Practice Problems for Midterm 1

Actual exam will be shorter! Calculator and crib sheet (8.5"x11", one side) allowed. Show work for full credit.

1) Fundamental quanta in solid-state physics

Two fundamental quanta were discovered in solid-state physics in the past 50 years. In cgs units, they are $hc/2e$ and h/e^2. Explain what these quanta are, and give derivations of their values and/or explain how they can be measured in the laboratory.

2) Fermi gas

Consider a system of N fermions of mass m moving freely in a box of volume V.

a) Derive the value of the Fermi wavevector k_F and the Fermi energy ε_F.

b) Compute the total energy of the system in its ground state.

c) What is the probability $f(\varepsilon)$ that a quantum energy level ε in the box will be occupied if the system is in thermal equilibrium at temperature T and chemical potential μ?

3) Boltzmann equation

The Boltzmann equation in the relaxation-time approximation is

$$\partial f/\partial t + \mathbf{v} \cdot \partial f/\partial \mathbf{r} + \mathbf{F} \cdot \partial f/\partial \mathbf{p} = -(f - f_0)/\tau,$$

where f_0 is a local equilibrium distribution and you may assume $\tau = \text{const}$.

Consider a bulk system of electrons subject to a constant electric field $\mathbf{E}$.

a) Which of the terms on the left-hand-side of the Boltzmann equation are zero in this case (and why)?

b) Solve for the nonequilibrium distribution $f_1 \equiv f - f_0$ to leading order in $\mathbf{E}$.

c) Show that the electric conductivity tensor for free electrons ($\mathbf{v} = \mathbf{p}/m$) is given by

$$\sigma_{ij} = \frac{ne^2\tau}{m}\delta_{ij}.$$

Note: The electron density and electrical current density are given in terms of the nonequilibrium distribution $f(\mathbf{r}, \mathbf{p}, t)$ by

$$n(\mathbf{r}, t) = \int \frac{d^3p}{h^3} f(\mathbf{r}, \mathbf{p}, t), \quad \mathbf{j}(\mathbf{r}, t) = -e \int \frac{d^3p}{h^3} \mathbf{v} f_1(\mathbf{r}, \mathbf{p}, t),$$

4) Wiedemann-Franz law for quantum conductors

Show that the Wiedemann-Franz law relating the thermal conductance κ to the electrical conductance G,

$$\frac{\kappa}{GT} = \frac{\pi^2}{3} \frac{k_B^2}{e^2},$$

holds also for pure quantum systems, for which the Onsager linear-response coefficients are given by

$$L^{(\nu)} = \frac{1}{h} \int dE \, (E - \mu_0)^\nu \, \mathcal{T}(E) \left(-\frac{\partial f_0}{\partial E} \right),$$

where $\mathcal{T}(E)$ is the quantum mechanical transmission probability through the (two-terminal) device. For simplicity, you may take $L^{(1)} = 0$.

Useful integral:

$$\int_{-\infty}^{\infty} dx \frac{x^2 e^x}{(e^x + 1)^2} = \frac{\pi^2}{3}.$$