Physics 460 Lecture 6.2

Electron in a uniform static magnetic field

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + qV$$

For static fields, a convenient gauge is the Coulomb gauge

$$\nabla \cdot \vec{A} = 0$$

Recall $[f(x), p_x] = i\hbar \frac{df}{dx}$

$$\Rightarrow \quad \vec{A}\cdot\vec{p} - \vec{p}\cdot\vec{A} = i\hbar \nabla\cdot\vec{A} = 0$$

The vector potential commutes with momentum in the Coulomb gauge.

Landau levels

While the quadratic term in $\vec{B}$ can be neglected for bounded systems, such as atoms, in weak fields, this is not true for free particles.

Consider a two-dimensional electron gas in the x-y plane in a field $\vec{B} = B\hat{z}$.

Instead of the symmetric gauge, it is convenient, for this problem to work in the <u>Landau gauge</u>

$$\vec{A} = (-yB, 0, 0)$$

$$H = \frac{1}{2m}\left(p_x - \frac{eBy}{c}\right)^2 + \frac{p_y^2}{2m}$$

$[p_x, H] = 0$, so seek sol'n of the form $\psi(x,y) = e^{ikx}\chi(y)$.

$$H\psi(x,y) = \frac{1}{2m}\left(\hbar k - \frac{eBy}{c}\right)^2\psi + \frac{p_y^2}{2m}\psi$$

$$E\chi(y) = \left[\frac{p_y^2}{2m} + \frac{1}{2m}\left(\hbar k - \frac{eBy}{c}\right)^2\right]\chi$$

$$E\chi(y) = \frac{p_y^2}{2m}\chi + \frac{e^2B^2}{2mc^2}\left(y - \frac{\hbar c k}{eB}\right)^2\chi$$

$$E\chi = \left[\frac{p_y^2}{2m} + \frac{m\Omega^2}{2}(y-y_0)^2\right]\chi$$

$$\boxed{\Omega = \frac{eB}{mc} = \text{cyclotron frequency}}$$

$$y_0 = \frac{\hbar c k}{eB}$$

$\Rightarrow$ shifted harmonic oscillator

$$\boxed{E_n = \hbar \Omega \left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots, \infty \quad \text{Landau levels}}$$

Degeneracy

Suppose system has dimensions L_x, L_y. Impose periodic BCs in x-direction.

$$e^{ikL_x} = 1 \Rightarrow k = \frac{2\pi n_x}{L_x}$$

Moreover, $0 \leq y_0 \leq L_y$

$$0 \leq \frac{\hbar c}{eB} \frac{2\pi n_x}{L_x} \leq L_y$$

$$0 \leq n_x \leq \frac{eB}{hc} L_x L_y$$

Total number of states in a given Landau level of energy $E_n = \hbar\Omega(n + 1/2)$ is

thus $N_n = \frac{eB}{hc} A$

$$N_n = \frac{BA}{\phi_0}, \quad \phi_0 = \frac{hc}{e}$$

(normal flux quantum)

