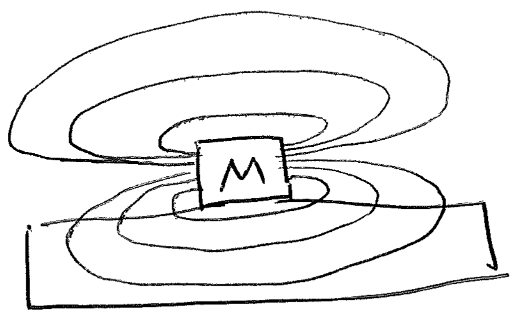
Physics 460

Lecture 1

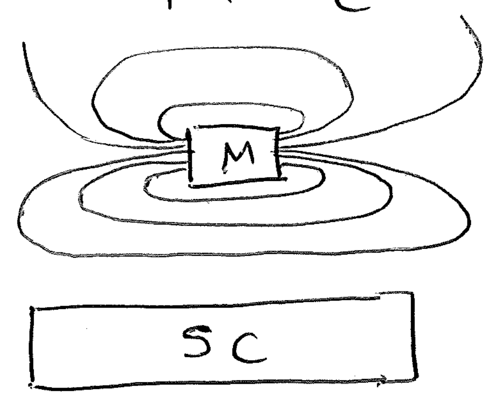
Meissner effect

$T > T_c$



magnetic field penetrates normal metal

$T < T_c$



magnetic field expelled from superconductor

Q: Can we understand the Meissner effect in terms of perfect conductivity?

Classically, $\quad m \dfrac{d\vec{v}}{dt} = q\vec{E}$

(charged particle of mass m in an electric field)

$$\vec{J}_e = n_s q \langle \vec{v} \rangle$$

$n_s = \dfrac{\#\text{ carriers}}{\text{volume}}$ $\qquad q = \text{charge}$

$$n_s q \frac{d\langle \vec{v} \rangle}{dt} = \frac{n_s q^2}{m} \vec{E}$$

$$\frac{d\vec{J}_e}{dt} = \frac{n_s q^2}{m} \vec{E}$$

$$\frac{d}{dt} \nabla \times \vec{J}_e = \frac{n_s q^2}{m} \nabla \times \vec{E}$$

But $\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ (Faraday's law)

$$\Rightarrow \frac{d}{dt}\left[\nabla \times \vec{J}_e + \frac{n_s q^2}{mc} \vec{B}\right] = 0$$

$$\nabla \times \vec{J}_e + \frac{n_s q^2}{mc} \vec{B} = \text{const.}$$

Currents flow only on the surface

of a perfect conductor, so in order to force $\vec{B}=0$ in the interior, we must have the integration constant in the above equation $= 0$.

$$\boxed{\nabla \times \vec{J}_e + \frac{n_s q^2}{mc}\vec{B} = 0 \quad \text{London equation}}$$

The London equation describes the Meissner effect. Perfect conductivity does not imply the Meissner effect, since a perfect conductor maintains a constant field in its interior, but this constant field need not be zero!

