m} \left( \frac{\hbar}{c} \vec{P} - \frac{e}{c} \vec{A} \right)^2 \psi + a \psi + b |\psi|^2 \psi = 0.$$
Varying $F^a$ with respect to $A^a$ gives the complex conjugate equation, and therefore nothing new.

Varying $F^a$ with respect to $\tilde{A}$ gives (see advanced texts on electromagnetism):

$$D \times \vec{B} = \frac{4\pi}{c} \vec{J},$$

where

$$\vec{J} = -\frac{i\hbar}{2m^*} \left( 4 \ast D^a - \ast D^a \right)$$

$$= \frac{e^2}{m^*c} |\psi|^2 \vec{A}$$

$$= \vec{J}_s$$

($\vec{J}_n = 0$ in equilibrium!). Thus we identify the parameter $m^*$ as the mass of a Cooper pair $= 2m_e$. 
The boundary conditions on \( \psi \) and \( \bar{A} \) are

\[
\hat{n} \cdot \left( \frac{\hbar}{i} \partial \psi - \frac{\varphi}{c} \bar{A} \psi \right) = 0
\]

and \( \bar{B} = \text{continuous} \).

The Ginzburg-Landau theory has two characteristic length scales:

1) Penetration depth. In a weak field, one can neglect the dependence of \( |\psi|^2 \) on \( \bar{B} \), and assume

\[
|\psi|^2 = \kappa (T_c-T)/\hbar \text{ throughout the sample. Then}
\]

\[
\hat{j} = \left( \frac{\hbar}{m^*} \partial \Theta - \frac{\varphi^2}{m^*c} \bar{A} \right) |\psi|^2
\]
and \( \nabla \times \nabla \times \mathbf{B} = \frac{4\pi}{c} \nabla \times \mathbf{J} \)

\[ \Rightarrow \nabla^2 \mathbf{B} = \frac{1}{\lambda^2} \mathbf{B} \]

\[ \lambda^2 = \frac{m^* c^2}{4\pi \hbar s \alpha^2} \]

\[ \lambda = \sqrt[4]{\frac{m^* c^2 b}{4\pi \hbar s^2 \alpha (T_c - T)}} \]

\( \text{l.c) correlation length} \)

In the absence of a magnetic field, let's rescale \( \psi \) by its value for a homogeneous superconductor.

\[ \frac{\psi}{\psi_0} = \psi \quad / \quad \psi_0 = \sqrt{\frac{\alpha (T_c - T)}{b}} \]
\[ -\xi^2 \partial^2 \psi - \psi + 141^2 \psi = 0 \]

\[ \xi(\xi) = \frac{\hbar}{\sqrt{2m \alpha (T_c - T)}} \]

The Ginzburg-Landau parameter is the temperature-independent ratio:

\[ K = \frac{\chi(T)}{\xi(T)} = \frac{m^* c}{\hbar 181} \sqrt{\frac{1}{2\pi}} \]

\( \xi \) characterizes the length-scale over which \( \psi \) varies in inhomogeneous problems.
Type-II superconductors

If \( k > \frac{1}{\sqrt{2}} \), it is possible for superconductivity to persist for \( H > H_c \) (thermodynamic critical field). Superconductivity is then destroyed at a higher field \( H_c^2 \) determined as follows.

Near \( H_c^2 \) is small, so we can neglect the cubic term in the non-linear Schrödinger equation for \( \Psi \):

\[
\frac{1}{2m^*} \left( \frac{i}{\hbar} \frac{\partial}{\partial x} - \frac{e}{c} A \right)^2 \Psi = -\alpha \Psi.
\]

We define \( H_c^2 \) as the largest
field for which this equation has a non-zero solution. This equation is just the Schrödinger equation for a particle of charge $e$ in a uniform magnetic field. The lowest eigenvalue is

$$\frac{\hbar \Omega_c}{2}, \text{ where } \Omega_c = \frac{eH_c^2}{m^*c}.$$ 

Thus

$$-a = |a| = \frac{19\hbar H_c^2}{2m^*c}.$$ 

$$\Rightarrow H_{c2} = \frac{2m^*c}{19\hbar} \times (T_c - T)$$
tor exhibiting a
state is called a
ductor and the
down magnetization
field \( H_{c1} \) lower
\( H_{c2} \) and \( H_{c3} \),
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Figure 5a Superconducting magnetization curves of annealed polycrystalline lead and lead-
indium alloys at 4.2 K. (A) lead; (B) lead-2.08 wt. percent indium; (C) lead-8.23 wt. percent
indium; (D) lead-20.4 wt. percent indium. (After Livingston.)

Figure 5b Stronger magnetic fields than any now contemplated in practical superconducting
devices are within the capability of certain Type II materials. These materials cannot be exploited,
however, until their critical current density can be raised and until they can be fabricated as finely
divided conductors. (Magnetic fields of more than about 20 teslas can be generated only in pulser,
and so portions of the curves shown as broken lines were measured in that way.)
But \( H_c = \sqrt{\frac{4\pi}{b}} \alpha (T_c - T) \).

So

\[
\frac{H_{c2}}{H_c} = \frac{2\mu_0 c}{181 t_b} \sqrt{\frac{b}{4\pi}} = \sqrt{2} K
\]

If \( K > \frac{1}{\sqrt{2}} \), then \( H_{c2} > H_c \).

In type-II superconductors, the interface between normal state and superconducting state has a negative surface tension. The Meissner effect is not complete in type-II superconductors. Above a lower-critical field \( H_{c1} \), magnetic field penetrates
the superconductor through the superconducting nodes of the superconducting order parameter. Consider the case $\lambda \gg 1$, which is technologically relevant:

\[ |4|^2 \]

\[ B(r) \]

$\lambda \gg 3$
For \( r > \delta \):

\[
\text{D} \times \vec{B} = \frac{4\pi}{c} \vec{J} = \frac{4\pi e g \tan s}{m^*} (\vec{A} \cdot \frac{\vec{A}}{\delta} - \frac{\vec{r}}{r c} \vec{A})
\]

\[\vec{A} + \lambda^2 \nabla \times \vec{B} = \frac{t c}{\delta} \nabla \theta \]

\[
\oint \vec{A} \cdot d\vec{l} + \lambda^2 \oint \nabla \times \vec{B} \cdot d\vec{l} = \frac{t c}{\delta} 2\pi s
\]

For a large enough loop, \( \nabla \times \vec{B} = 0 \).

\[
\oint \vec{A} \cdot d\vec{l} = \Phi = \frac{hc}{2e} s = \phi_0 s.
\]

Thus, the magnetic flux through such a vortex is quantized. Typically \( s = 1 \), so \( \Phi = \phi_0 \).
The lower-critical field \( \mathcal{C}_1 \) is the minimum field to nucleate a vortex:

\[
\phi_0 \approx \pi x^2 H_{c_1}
\]

\[
H_{c_1} \approx \frac{\phi_0}{\pi x^2} \sim \frac{H_c}{k}
\]

One can also write

\[
H_{c_2} = \frac{\phi_0}{2\pi \xi^2}. \quad \text{For} \quad H_{c_1} < H < H_{c_2},
\]

one has an Abrikosov lattice of magnetic flux tubes/vortices (show image). \( H_{c_2} \) corresponds to the minimum spacing between vortices \( \sim \xi \).
Phase diagram

\[ H \]

\[ \text{mixed state} \]

\[ H_{c2} \]

\[ H_{c1} \]

\[ \text{normal state} \]

\[ T \]

Pinning of vortices by defects still gives perfect conductivity in mixed phase.

\[ \Rightarrow \text{superconducting magnets.} \]