Filling factor ν

For a total # N of electrons, the total # ν of Landau levels populated is

$$\nu = \frac{N}{N_n} = \frac{N\phi_0}{BA}$$

$$= \frac{\text{\# electrons}}{\text{flux quantum}}$$

Quantum Hall Effect

$$\frac{h}{e^2} = 25.8\,k\Omega$$

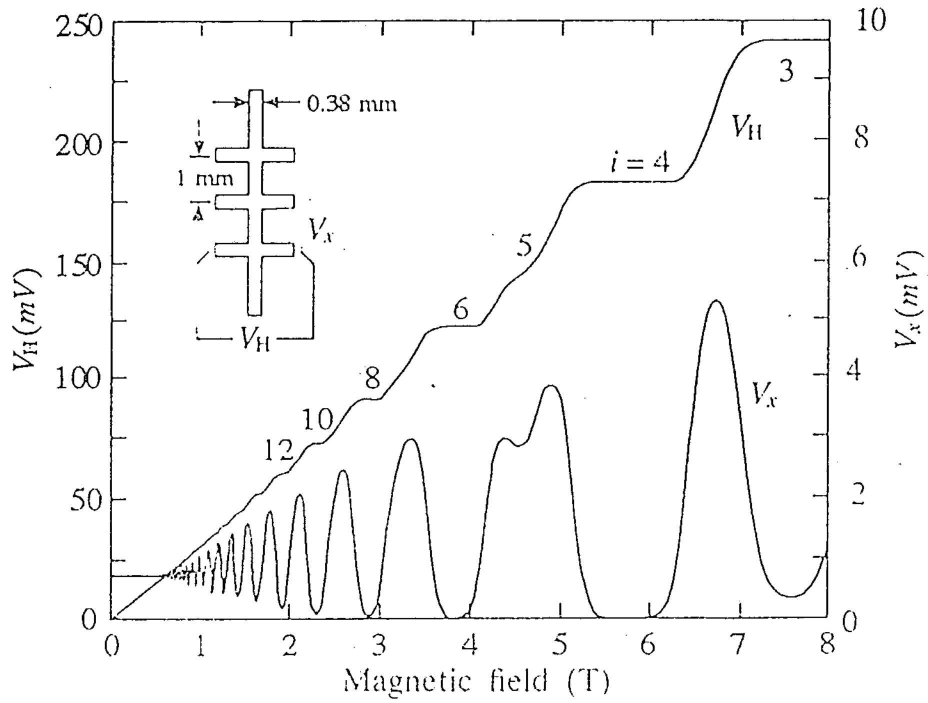


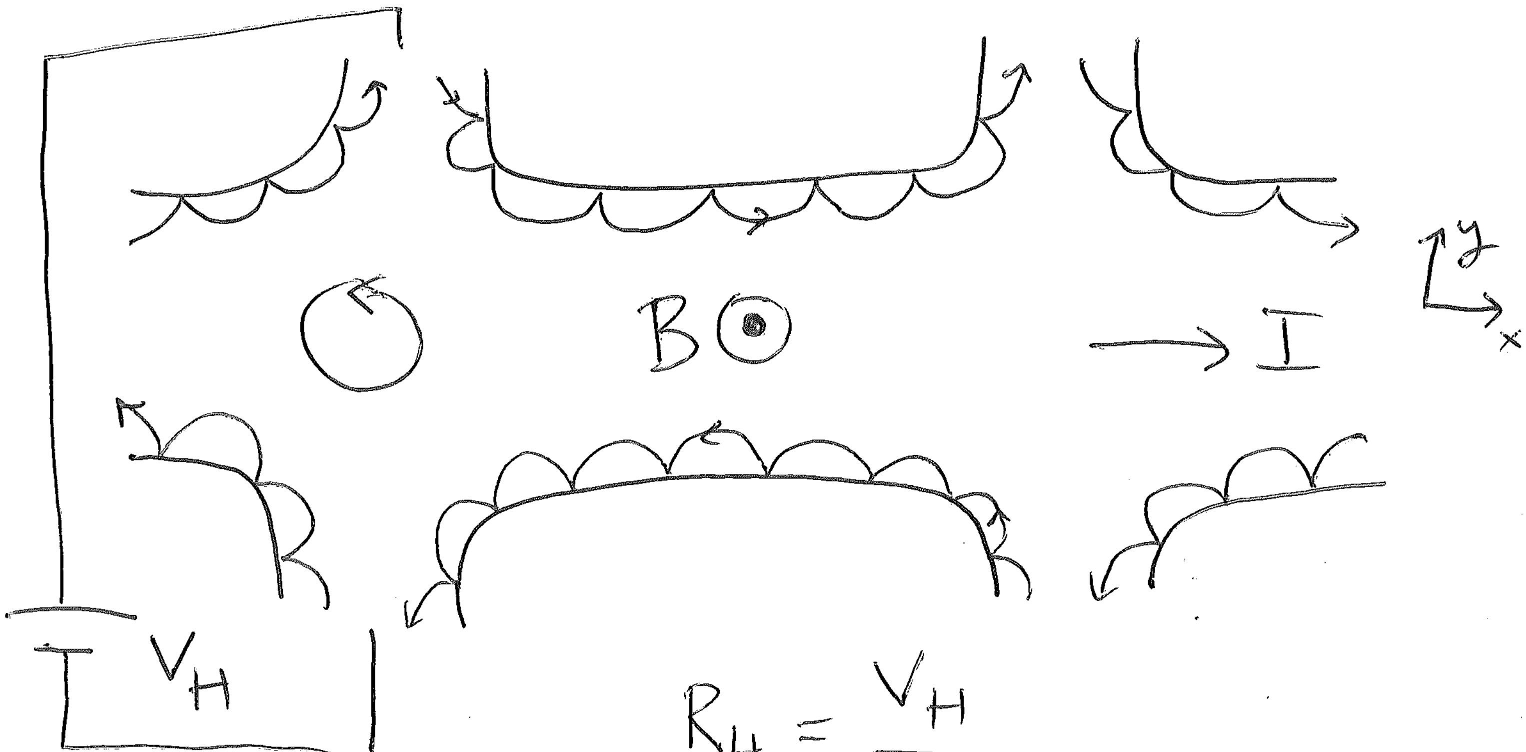
Fig. 1.4.2. Measured longitudinal and transverse voltages for a modulation-doped GaAs film at $T = 1.2$ K ($I = 25.5$ μA). Reproduced with permission from Fig. 1 of M. E. Cage, R. F. Dziuba and B. F. Field (1985), *IEEE Trans. Instrum. Meas.* IM-34, 301. © 1985 IEEE