TABLE I: MASTER TABLE OF BROKEN-SYMMETRY PHENOMENA

Phenomenon	High Phase	Low Phase	Order Parameter (constant of motion)		Order-Parameter Dimensionality $T \to 0$	T_c	Common Transition Type (Can Always Be First Order	"Goldstone Bosons" (or "Higgs" Bosons)	Fluctuations	Collective Hydrodynamic Modes	Generalized Rigidity Phenomenon	Long-Range Forces Due to General. Rigidity	Singularities
Ferroelectricity (Pyroelectricity)	Non-Polar crystal	Polar crystal	$\vec{P}$	(no)	1	1 or 3	2nd or 1st nearly 2nd	no (optical phonons)	Soft Modes	No	Ferroelectric hysteresis	No	Domain walls (thin)
Ferromagnetism	Paramagnet	Ferromagnet	$\vec{M}$	(yes)	1, often ≈ 3	1 or 3	2nd (time reversal)	Spin waves, one branch $\omega \propto k^2 (+\text{const})$	Spin waves	No ?	Permanent magnetism: hysteresis	Suhl-Nakamura	Domain walls (very mobile, $\approx$ 3 dim.)
Antiferromagnetism	Paramagnet	Antiferromagnet	$\vec{M}_{\text{sublattice}}$	(no)	1, often ≈ 3	1 or 3	2nd (or first)	Spin waves, 2 branches $\omega \propto \sqrt{k^2 + \text{const}}$ ("Fermions" in metal case?)	Spin waves (diffusion?)	No?	subtle effects in A.F. resonance	Suhl-Nakamura	Domain walls
Superconductivity	Normal Metal	Superconductor	$\langle \psi^*_\sigma \psi^*_{-\sigma} \rangle = Fe^{i\varphi}$	(no)	2	2	2nd (Gauge; no 3rd order terms)	no (plasmons)	diffusive fluctuations of gap	Mostly not	super conductivity	No: Penetration depth	Flux lines (or normal domains in Type I)
He II	Normal liquid	Superfluid	$\langle \psi \rangle$	(no)	2	2	2nd (Gauge; no 3rd order terms)	phonons (1 branch)	diffusive fluctuations of $\langle \psi \rangle$	2nd sound, especially	superfluidity	Yes, vortex lines unscreened	vortex lines
Nematic liquid crystal	Normal liquid	oriented liquid	$\overleftrightarrow{\|d\|}$ (directrix)	(no)	3 {complex topology}	3	2nd	not stable at T = O (yes in principle)	Yes	usually overdamped	various peculiar properties: orientation elasticity	Yes	disclinations, points
Cholesteric, Smectic liquid crystal	Nematic	Density wave	$\rho(Q)$	(no)	>3 {complex topology}	>3	2nd or 1st	″	Yes	″	″	Yes	disclinations, points and dislocations
Crystal	liquid	solid	ρ_G, all G on recip. lattice	(no)	3(2 orient, 1 phase) at least	3	1st	yes: 3 kinds 2 transverse 1 longitudinal	phonons	2nd sound in solid, etc.	rigidity	Yes: elasticity effects	dislocations, grain boundaries, points (vacancies, interstitials)
He 3	normal liquid	anisotropic superfluid	$d_{ij} = \langle \psi\psi \rangle_{M_L, M_S}$	(no)	at least 3 {complex topology}	18	2nd	yes, several kinds	yes, complex	probably	superfluidity and orientation elasticities	Yes	Vortex lines, disgyrations, point defects
CDW's	normal electron gas (2-dim.)	Incommensurate Density wave → commensurate	ρ_G G on triangular superlattice		2	2	1st	Yes, phasons		Yes	Yes, $NbSe_3$ sliding CDW's	?	discommensurations, dislocations

The Meissner effect is a quantum effect, which occurs in superconductors, but would not occur in a perfect classical conductor.

In order to see why the constant in the London equation is zero, we need to see how the Schrödinger equation is modified in the presence of an (electro)magnetic field:

Maxwell's equations: (cgs units)

$$\nabla \cdot \vec{B} = 0$$
$$\nabla \cdot \vec{E} = 4\pi \rho_e$$
$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J}_e + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$
$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

Vector and scalar potentials

$$\vec{B} = \nabla \times \vec{A}, \qquad \vec{A}(\vec{r},t) = \text{vector potential}$$

$$\vec{E} = -\nabla V - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \qquad V = \text{scalar potential}$$

Force on a charged particle

(classical):

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

Gauge invariance:

$$\vec{A}' = \vec{A} + \nabla f(\vec{r},t)$$

$$V' = V - \frac{1}{c}\frac{\partial f}{\partial t}$$

$$\vec{B}' = \vec{B}, \qquad \vec{E}' = \vec{E}$$

Similar symmetry in QM

$$\psi'(\vec{r},t) = e^{i\theta(\vec{r},t)}\psi(\vec{r},t)$$

$\rho(\vec{r},t) = |\psi|^2$ unchanged

But $\vec{P}\psi' = e^{i\theta}(\vec{P} + \hbar \nabla\theta)\psi.$

(Recall $\vec{P} = \frac{\hbar}{i}\nabla$ = momentum operator.)

