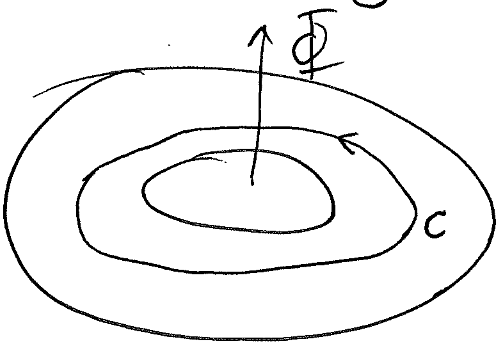
Phys 460 Practice Problem solutions

1) a) $\frac{hc}{2e} = \phi_0 =$ superconducting flux quantum

b) $\frac{h}{e^2} = 25{,}812.8\ \Omega =$ electrical resistance quantum

a) SC ring



In body of sc,

$$\Psi = \sqrt{n_s}\, e^{i\theta}$$

$$\vec{J} = \frac{1}{m}\,\mathrm{Re}\left\{\Psi^*\left(\vec{p} - \frac{q}{c}\vec{A}\right)\Psi\right\}$$

$$= \frac{1}{m}\left(\hbar\nabla\theta - \frac{q}{c}\vec{A}\right)n_s = 0$$

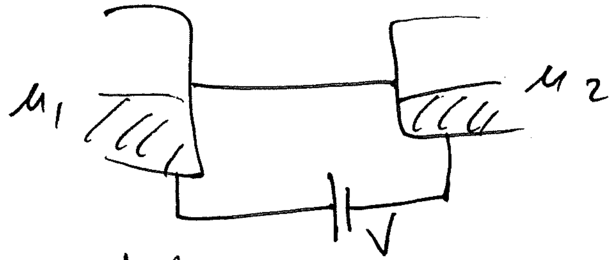
$$\Rightarrow \quad \nabla\theta = \frac{q}{\hbar c}\vec{A} \qquad \oint_c \nabla\theta\cdot d\vec{l} = \frac{q}{\hbar c}\oint_c \vec{A}\cdot d\vec{l}$$

$$2\pi s = \frac{q}{\hbar c}\Phi \quad , \quad s\in\mathbb{Z}$$

$$\Phi = \frac{2\pi\hbar c}{q}s = \frac{hc}{q}s = \frac{hc}{2e}s$$

Perfect 1D wire $\qquad$ e = electron charge

b)



$$\mu_1 - \mu_2 = eV$$

$$I = e\frac{dN}{dt} \qquad d^2N = \frac{dx\,dp}{h}$$

$$\frac{dN}{dt} = \frac{1}{h}\frac{dx}{dt}\int_{p_{F_2}}^{p_{F_1}} dp \cong \frac{v_F}{h}\left(p_{F_1} - p_{F_2}\right)$$

$$\frac{dN}{dt} \simeq \frac{v_F}{h}\frac{\mu_1 - \mu_2}{\frac{\partial \varepsilon}{\partial p}} = \frac{v_F}{h}\frac{\mu_1 - \mu_2}{v_F}$$

$$I = e\frac{dN}{dt} = e\frac{\mu_1 - \mu_2}{h} = \frac{e^2}{h}V; \quad R = \frac{h}{e^2}$$

---

2) $$N = 2\int \frac{d^3x\,d^3p}{h^3}\,\theta\!\left(\varepsilon_F - \frac{\vec{p}^{\,2}}{2m}\right)$$

spin

$$N = \frac{2V}{h^3}\frac{4\pi}{3}p_F^3 = \frac{8\pi V}{3\,h^3}\hbar^3 k_F^3 = \frac{V k_F^3}{3\pi^2}$$

a) $$k_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3} \qquad \varepsilon_F = \frac{\hbar^2 k_F^2}{2m}$$

b) $$E_0 = 2\int \frac{d^3x\, d^3p}{h^3} \frac{\vec{p}^{\,2}}{2m}\, \theta\left(\varepsilon_F - \frac{\vec{p}^{\,2}}{2m}\right)$$

$$= \frac{2V}{h^3} \int_0^{p_F} 4\pi p^2 \frac{p^2}{2m}\, dp$$

$$= \frac{4\pi V}{m h^3} \int_0^{p_F} p^4\, dp = \frac{4\pi V}{m h^3} \frac{p_F^5}{5}$$

$$= \frac{8\pi V}{5h^3} \frac{p_F^2}{2m} \hbar^3 k_F^3 = \frac{3}{5} N \varepsilon_F$$

c) $$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1}, \quad \beta = \frac{1}{k_B T}$$