$$R_H = \frac{h}{\cancel{N}e^2}$$

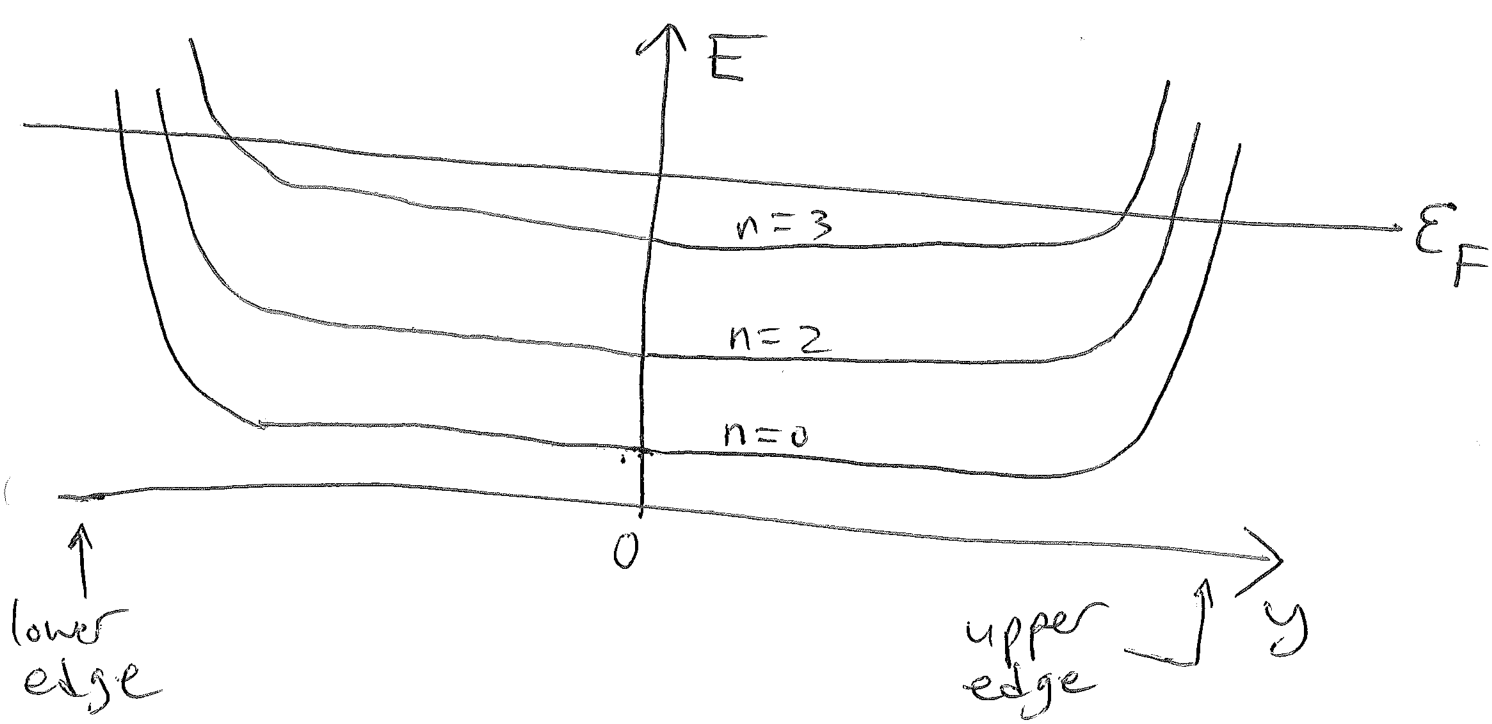
classically: $R_H = \frac{B}{ne}$

Quantum Hall effect

Current carried by quantized "edge states":



$$R_H = \frac{V_H}{I}$$



Group velocity

$$V_x^{(n)} = \frac{1}{\hbar} \frac{\partial E_n}{\partial k_x}$$

Recall $\quad y_0 = \frac{\hbar c\, k_x}{eB}$

$$V_x^{(n)} = \frac{1}{\hbar} \frac{\hbar c}{eB} \frac{\partial E_n}{\partial y_0}$$

$$E_n(y_0) \simeq \hbar \Omega \left(n + \tfrac{1}{2}\right) + U(y_0)$$

$U(y)$ = confining potential

$$V_x^{(n)} \simeq \frac{c}{eB} \frac{\partial U}{\partial y_0}$$

approx. indep. of quantum # n

states at upper edge travel right, lower go left.

These edge states form ideal one-dimensional channels. The # of channels is equal to the # of Landau levels lying below $\varepsilon_F \Rightarrow \nu$ channels.