Define $\vec{P}' = \frac{\hbar}{i}\nabla - \hbar\nabla\theta$

Then $\vec{P}'\psi' = e^{i\theta}\vec{P}\psi$

$\vec{J} = \mathrm{Re}\left\{\psi^* \frac{\vec{P}}{m}\psi\right\} = \mathrm{Re}\left\{(\psi')^* \frac{\vec{P}'}{m}\psi'\right\}$

Physical observables (e.g. ρ, $\vec{J}$) are unchanged under the transformation

$$\psi \to e^{i\theta(\vec{r},t)}\psi$$

$$\vec{P} \to \vec{P} - \hbar\nabla\theta$$

Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + q V(\vec{r},t)\, \psi$$

$$\psi = e^{-i\theta} \psi'$$

$$i\hbar \frac{\partial \psi'}{\partial t} + \hbar \frac{\partial \theta}{\partial t} \psi' = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \hbar \nabla\theta\right)^2 \psi' + q V(\vec{r},t)\, \psi'$$

$$i\hbar \frac{\partial \psi'}{\partial t} = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \hbar\nabla\theta\right)^2 \psi' + q\left(V - \frac{\hbar}{q}\frac{\partial \theta}{\partial t}\right)\psi'$$

Looks like a gauge transformation with

$$f(\vec{r},t) = \frac{\hbar c}{q}\, \theta(\vec{r},t)$$

If we introduce the "kinetic momentum"

$$\vec{P}_{kin} = \frac{\hbar}{i}\nabla - \frac{q}{c}\vec{A},$$

then under a gauge transformation

$$\vec{P}'_{kin} = \vec{P}_{kin} - \frac{q}{c}\nabla f = \vec{P}_{kin} - \hbar\nabla\theta \checkmark$$

The gauge-invariant form of Schrödinger's equation is

$$\boxed{i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{q}{c}\vec{A}(\vec{r},t)\right)^2\psi + q\,V(\vec{r},t)\,\psi}$$

Cf. Classical Hamiltonian:

$$H = \frac{1}{2m}\left(\vec{P} - \frac{q}{c}\vec{A}\right)^2 + q\,V,$$

$$m\vec{v} = \vec{P} - \frac{q}{c}\vec{A}$$

$\vec{p}$ = "canonical momentum"

Quantum mechanically, the electric current is

$$\vec{J}_e = q \, \mathrm{Re}\{\psi^* \vec{v} \, \psi\}$$

$$\equiv \frac{q}{m} \mathrm{Re}\{\psi^* (\vec{P} - \tfrac{q}{c}\vec{A}) \, \psi\}$$

$$\boxed{\vec{J}_e = \frac{q}{m} \mathrm{Re}\left\{\psi^* \left(\frac{\hbar}{i}\nabla - \frac{q}{c}\vec{A}\right)\psi\right\}}$$

In a superconductor, the charge carriers condense into a single, macroscopic wave function

$$\psi_s(\vec{r}) = \sqrt{n_s(\vec{r})} \, e^{i\theta(\vec{r})}.$$

$$n_s(\vec{r}) = |\psi_s(\vec{r})|^2$$

$$\Rightarrow \vec{J}_e = \frac{n_s q}{m}\left(\hbar \nabla\theta - \frac{q}{c}\vec{A}\right)$$

$$\nabla \times \vec{J}_e = -\frac{n_s q^2}{mc} \nabla \times \vec{A}$$

$$\nabla \times \vec{J}_e + \frac{n_s q^2}{mc} \vec{B} = 0 \qquad !$$

London equation follows trivially from QM def. of current, provided all carriers are in the same wavefunction $\psi_s(\vec{r})$.

Q: If electrons are fermions, and must obey the Pauli exclusion principle, how can they all have the same wave function in a superconductor?!

A: They don't! The charge

carriers in a SC are "Cooper pairs" of electrons, with charge $q = -2e$ and mass $m = 2m_e$. Such pairs of electrons are bosons, and can condense into a single "ground state" wave function.