---

3) a) $\frac{\partial f}{\partial t} = 0$ because $\vec{E} = \text{const.}$

$\frac{\partial f}{\partial \vec{r}} = 0$ because $\vec{E} = \text{const.}$

b) $\vec{F} = e\vec{E}$ $\qquad e\vec{E} \cdot \frac{\partial f}{\partial \vec{p}} = -\frac{f_1}{\tau}$

$f_1 \simeq -e\tau \vec{E} \cdot \frac{\partial f_0}{\partial \vec{p}}$ $\qquad \frac{\partial f_0}{\partial \vec{p}} = \frac{\partial f_0}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial \vec{p}}$

$$\frac{\partial \varepsilon}{\partial \vec{p}} = \vec{v}$$

$$\boxed{f_1 \simeq + e\tau \vec{E} \cdot \vec{v} \left(-\frac{\partial f_0}{\partial \varepsilon}\right)}$$

c) $$\vec{J} = e \int \frac{d^3p}{h^3} \vec{v} f_1 = e^2\tau \int \frac{d^3p}{h^3} \vec{v} (\vec{E}\cdot\vec{v}) \left(-\frac{\partial f_0}{\partial \varepsilon}\right)$$

$$= e \int \frac{d^3p}{h^3} \vec{v} \left(-e\tau \vec{E} \cdot \frac{\partial f_0}{\partial \vec{p}}\right)$$

$$= -\frac{e^2\tau}{m} \int \frac{d^3p}{h^3} \vec{p}\, \vec{E} \cdot \frac{\partial f_0}{\partial \vec{p}}$$

$$J_i = \sum_j \sigma_{ij} E_j$$

$$\sigma_{ij} = -\frac{e^2\tau}{m} \int \frac{d^3p}{h^3} p_i \frac{\partial f_0}{\partial p_j}$$

$\downarrow$ integrate by parts

$$\sigma_{ij} = \frac{e^2\tau}{m} \underbrace{\int \frac{d^3p}{h^3} f_0(\vec{p})}_{n} \delta_{ij} = \frac{ne^2\tau}{m} \delta_{ij}.$$

4) $$G = e^2 L^{(0)}$$

$$K \simeq \frac{L^{(2)}}{T} \qquad \text{(neglecting } L^{(1)}\text{)}$$

$$\frac{K}{GT} = \frac{L^{(2)}}{e^2 T^2 L^{(0)}}$$

$$L^{(0)} = \frac{1}{h} \int dE \, \mathcal{T}(E) \left(-\frac{\partial f_0}{\partial E}\right) \simeq \frac{\mathcal{T}(\varepsilon_F)}{h}$$

$$L^{(2)} = \frac{1}{h} \int dE \, (E-\mu_0)^2 \, \mathcal{T}(E) \left(-\frac{\partial f_0}{\partial E}\right)$$

let $x = \beta(E-\mu_0)$

$$L^{(2)} = \frac{(k_B T)^2}{h} \int_{-\beta\mu_0}^{\infty} dx \; x^2 \, \mathcal{T}\left(\mu_0 + \frac{x}{\beta}\right) \left(\frac{e^x}{(e^x+1)^2}\right)$$

$$\simeq \frac{(k_B T)^2}{h} \mathcal{T}(\mu_0) \int_{-\infty}^{\infty} dx \, \frac{x^2 e^x}{(e^x+1)^2} = \frac{\pi^2}{3} \frac{(k_B T)^2}{h} \mathcal{T}(\mu_0)$$

$$\frac{K}{GT} \cong \frac{\pi^2}{3} \frac{k_B^2}{e^2}$$