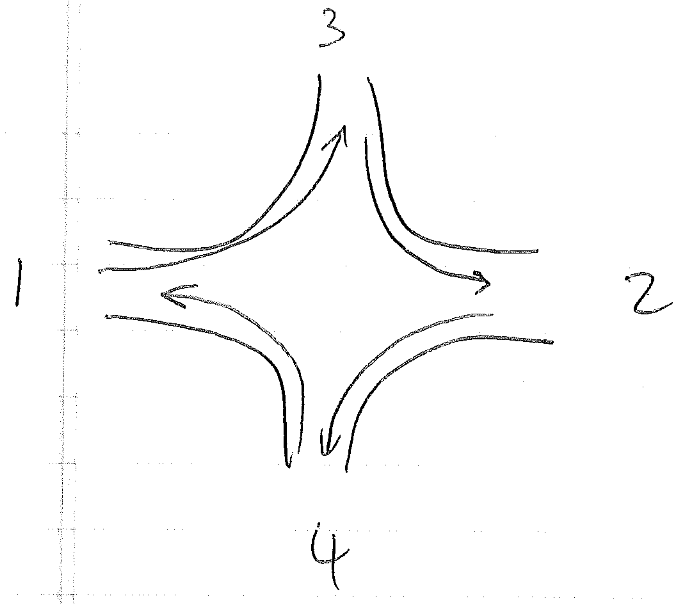
$$R_H = \frac{V_H}{I} = \frac{h}{\nu e^2}$$

Recall degeneracy $N_n = \frac{eBA}{hc}$

$$\nu = \mathrm{Int}\left\{ \frac{N}{N_n} \right\} = \mathrm{Int}\left\{ \rho_{2D} \frac{hc}{eB} \right\}$$

Classical limit $\nu \approx \rho_{2D} \frac{hc}{eB}$

$$R_H \approx \frac{h}{\nu e^2} \approx \frac{B}{\rho_{2D}\, ec} \quad \checkmark$$



$$I_4 = \frac{\nu e}{h}(\mu_2 - \mu_4) = 0 \quad \text{(voltage probe)}$$

$$\Rightarrow \mu_4 = \mu_2$$

$$I_3 = \frac{\nu e}{h}(\mu_1 - \mu_3) = 0 \quad \text{(voltage probe)}$$

$$\Rightarrow \mu_3 = \mu_1$$

$$I_2 = \frac{\nu e}{h}(\mu_3 - \mu_2) = \frac{\nu e}{h}\underbrace{(\mu_3 - \mu_4)}_{eV_H}$$