Penetration depth

At the surface of a SC, the currents which screen out magnetic fields from the interior flow, and the magnetic field can penetrate a short distance.

Combining the London equation $\nabla \times \vec{J}_e = -\frac{n_s q^2}{mc} \vec{B}$

and Ampere's law (for static fields)

$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J}_e$ gives

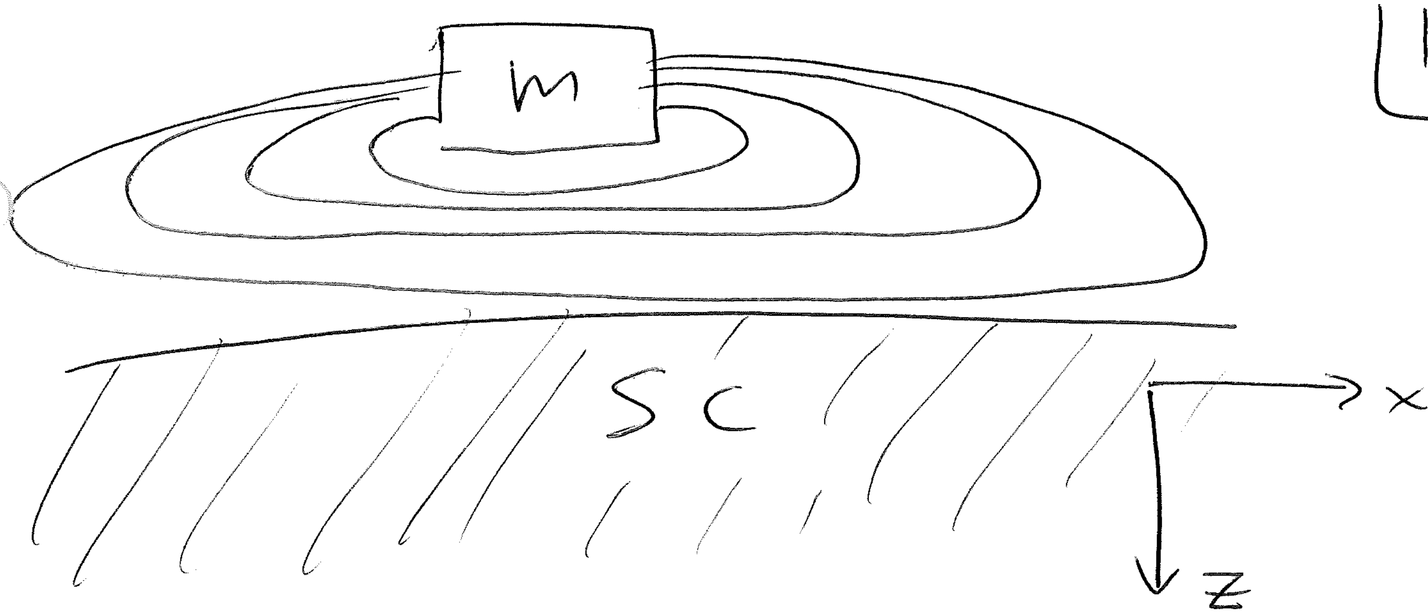
$$\nabla \times (\nabla \times \vec{B}) = -\nabla^2 \vec{B} = \frac{4\pi}{c} \nabla \times \vec{J}_e$$

$$\Rightarrow \quad \nabla^2 \vec{B} = \frac{4\pi n_s q^2}{mc^2} \vec{B}$$

Define the London penetration depth

$$\lambda_L = \sqrt{\frac{mc^2}{4\pi n_s q^2}}$$

$$\nabla^2 \vec{B} = \lambda_L^{-2} \vec{B}$$



Neglecting edge effects,

$$\vec{B}(\vec{r}) = \hat{x}\, B_x(z)$$

$$\frac{d^2 B_x}{dz^2} = \frac{1}{\lambda_L^2} B_x(z)$$

sol'n: $B_x(z) = B_x(0)\, e^{-z/\lambda_L}$

$\Rightarrow$ Field penetrates exponentially, with a decay length λ_L.